A DIRECTIONAL IMPLICIT FUNCTION THEOREM FOR QUASIDIFFERENTIABLE FUNCTIONS

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PREFACE

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In this paper the authors consider problems related to deriving analogous theorems in quasidifferential calculus.

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1. INTRODUCTION

In this paper we consider problems related to the derivation of analogues in quasidifferential calculus to the implicit and inverse function theorems of classical differential calculus.

Let us first recall some definitions. A function ϕ defined and finite on an open set S of E_m is called quasidifferentiable at $x \in S$ if it is directionally differentiable at x and if there exists a pair of convex compact sets $\underline{\partial}\phi(x) \subset E_m$ and $\overline{\partial}\phi(x) \subset E_m$ such that for any $g \in E_m$

$$\frac{\partial\phi(x)}{\partial g} \equiv \lim_{\alpha\to+0} \frac{1}{\alpha} [\phi(x+\alpha g)-\phi(x)] = \max_{v \in \underline{\partial}\phi(x)} (v,g) + \min_{w \in \overline{\partial}\phi(x)} (w,g).$$

The pair $D\phi(x) = [\underline{\partial}\phi(x), \overline{\partial}\phi(x)]$ is called a *quasidifferential* of ϕ at x; it is not unique. The properties of quasidifferentiable functions were first investigated in [1-3]. This work led to the development of quasidifferential calculus, which is a generalization of classical differential calculus (see, e.g.,[4-6]). Some extensions to Banach spaces are discussed in [4] and [5].

The implicit and inverse function theorems of classical differential calculus represent an essential element in the structure of the calculus and have important applications. The problem of deriving analogous theorems in quasidifferential calculus was introduced and briefly examined in [5,6]. In the present paper we continue our study of this problem.

2. AN IMPLICIT FUNCTION THEOREM

Let $z = [x,y]$, $x \in E_m$, $y \in E_n$, and let the functions $f_i(z)$ $(i \in 1:n)$ be finite quasidifferentiable on E_{m+n}.

Consider the following system:

$$f_i(x,y) = 0 \quad \forall i \in 1:n \quad .$$

This can be rewritten in the form

$$f(z) = 0 \tag{1}$$

where

$$f = (f_i,\ldots,f_n), \; 0 \in E_n \quad .$$

The problem is to find a function $y(x)$ such that

$$f_i(x,y(x)) = 0 \qquad \forall x \in 1:n, \; \forall x \in E_n \quad .$$

Unfortunately we cannot solve this very general formulation of the problem for an arbitrary quasidifferentiable system of type (1). But what we shall try to do is to solve this problem for a given direction $g \in E_m$. We shall call this a *directional implicit function problem.*

Suppose that $z_0 = [x_0,y_0]$ is a solution of system (1), i.e.,

$$f_i(z_0) = 0 \quad \forall i \in 1:n$$

Consider the system of equations

$$f(x_0 + \alpha g,\ y(\alpha)) = 0 \tag{2}$$

where $\alpha > 0$.

Since the functions f_i are quasidifferentiable for any $q \in E_n$, we have from (1)

$$f_i(x_0 + \alpha g,\ y_0 + \alpha q) = f_i(x_0, y_0) + \alpha \frac{\partial f_i(z_0)}{\partial [g,q]} + o_i(\alpha,q) = \alpha \frac{\partial f_i(z_0)}{\partial [g,q]} + o_i(\alpha,q) \tag{3}$$

where

$$\frac{\partial f_i(z_0)}{\partial [g,q]} = \max_{v_i \in \underline{\partial} f_i(z_0)} [(v_{1i},g)+(v_{2i},q)] + \min_{w_i \in \overline{\partial} f_i(z_0)} [(w_{1i},g) + (w_{2i},q)] \ . \tag{4}$$

Here $Df_i(z) = [\underline{\partial} f_i(z), \overline{\partial} f_i(z)]$ is a quasidifferential of f_i at z; $\underline{\partial} f_i(z) \subset E_{m+n}$, $\overline{\partial} f_i(z) \subset E_{m+n}$ are respectively sub- and superdifferentials of f_i at z (convex compact sets); $v_i = [v_{1i}, v_{2i}]$, and $w_i = [w_{1i}, w_{2i}]$.

Let $q_0 \in E_n$ be a solution to the quasi-linear system

$$\frac{\partial f_i(z_0)}{\partial [\hat{g}, q_0]} = 0 \qquad \forall i \in 1:n \ . \tag{5}$$

Suppose that in (3)

$$\frac{o_i(\alpha,q)}{\alpha} \xrightarrow[\alpha \to +0]{} 0 \tag{6}$$

uniformily with respect to

$$q \in S_\delta(q_0) = \{q \in E_n \mid \|q-q_0\| \leq \delta\},$$

where $\delta>0$ is fixed.

