A NONTECHNICAL PRESENTATION OF VIABILITY THEORY

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> Everything that exists in the Universe
> is due to chance and necessity.
>
> Democritus

We often observe that potential users of mathematical metaphors* believe that a deterministic or stochastic framework is a prerequisite for the use of mathematical techniques. This is one of the reasons put forward for the rejection of mathematics in the so-called "soft sciences" such as economics.

A very severe misunderstanding lies at the heart of this belief. As in the physical sciences, specialists in the soft sciences wish mathematicians to construct predictive models for them. But only in dynamical systems (i.e., evolutionary systems) which can evolve at will from any starting point can predictions have any real meaning -- and to estimate them requires not only that experimentation should be possible but also that some knowledge of the future environment of such systems should be available.

Obviously, this is not the case in many of the "macrosystems" arising in economics and the social sciences. To study such macrosystems we should take into account not only:

(1) our ignorance of the future environment of the system

but also:

(2) the absence of determinism (including the impossibility of a comprehensive description of the dynamics of the system)

(3) our ignorance of the laws relating certain controls to the states of this system

(4) the variety of dynamics available to the system.

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* Mathematical metaphors. Like other means of communication (languages, painting, music, etc.) mathematics provides metaphors that can be used to explain a given phenomenon by associating with it some other phenomenon that is more familiar, or at least is believed to be more familiar. This feeling of familiarity, individual or collective, inborn or acquired through education, is responsible for the inner conviction that this phenomenon is understood.

The first task of Viability Theory is to describe the evolution of dynamical systems such that, at each instant, the velocity depends in a multivalued way (i.e., nondeterministically) upon

(1) the present state (or history) of the system
(2) various regulatory controls

and to study their mathematical properties.

This is done by means of <u>differential inclusions</u> (with or without memory), as opposed to <u>differential equations</u>, which assume that the velocity depends in a unique way upon the current state of the system.

We assume that the regulatory controls have a high inertia and change only under the most severe conditions. Naturally, we also expect such dynamical systems to possess many possible trajectories due to the lack of determinism and the choice among several regulatory controls.

The first questions addressed by Viability Theory deal with the mathematical <u>structure</u> of these sets of trajectories. Once these questions have been answered, the problem of <u>selecting</u> trajectories arises.

<u>Optimal Control Theory</u> furnishes one class of selection procedures in which a cost is associated with each trajectory and this cost is then minimized. Implicit assumptions include:

(1) the existence of a decision maker operating the controls of the system (there may be more than one decisionmaker in a game-theoretical setting)

(2) the availability of information (deterministic or stochastic) on the future of the system; this is necessary to define the costs associated with the trajectories

(3) that decisions (even if they are conditional) are taken once and for all at the starting point.

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* (cont'd)

The construction of mathematical metaphors naturally requires autonomous development in the field responsible for providing theories to be associated with unexplained phenomena: this is the domain of pure mathematics. The development of the mathematical sciences obeys its own logic, as in other fields such as literature, music, painting, etc. In all these areas, aesthetic satisfaction is both an aim to be achieved and a signal by which successful work can be recognized. (In all these domains, too, fashion -- or social concensus -- influences the aesthetic criteria by which the work is judged.)

Viability Theory proposes another class of selection methods in which we choose only the trajectories that, at each instant, obey given restrictions known as viability constraints. These constraints determine a region of state space, called the viability domain; viable trajectories are those lying entirely within the viability domain. The viability domain can depend upon time, the present state or history of the system, the regulatory controls, and so on.

Viability Theory makes explicit the necessary and sufficient conditions for the existence of at least one viable trajectory starting from any viable initial state. It also provides the feedbacks (concealed in both the dynamics and the viability constraints) which relate the state of the system to the controls. These feedbacks are not necessarily deterministic: they are set-valued maps associating a subset of controls with each state of the system. We observe that the larger these subsets of controls are, the more flexible -- and, thus, the more robust -- the regulation of the system will be.

Viability Theory shows that as long as the state of the system lies within the viability domain (but not on the boundary), any regulatory control will work and, therefore, that the system can maintain the control inherited from the past. (The regulatory control remains constant, or changes very slowly, even though the state may evolve quite rapidly.)

What happens when the state reaches the boundary of the viability domain? If the chosen velocity is "inward" in the sense that it pushes the trajectory back into the interior of the domain, then we can still keep the same regulatory control.