Is it possible to find a vector function $\tau(\alpha)$ with $\alpha_0>0$ such that

$$f_i(x_0 + \alpha g,\ y_0 + \alpha[q_0 + \tau(\alpha)]) = 0 \qquad \forall i \in 1:n,\ \alpha \in [0,\alpha_0] \quad (7)$$

where $\tau(\alpha) \in E_n \qquad \forall \alpha \in [0,\alpha_0]$?

Take $\varepsilon \geq 0$ and introduce the sets

$$\underline{R}_{i\varepsilon} = \{v_i \in \underline{\partial} f_i(z_0) \mid (v_{1i},g) + (v_{2i},q_0) \geqslant$$

$$\geqslant \max_{v_i \in \underline{\partial} f_i(z_0)} [(v_{1i},g) + (v_{2i},q_0)] - \varepsilon\},$$

$$\overline{R}_{i\varepsilon} = \{w_i \in \overline{\partial} f_i(z_0) \mid w_{1i},g) + (w_{2i},q_0) \leqslant$$

$$\leqslant \min_{w_i \in \overline{\partial} f_i(z_0)} [(w_{1i},g) + (w_{2i},q_0)] + \varepsilon\},$$

$$\underline{R}_i(\tau) = \{v_i \in \underline{\partial} f_i(z_0) \mid (v_{1i},g) + (v_{2i},q_0 + \tau) =$$

$$= \max_{v_i \in \underline{\partial} f_i(z_0)} [(v_{1i},g) + (v_{2i},q_0 + \tau)]\},$$

$$\overline{R}_i(\tau) = \{w_i \in \overline{\partial} f_i(z_0) \mid (w_{1i},g) + (w_{2i},q_0 + \tau) =$$

$$= \min_{w_i \in \overline{\partial} f_i(z_0)} [(w_{1i},g) + (w_{2i},q_0 + \tau)]\} .$$

It is clear that all these sets depend on z_0, g, q_0. Note that mappings $\underline{R}_i(\tau)$ and $\overline{R}_i(\tau)$ are upper-semicontinuous (i.e., closed) and that for any $\varepsilon>0$ there exists a $\delta_1>0$ such that

$$\delta_1 = \delta_1(\varepsilon) < \delta,$$

$$\underline{R}_i(\tau) \subset \underline{R}_{i\varepsilon},\ \overline{R}_i(\tau) \subset \overline{R}_{i\varepsilon} \qquad \forall i \in 1:n,\ \forall \tau \in \bar{S}_{\delta_1}(0) \tag{8}$$

From (4)

$$\frac{\partial f_i(z_0)}{\partial [g, q_0+\tau]} = (v_{1i}(\tau), g) + (v_{2i}(\tau), q_0 + \tau) + (w_{1i}(\tau), g) +$$

$$+ (w_{2i}(\tau), q_0 + \tau) = (v_{2i}(\tau) + w_{2i}(\tau), \tau) + r_{1i}(\tau) \quad .$$

where

$$r_{1i}(\tau) = (v_{1i}(\tau), g) + (v_{2i}(\tau), q_0) + (w_{1i}(\tau), g) + (w_{2i}(\tau), q_0),$$

$$v_i(\tau) = [v_{1i}(\tau), v_{2i}(\tau)] \in \underline{R}_i(\tau) \quad ,$$

$$w_i(\tau) = [w_{1i}(\tau), w_{2i}(\tau)] \in \overline{R}_i(\tau) \quad .$$

Since $\underline{R}_i(\tau)$ and $\overline{R}_i(\tau)$ are upper-semicontinuous, if $\tau_s \xrightarrow[s\to\infty]{} 0$, $v_i(\tau_s) \xrightarrow[s\to\infty]{} v_i$, and $w_i(\tau_s) \xrightarrow[s\to\infty]{} w_i$ then

$$v_i \in \underline{R}_i(0), \qquad w_i \in \overline{R}_i(0) \quad .$$

This means that

$$r_{1i}(0) = \frac{\partial f_i(z_0)}{\partial [g, q_0]}$$

and $r_{1i}(\tau)$ are continuous. From (5) it follows that

$$r_{1i}(0) = 0 \qquad \forall i \in 1:n \quad . \tag{9}$$

Thus, from (3)

$$f_i(x_0 + \alpha g,\ y_0 + \alpha(q_0 + \tau)) = \alpha[(v_{2i}(\tau) + w_{2i}(\tau), \tau) + r_i(\alpha, \tau)]$$

where $r_i(\alpha, \tau) = r_{1i}(\tau) + \dfrac{o_i(\alpha, q_0+\tau)}{\alpha}$.