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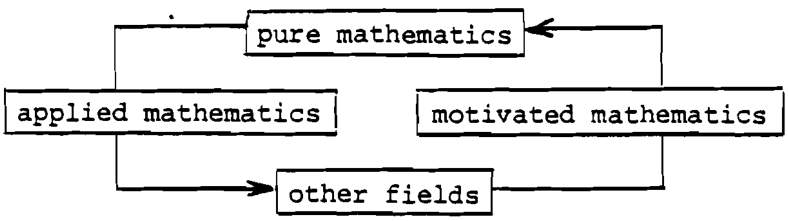
* (cont'd)

We have already described a mathematical metaphor as a means of relating mathematical theory with certain phenomena. This association can be developed in two ways. The first possibility is to look for a mathematical theory which can be linked as closely as possible with the phenomenon under consideration. This is usually regarded as the domain of applied mathematics.



However, if the chosen velocity is "outward", we are in a period of crisis and must either:

(1) find another regulatory control such that the new associated velocity pushes the trajectory back into the interior of the viability domain

or:

(2) operate on the viability domain, enlarging it in such a way that the state of the system lies in the interior of the new viability domain.

When these two strategies for "structural change" fail, the trajectory "dies" i.e., it is no longer viable (see Figure 1).

Viability Theory also reveals a division of the viability domain into "cells"; each cell is the subset of viable states which can be regulated by a given control. To pass from one cell to another requires the control to be changed. The boundaries of these cells signal the need for structural change (see Figure 2)

Viability Theory cannot be said to provide deterministic mathematical metaphors, since there may be many feasible solutions, but, on the other hand, it does have the virtue of showing that certain trajectories are not viable.

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* (cont'd)

However, it is also possible to approach the problem from the opposite direction. Other fields provide mathematicians with metaphors, by suggesting new concepts and lines of argument, by giving some inkling of possible solutions, or by developing new modes of intuition: and this is the domain of what can be called "motivated mathematics".

For the time being, Viability Theory lies within the domain of motivated mathematics: and it still may not provide an ideal description of the evolution of macrosystems. It is possible that potential users (economists, biologists) are disappointed or discouraged by the results obtained so far -- it is still too early for Viability Theory to be "applied" in the engineering sense. Nevertheless, the motivation provided by the study of macrosystems is of benefit to mathematicians in that it renews and enriches the theory of dynamical systems and differential equations.