Consider the functions

$$F_{i\alpha}(\tau) = (v_{2i}(\tau) + w_{2i}(\tau), \tau) + r_i(\alpha, \tau) \quad . \tag{10}$$

Here $v_{2i}(\tau) \in \widetilde{V}_{2i}(\tau)$, $w_{2i}(\tau) \in \widetilde{W}_{2i}(\tau)$, where

$$\widetilde{V}_{2i}(\tau) = \{v_{2i} \mid \exists v_{1i} \in E_m : [v_{1i}, v_{2i}] \in \underline{R}_i(\tau)\},$$

$$\widetilde{W}_{2i}(\tau) = \{w_{2i} \mid \exists w_{1i} \in E_m : [w_{1i}, w_{2i}] \in \overline{R}_i(\tau)\} \quad .$$

The mappings $v_{1i}(\tau)$ and $w_{2i}(\tau)$ are upper-semicontinuous. Now introduce the set $M(\tau)$ of matrices such that $A \in M(\tau)$ if A is a matrix with i-th row $[v_{2i}(\tau) + w_{2i}(\tau)]^T$ where

$$v_{2i}(\tau) \in \widetilde{V}_{2i}(\tau) \text{ and } w_{2i}(\tau) \in \widetilde{W}_{2i}(\tau) \quad .$$

For any fixed τ the set $M(\tau)$ is convex and upper-semicontinuous.

Let us denote by M_ε (where $\varepsilon \geqslant 0$) the set of matrices defined as follows:

$$M_\varepsilon = \left\{ A = \begin{pmatrix} A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix} \; A_i = [v_{2i} + w_{2i}]^T,\; v_{2i} \in \underline{R}_{i\varepsilon}, w_{2i} \in \overline{R}_{i\varepsilon} \quad \forall i \right\} \quad .$$

From (8) it is clear that

$$M(\tau) \subset M_\varepsilon \qquad \forall \tau \in S_{\delta_1}(0) \quad . \tag{11}$$

Note that if $\delta_1 = \delta_1(\varepsilon)$ in (11) then (8) is satisfied.

Theorem 1. If for some $\varepsilon > 0$ we have

$$\min_{A \in M_\varepsilon} \det A > 0 \tag{12}$$

then for α positive and sufficiently small there exists a solution $\tau(\alpha)$ to system (7) or, equivalently, to the system

$$F_{i\alpha}(\tau) = 0 \qquad \forall i \in 1:n$$

Proof. Let us construct the mapping

$$M^{-1}(\tau)r(\alpha,\tau) = \phi_\alpha(\tau)$$

where

$$M^{-1}(\tau) = \{B = A^{-1}(\tau)\,|\,A(\tau) \in M(\tau)\} \quad .$$

From (11) and (12) it follows that $\phi_\alpha(\tau)$ is upper-semicontinuous (for any fixed $\alpha \in [0,\alpha_0]$) in $\tau \in S_{\delta_1}(0)$ and that

$$\phi_\alpha(S_{\delta_1}(0)) \subset S_{\delta_1}(0) \quad .$$

This means that all of the conditions of the Kakutani theorem (see [7,8]) are satisfied and therefore there exists one point $\tau(\alpha)$ which is a fixed point of the mapping $\phi_\alpha(\tau)$:

$$\phi_\alpha(\tau(\alpha)) = \tau(\alpha) \quad .$$

From (6) and (9) it is also clear that

$$\tau(\alpha) \xrightarrow[\alpha \to 0]{} 0 \quad .$$

Now from the above equation and (10) it follows that

$$F_{i\alpha}(\tau(\alpha)) = 0 \quad . \qquad \text{Q.E.D.}$$

Corollary. If q_0 is a solution to (5) and the condition (12) of Theorem 1 is satisfied then system (2) has a solution $y(\tau)$ defined on $[0,\alpha_0]$ (where $\alpha_0>0$) such that

$$y'_+(0) \equiv \lim_{\alpha\to+0} \frac{1}{\alpha}\,[y(\alpha) - y(0)] = q_0 \quad .$$

We shall call Theorem 1 a *directional implicit function theorem.*

Of course, there could be several solutions to (5), or none at all.

It is important to be able to solve systems of equations of the form

$$\max_{v_i \in \sigma_{1i}} [(v_{1i},g) + (v_{2i},q)] + \min_{w_i \in \sigma_{2i}} [(w_{1i},g) + (w_{21},q)] =$$

$$= b_i \qquad \forall i \in 1:n$$

where $v_i = [v_{1i}, v_{2i}]$, $w_i = [w_{1i}, w_{2i}]$, and $\sigma_{1i} \subset E_{m+n}$ and $\sigma_{2i} \subset E_{m+n}$ are convex compact sets.

We shall call systems of this type *quasilinear.*

In some cases (for example, if σ_{1i} and σ_{2i} are convex hulls of a finite number of points) the problem of solving quasilinear systems can be reduced to that of solving several linear systems of algebraic equations (we shall illustrate this later on).