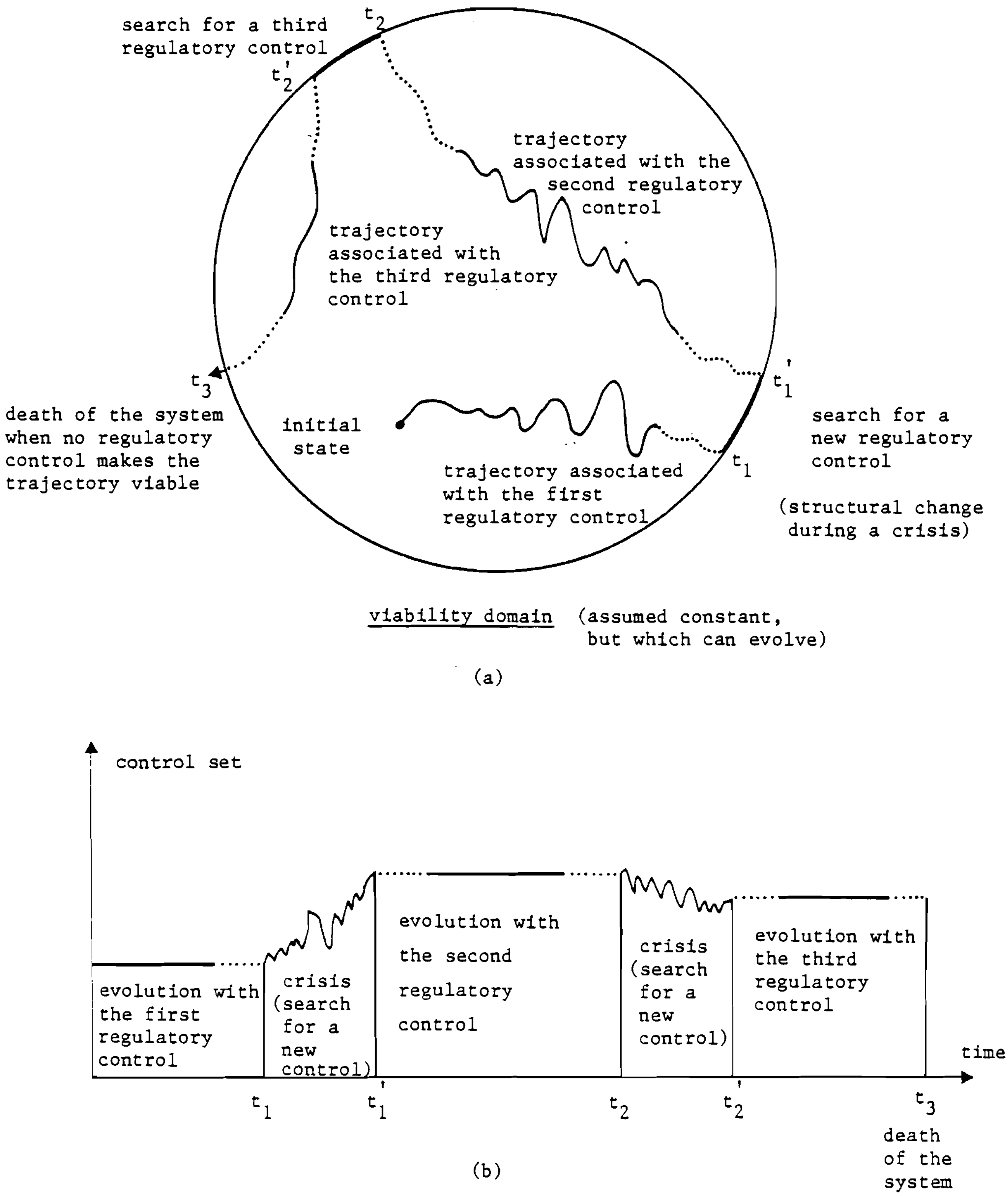

Figure 1. (a) Evolution of the state (in the state space);
(b) evolution of the control.

The dotted lines represent the parts of the trajectories close to the boundary of the viability domain which disappear after a certain period of time.

We should also mention another aspect of Viability Theory -- that concerned with complexity and robustness. It may be observed that the state of the system becomes increasingly robust the further it is from the boundary of the viability domain. Therefore, after some time has elapsed, only the parts of the trajectories furthest away from the viability boundary will remain. This fact may explain the apparent discontinuities ("missing links") and hierarchical organization arising from evolution in certain systems.*

In summary, the <u>main purpose of Viability Theory is to explain the evolution of a system, given feasible dynamics and constraints, and to reveal the concealed feedbacks which allow it to be regulated</u>. This involves the use of a policy, <u>opportunism</u>, which enables the system to conserve viable trajectories that its <u>lack of determinism</u> -- the availability of several feasible velocities -- makes possible. This provides a mathematical metaphor of the deeply intuitive statement of Democritus "Everything that exists in the Universe is due to chance and necessity".

Viability Theory can be adopted in many problems. Here we shall illustrate how it can account for the evolution of prices as a mechanism for the decentralization of a simple economic system.

Consumers must, at each instant, share a consumption bundle constrained to evolve in a set of available (scarce) commodities. This series of allocations determines a viable trajectory.

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*For the first time, excavations at Kenya's Lake Turkana have provided clear fossil evidence of evolution from one species to another. The rock strata there contain a series of fossils that show every small step of an evolutionary journey that seems to have proceeded in fits and starts. Peter Williamson of Harvard University examined 3,300 fossils showing how thirteen species of molluscs changed over several million years. What the record indicated was that the animals stayed much the same for immensely long stretches of time. But twice, about 2 million years ago and then again 700,000 years ago, the pool of life seemed to explode -- set off, apparently, by a drop in the lake's water level. In an instant of geologic time, as the changing lake environment allowed new types of molluscs to win the race for survival, all of the species evolved into varieties sharply different from their ancestors. That intermediate forms appeared so quickly, with new species suddenly evolving in 5,000 to 50,000 years after millions of years of constancy, challenges the traditional theories of Darwin's disciples since the fossils of Lake Turkana don't record any gradual change; rather, they seem to reflect eons of stasis interrupted by brief evolutionary "revolutions". (See Palaeontological documentation of speciation in Cenozoic Molluscs from Turkana Basin, by P.G. Williamson, <u>Nature</u>, Vol. 293, (1981), p. 437.)