3. AN INVERSE FUNCTION THEOREM

Now let us consider a special case of the problem, namely, where system (1) is of the form

$$x + \phi(y) = 0 \tag{13}$$

i.e.,

$$x^{(i)} + \phi_i(y) = 0 \qquad \forall_i \in 1:n,$$

where

$$x = (x^{(1)},\ldots,x^{(n)}) \in E_n, \quad y = (y^{(1)},\ldots,y^{(n)}) \in E_n$$

and the functions ϕ_i are quasidifferentiable on E_n.

Suppose that $z_0 = [x_0,y_0] \in E_{2n}$ is a solution to (13), i.e,

$$x_0 + \phi(y_0) = 0 \quad .$$

Choose and fix any direction $g \in E_n$. We now have to consider two questions:

1. What conditions are necessary for the existence of a positive α_0 and a continuous vector function $y(\alpha)$ such that the expressions

$$y(0) = y_0, \; x_0 + \alpha g + \phi(y(\alpha)) = 0 \qquad \forall \alpha \in [0,\alpha_0] \tag{14}$$

are satisfied?

2. If $y(\alpha)$ exists does

$$y'_+(0) \equiv \lim_{\alpha \to +0} \frac{1}{\alpha} [y(\alpha) - y(0)]$$

necessarily exist?

To answer these questions we turn to Theorem 1 and its corollary. Let $D\phi_i(y) = [\underline{\partial}\phi_i(y), \overline{\partial}\phi_i(y)]$ be a quasidifferential of ϕ_i at y. We then have

$$\phi_i(y_0 + \alpha q) = \phi_i(y_0) + \alpha \left[\max_{v_i \in \underline{\partial}\phi_i(y_0)} (v_i, q) + \min_{w_i \in \overline{\partial}\phi_i(y_0)} (w_i, q) \right] + o_i(\alpha, q) \quad . \tag{15}$$

In this case equation (4) takes the form

$$\max_{v_i \in \underline{\partial}\phi_i(y_0)} (v_i, q) + \min_{w_i \in \underline{\partial}\phi_i(y_0)} (w_i, q) = -g_i \qquad \forall i \in 1:n \quad . \tag{16}$$

Suppose that $q_0 \in E_n$ is a solution to (16) and that in (15)

$$\frac{o_i(\alpha, q)}{\alpha} \xrightarrow[\alpha \to +0]{} 0$$

uniformaly with respect to $q \in S_\delta(q_0)$.

We now introduce the sets

$$\underline{R}_{i\varepsilon} = \{v_i \in \underline{\partial}\phi_i(y_0) \mid (v_i, q) \geqslant \max_{v_i \in \underline{\partial}\phi_i(y_0)} (v_i, q) - \varepsilon\} \quad ,$$

$$\overline{R}_{i\varepsilon} = \{w_i \in \overline{\partial}\phi_i(y_0) \mid (w_i,q) \leqslant \min_{w_i \in \overline{\partial}\phi_i(y_0)} (w_i,q) + \varepsilon\} \quad .$$

Let M_ε be a set of matrices defined as follows:

$$M_\varepsilon = \left\{ A = \begin{pmatrix} A_i \\ \vdots \\ A_n \end{pmatrix} \quad A_i = [v_i + w_i]^T, \; v_i \in \underline{R}_{i\varepsilon}, \; w_i \in \overline{R}_{i\varepsilon} \quad \forall i \right\}$$

where $\varepsilon \geqslant 0$.

Theorem 2. If for some $\varepsilon > 0$ we have

$$\min_{A \in M_\varepsilon} \det A > 0 \tag{17}$$

then there exist an $\alpha_0 > o$ and a continuous vector function $y(\alpha)$ such that

$$y(0) = y_0, \; x_0 + \alpha g + \phi(y(\alpha)) = 0$$

and

$$y'_+(0) = q_0 \quad .$$

Remark 1. In the case where each of the sets $\underline{\partial}\phi_i(y_0)$ and $\overline{\partial}\phi_i(y_0)$ (for all values of i) is a convex hull of a finite number of points, it can be shown that Theorem 2 is valid if (17) holds for $\varepsilon = 0$. An analogous result can also be obtained for Theorem 1.

Remark 2. Suppose that $[x_0; y_0]$ is a solution to (14). Then to solve the directional inverse function problem it is necessary to find all the solutions to (16) and check whether condition (17) is satisfied.

As an illustration of Theorem 2 and the use of the technique outlined above we shall now present a simple example.

Example. Let $x = (x^{(1)}, x^{(2)}) \in E_2$, $y = (y^{(1)}, y^{(2)}) \in E_2$, $x_0 = (0,0)$, $y_0 = (0,0)$.