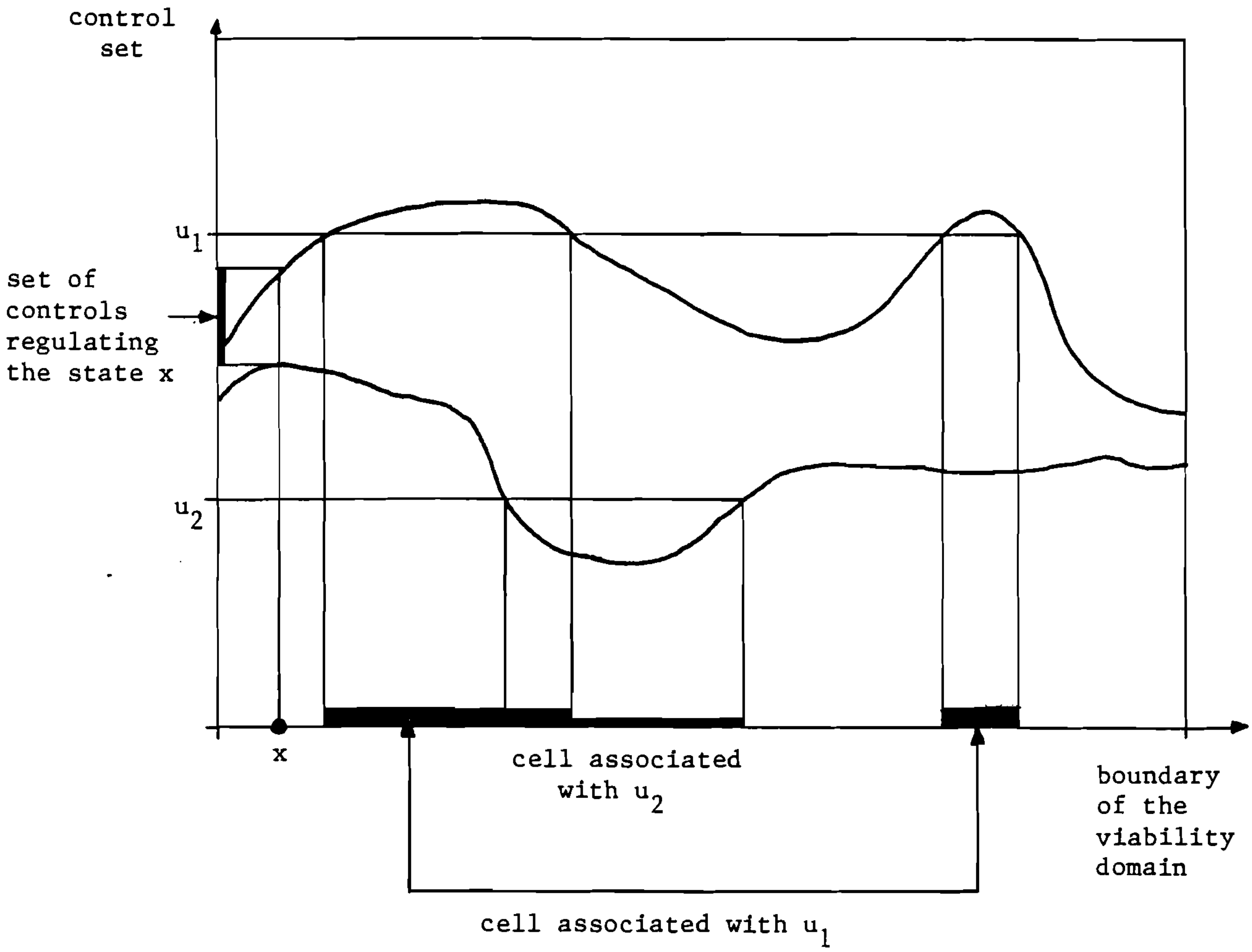

Figure 2. Graph of the Feedback map and division of the state space into cells.

There are several regulatory mechanisms which would yield viable trajectories. For instance, regulation could be achieved by queuing when shortages occur, i.e., when the total consumption leaves the set of available resources. A second mechanism involves the introduction of one or more fictitious goods for which the scarcity constraint can be transgressed. These are essentially fiduciary goods which, unlike physical goods, are limited only by measures dictated by the trust (or rather, the tolerance) of the set of consumers. The disequilibrium which cannot exist in physical goods can then be transferred to the fiduciary goods.

A third mechanism uses prices as a means of control. We may observe that these three mechanisms can be combined in various ways; we can even state that the first two are correcting mechanisms for the third when it is wrongly implemented.

Let us consider only the third mechanism, as an example. A consumer is assumed to be an automaton represented by a <u>change function</u> which associates with every act of consumption and every price the velocity with which he changes his consumption. (This dynamical representation of a consumer is not the usual mathematical representation in which the consumer is assumed to maximize his utility function.) These automata associate trajectories (some of which are not necessarily viable) with every evolution of prices. We must then consider the question of whether there exists a price evolution such that the associated trajectories are viable. The answer is yes when the consumers -- the change functions -- are forbidden to "spend more than they earn."

Curiously enough, the same assumptions also imply the existence of an equilibrium, a level of prices and consumption for each consumer which is viable and does not change.

Viability Theory tells us that prices evolve through a set-valued feedback map associating a subset of prices which regulates the market with every allocation of an available commodity. In the free-market framework, Adam Smith's invisible hand chooses a price via this map, while in the planned economy the planning bureau has to "compute" a single-valued feedback map, which is selected from the set-valued feedback map.

References

There are only a few technical papers dealing with this young theory. Most of the mathematical results are presented in the book "Differential Inclusions" by J.-P. Aubin and A. Cellina, published by Springer-Verlag, and due to appear in 1983. The main results are summarized in the IIASA Working Paper WP-82-51 "Differential Inclusions and Viability Theory" by the same authors.