Consider the following system of equations:

$$\left.\begin{aligned} x^{(1)} + |y^{(1)}| - 2|y^{(2)}| &= 0 \\ x^{(2)} + |y^{(1)}-y^{(2)}| &= 0 \quad . \end{aligned}\right\} \qquad (18)$$

This system is simple enough to be solved directly. It is not difficult to derive the following solutions:

1.
$$\left.\begin{aligned} y^{(1)} &= x^{(1)} - 2x^{(2)} \\ y^{(2)} &= x^{(1)} - x^{(2)} \end{aligned}\right\} \qquad (19)$$

if

$$y \in \Omega_1 = \{y = (y^{(1)}, y^{(2)}) \,|\, y^{(1)} \geqslant 0,\ y^{(2)} \geqslant 0,\ y^{(1)} - y^{(2)} \geqslant 0\} \quad ;$$

2.
$$\left.\begin{aligned} y^{(1)} &= x^{(1)} + 2x^{(2)} \\ y^{(2)} &= x^{(1)} + x^{(2)} \end{aligned}\right\} \qquad (20)$$

if

$$y \in \Omega_2 = \{y = (y^{(1)}, y^{(2)}) \,|\, y^{(1)} \geqslant 0,\ y^{(2)} \geqslant 0,\ y^{(1)} - y^{(2)} \leqslant 0\} \quad ;$$

3.
$$\left.\begin{aligned} y^{(1)} &= -\frac{1}{3} x^{(1)} - \frac{2}{3} x^{(2)} \\ y^{(2)} &= -\frac{1}{3} x^{(1)} + \frac{1}{3} x^{(2)} \end{aligned}\right\} \qquad (21)$$

if

$$y \in \Omega_3 = \{y = (y^{(1)}, y^{(2)}) \,|\, y^{(1)} \geqslant 0,\ y^{(2)} \leqslant 0,\ y^{(1)} - y^{(2)} \geqslant 0\} \quad ;$$

4.
$$\left.\begin{aligned} y^{(1)} &= \frac{1}{3} x^{(1)} + \frac{2}{3} x^{(2)} \\ y^{(2)} &= \frac{1}{3} x^{(1)} - \frac{1}{3} x^{(2)} \end{aligned}\right\} \qquad (22)$$

if

$$y \in \Omega_4 = \{y = (y^{(1)}, y^{(2)}) \,|\, y^{(1)} \leqslant 0,\ y^{(2)} \geqslant 0,\ y^{(1)} - y^{(2)} \leqslant 0\} \quad ;$$

5.
$$\left.\begin{aligned} y^{(1)} &= -x^{(1)} - 2x^{(2)} \\ y^{(2)} &= -x^{(1)} - x^{(2)} \end{aligned}\right\} \quad (23)$$

if

$$y \in \Omega_5 = \{y = (y^{(1)}, y^{(2)}) \mid y^{(1)} \leqslant 0, y^{(2)} \leqslant 0, \ y^{(1)} - y^{(2)} \geqslant 0\}.$$

6.
$$\left.\begin{aligned} y^{(1)} &= -x^{(1)} + 2x^{(2)} \\ y^{(2)} &= -x^{(1)} + x^{(2)} \end{aligned}\right\} \quad (24)$$

if

$$y \in \Omega_6 = \{y = (y^{(1)}, y^{(2)}) \mid y^{(1)} \leqslant 0, \ y^{(2)} \leqslant 0, \ y^{(1)} - y^{(2)} \leqslant 0\}$$

In this example it is obvious that $[x_0, y_0]$ satisfies (18). Now consider the (arbitrarily chosen) four directions $g_1 = (1,0)$, $g_2 = (-1,0)$, $g_3 = (1,1)$, $g_4 = (-1,-1)$.

For g_1 we have $x_0 + \alpha g_1 = (\alpha, 0)$. We now look at each of the possible solutions in turn.

From (19), $y_{11}(\alpha) = (\alpha,\alpha) \in \Omega_1 \ \ \forall\alpha \geqslant 0$, i.e., $y_{11}(\alpha)$ satisfies (14) for all $\alpha \geqslant 0$ and therefore $y_{11}(\alpha)$ is a directional inverse function of (13) in the direction g_1 and

$$y'_{11+}(0) = (1,1) = q_{01} \quad .$$

Solution (20) yields the same directional inverse function as (19).

From (21) we obtain $y_{13}(x) = (-\frac{1}{3}\alpha, -\frac{1}{3}\alpha)$; in this case $y_{13} \notin \Omega_3 \ \ \forall\alpha>0$ and therefore $y_{13}(\alpha)$ is not a directional inverse function of (13) in the direction g_1.

From (22) $y_{14}(\alpha) = (\frac{1}{3}\alpha, \frac{1}{3}\alpha) \notin \Omega_4 \ \ \forall\alpha>0$, and therefore $y_{14}(\alpha)$ is also not a directional inverse function of (13) in the direction g_1.

Solutions (23) and (24) yield the functions $y_{15}(\alpha) = y_{16}(\alpha) = (-\alpha,-\alpha)$, where

$$y_{15}(\alpha) \in \Omega_5 \text{ and } y_{16}(\alpha) \in \Omega_6 \quad \forall\alpha \geqslant 0 \quad .$$

Thus $y_{15}(\alpha) = y(\alpha)$ is a directional inverse function of (13) in the direction g_1 and $y'_{15+}(0) = (-1,-1) = q_{05}$.
Thus there are two directional inverse functions of (13) in the direction g_1 : $y_{11}(\alpha) = (\alpha,\alpha)$ and $y_{15}(\alpha) = (-\alpha, -\alpha)$.

Now let us consider $g_2 = (-1,0)$. From (19),

$$y_{21}(\alpha) = (-\alpha, -\alpha) \notin \Omega_1 \quad \forall\alpha>0,$$

and therefore $y_{21}(\alpha)$ is not a directional inverse function of (13) in the direction g_2. In the same way we obtain (for $\alpha>0$):

$$y_{22}(\alpha) = (-\alpha,-\alpha) \notin \Omega_2,$$

$$y_{23}(\alpha) = (\tfrac{1}{3}\alpha, \tfrac{1}{3}\alpha) \notin \Omega_3, \; y_{24}(\alpha) = (-\tfrac{1}{3}\alpha, -\tfrac{1}{3}\alpha) \notin \Omega_4.$$

$$y_{25}(\alpha) = (\alpha,\alpha) \notin \Omega_5, \; y_{26}(\alpha) = (\alpha,\alpha) \notin \Omega_6.$$

This means that there is no directional inverse function of the system(13) in the direction g_2.

The same is also true for the direction $g_3 = (1,1)$ since for $\alpha> 0$

$$y_{31}(\alpha) = (-\alpha,0) \notin \Omega_1, \; y_{32} = (3\alpha, 2\alpha) \notin \Omega_2,$$

$$y_{33}(\alpha) = (-\alpha,0) \notin \Omega_3, \; y_{34}(\alpha) = (\alpha,0) \notin \Omega_4,$$

$$y_{35}(\alpha) = (-3\alpha, -2\alpha) \notin \Omega_5, \; y_{36}(\alpha) = (\alpha,0) \notin \Omega_6 \quad .$$

In the same way we find that there are two directional inverse functions of (13) in the direction $g_4 = (-1,-1)$:

$$y_{41}(\alpha) = y_{43}(\alpha) = (\alpha,0), \qquad y_{44}(\alpha) = y_{46}(\alpha) = (-\alpha,0),$$

and

$$y'_{41+}(0) = y'_{43+}(0) = (1,0), \qquad y'_{44+}(0) = y'_{46+}(0) = (-1,0) .$$

Now let us solve the problem again using the results of Theorem 2. System (18) can be rewritten in the following form (see (13)):

$$x + \phi(y) = 0$$

where $\phi = (\phi_1, \phi_2)$, $\phi_1(y) = |y^{(1)}| - 2|y^{(2)}|$, and $\phi_2(y) = |y^{(1)} - y^{(2)}|$.

The functions ϕ_1 and ϕ_2 are quasidifferentiable. We first find their quasidifferentials at $y_0 = (0,0)$:

$$D\phi_1(y_0) = [\underline{\partial}\phi_1(y_0), \overline{\partial}\phi_1(y_0)], \; D\phi_2(y_0) = [\underline{\partial}\phi_2(y_0), \overline{\partial}\phi_2(y_0)] \tag{25}$$

where

$$\underline{\partial}\phi_1(y_0) = \{v = (v^{(1)}, v^{(2)}) \mid v^{(1)} \in [-1,1], \; v^{(2)} = 0\} =$$

$$= \mathrm{co}\{(-1,0), (1,0)\} \quad ,$$

$$\overline{\partial}\phi_1(y_0) = \{w = (w^{(1)}, w^{(2)}) \mid w^{(1)} = 0, \; w^{(2)} \in [-2,2]\} =$$

$$= \mathrm{co}\{(0,-2), \; (0,2)\} \quad ,$$

$$\underline{\partial}\phi_2(y_0) = \mathrm{co}\{(-1,1), \; (1,-1)\}, \; \overline{\partial}\phi_2(y_0) = \{(0,0)\} \quad .$$

For any fixed $g = (g^{(1)}, g^{(2)})$, we have to solve (16) and find $q_0 = (q_0^{(1)}, q_0^{(2)})$. From (25) and (16) we obtain the system

$$\max_{v_1^{(1)} \in [-1,1]} v_1^{(1)} q^{(1)} + \min_{w_1^{(2)} \in [-2,2]} w_1^{(2)} q^{(2)} = -g^{(1)} \tag{26}$$

$$\max_{v_2 \in \mathrm{co}\{(-1,1),(1,-1)\}} (v_2, \; q) = -g^{(2)} \quad .$$

In general we cannot solve (16) but if $\underline{\partial}\phi_i$ and $\overline{\partial}\phi_i$ are convex hulls of a finite number of points, as is the case here, we can solve (26) by considering the following eight linear systems of algebraic equations:

1. $$\left.\begin{array}{l} -q^{(1)} - 2q^{(2)} = -g^{(1)} \\ -q^{(1)} + q^{(2)} = -g^{(2)} \end{array}\right\} \qquad (27)$$

2. $$\left.\begin{array}{l} -q^{(1)} - 2q^{(2)} = -g^{(1)} \\ q^{(1)} - q^{(2)} = -g^{(2)} \end{array}\right\} \qquad (28)$$

3. $$\left.\begin{array}{l} -q^{(1)} + 2q^{(2)} = -g^{(1)} \\ -q^{(1)} + q^{(2)} = -g^{(2)} \end{array}\right\} \qquad (29)$$

4. $$\left.\begin{array}{l} -q^{(1)} + 2q^{(2)} = -g^{(1)} \\ q^{(1)} - q^{(2)} = -g^{(2)} \end{array}\right\} \qquad (30)$$

5. $$\left.\begin{array}{l} q^{(1)} - 2q^{(2)} = -g^{(1)} \\ -q^{(1)} + q^{(2)} = -g^{(2)} \end{array}\right\} \qquad (31)$$

6. $$\left.\begin{array}{l} q^{(1)} - 2q^{(2)} = -g^{(1)} \\ q^{(1)} - q^{(2)} = -g^{(2)} \end{array}\right\} \qquad (32)$$

7. $$\left.\begin{array}{l} q^{(1)} + 2q^{(2)} = -g^{(1)} \\ -q^{(1)} + q^{(2)} = -g^{(2)} \end{array}\right\} \qquad (33)$$

8. $$\left.\begin{array}{l} q^{(1)} + 2q^{(2)} = -g^{(1)} \\ q^{(1)} - q^{(2)} = -g^{(2)} \end{array}\right\} \qquad (34)$$

The systems (27) - (34) are all nondegenerate and thus solutions exist for any $g \in E_2$.

Take $g_1 = (1,0)$. Solving (27) - (34) we obtain four different vectors:

$$q_{11} = (\tfrac{1}{3}, \tfrac{1}{3}) \quad \text{(from (27) and (28))} \quad ,$$

$$q_{12} = (-1,-1) \quad \text{(from (29) and (30))} \quad ,$$

$$q_{13} = (1,1) \quad \text{(from (31) and (32))} \quad ,$$

$$q_{14} = (-\tfrac{1}{3}, -\tfrac{1}{3}) \quad \text{(from (33) and (34))} \quad .$$

Now it is necessary to check which of the values of q_{1i} are solutions to (26), i.e., satisfy

$$\max_{v_1^{(1)} \in [-1,1]} v_1^{(1)} q^{(1)} + \min_{w_1^{(2)} \in [-2,2]} w_1^{(2)} q^{(2)} = -1$$

$$\max_{v_2 \in co\{(-1,1),\ (1,-1)\}} (v_2, q) = 0 \quad . \tag{35}$$

A quick check shows that only vectors q_{12} and q_{13} satisfy (35). For $q_{12} = (-1,-1)$ we have

$$\underline{R}_{10} = \{v_1 \in \underline{\partial}\phi_1(y_0) \mid (v_1, q_{12}) = \min_{v_1 \in \underline{\partial}\phi_1(y_0)} (v_1, q_{12})\} = \{(-1,0)\},$$

$$\overline{R}_{10} = \{w_1 \in \overline{\partial}\phi_1(y_0) \mid (w_1, q_{12}) = \min_{w_1 \in \overline{\partial}\phi_1(y_0)} (w_1, q_{12})\} = \{(0,2)\} \quad ,$$

$$\underline{R}_{20} = \{v_2 \in \underline{\partial}\phi_2(y_0) \mid (v_2, q_{12}) = \max_{v_2 \in \underline{\partial}\phi_2(y_0)} (v_2, q_{12})\} =$$

$$= co\{(-1,1), (1,-1)\} \quad ,$$

$$\overline{R}_{20} = \{w_2 \in \overline{\partial}\phi_2(y_0) \mid (w_2,q_{12}) = \min_{w_2 \in \overline{\partial}\phi_2(y_0)} (w_2,q_{12})\} = \{(0,0)\} .$$

Then

$$M_0 = \left\{ A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \middle| a_i = [v_i + w_i]^T,\ v_i \in \underline{R}_{i0},\ w_i \in \overline{R}_{i0} \right\} = co\{A_1, A_2$$

where $A_1 = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$.

Note that

$$A_0 = \tfrac{1}{2}A_1 + \tfrac{1}{2}A_2 = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \in M_0$$

and $\det A_0 = 0$, i.e., condition (17) is not satisfied. But we should remember that condition (17) was originally introduced to deal with $O_i(\alpha,q)$ in (15) and that it is a sufficient, not necessary condition for the existence of a directional inverse. In our case $O_1(\alpha,q) = 0$ for all values of i and q, and therefore there must be a vector function $y_{12}(\alpha)$ such that

$$x_0 + \alpha g_1 + \phi(y_{12}(\alpha)) = 0 \quad \forall\alpha \geqslant 0$$

and

$$y'_{12+}(0) = q_{12} = (-1, -1) .$$

Moving to $q_{13} = (1,1)$, and following the same line of argument we deduce the existence of a vector function $y_{13}(\alpha)$ such that

$$x_0 + \alpha g_1 + \phi(y_{13}(\alpha)) = 0 \quad \forall\alpha \geqslant 0$$

and

$$y'_{13+}(0) = q_{13} = (1,1) .$$

Thus there are two solutions to (15) for $g_1 = (1,0)$: $y_{12}(\alpha)$ and $y_{13}(\alpha)$. This duplicates the result obtained earlier.

For $g_2 = (-1,0)$ we again arrive at the four vectors

$$q_{21} = (\frac{1}{3}, \frac{1}{3}),\ q_{22} = (-1, -1),\ q_{23} = (1,1),\ q_{24} = (-\frac{1}{3}, -\frac{1}{3})$$

calculated previously as solutions to (27) - (34); however, none of them satisfies (26). Thus the system (13) has no directional inverse function in the direction g_2.

For $g_3 = (1,1)$, solving the systems (27) - (34) yields six different vectors:

$$q_{31} = (1,0) \quad \text{(from (27) and (29))},$$

$$q_{32} = (-1,0) \quad \text{(from (32) and (34))},$$

$$q_{33} = (-\frac{1}{3}\ \frac{2}{3}) \quad \text{(from (28))},$$

$$q_{34} = (\frac{1}{3}\ -\frac{2}{3}) \quad \text{(from (33))},$$

$$q_{35} = (-3, -2) \quad \text{(from (30))},$$

$$q_{36} = (3,2) \quad \text{(from (31))} .$$

These values should then be tested by substituting them into (26) (for $q^{(1)} = 1$, $q^{(2)} = 1$). We find that none of these six vectors satisfies (26), and therefore system (13) has no directional inverse function in the direction g_3.

For $g_4 = (-1,-1)$ we obtain the same six vectors as for g_3

$$q_{41} = (1,0),\ q_{42} = (-1,0),\ q_{43} = (-\frac{1}{3}, \frac{2}{3}),\ q_{44} =$$

$$= (\frac{1}{3}, -\frac{2}{3}),\ q_{45} = (-3,-2),\ q_{46} = (3,2) ,$$

but now $q_{41} = (1,0)$ and $q_{42} = (-1,0)$ satisfy (26) (the four other vectors still do not). Condition (17) does not hold but it is not essential to invoke Theorem 2 in this case since in (15)

$$o_i(\alpha,q) = 0 \quad \forall\alpha \geqslant 0, \quad \forall i \in 1:2 \quad .$$

Thus for the direction $g_4 = (-1,-1)$ there are two directional inverse functions $y_{41}(\alpha)$ and $y_{42}(\alpha)$ such that

$$x_0 + \alpha g_4 + \phi(y_{41}(\alpha)) = 0 \quad \forall\alpha \geqslant 0,$$

$$x_0 + \alpha g_4 + \phi(y_{42}(\alpha)) = 0 \quad \forall\alpha \geqslant 0,$$

and

$$y'_{41+}(0) = (1,0), \; y'_{42+}(0) = (-1,0) \quad .$$

This again duplicates the results obtained earlier.

REFERENCES

1. V.F. Demyanov and A.M. Rubinov. On quasidifferentiable functionals. Soviet Math. Dokl, Vol. 21(1), 1980, pp. 14-17.
2. V.F. Demyanov and L.N. Polyakova. Minimization of a quasi-differentiable function on a quasidifferentiable set. USSR Comput. Math.and Math. Phys., Vol 20(4), 1980, pp.34-43.
3. L.N. Polyakova. Necessary conditions for an extremum of quasidifferentiable functions. Vestnik Leningrad Univ. Math., Vol. 13, 1981, pp. 241-247.
4. V.F. Demyanov and A.M. Rubinov. On quasidifferentiable mappings. Math. Operationsforsch u. Statist., Ser. Optimization, vol. 14(1), 1983, pp. 3-21.
5. V.F. Demyanov (Ed.) Nonsmooth Problems of Optimization Theory and Control. Leningrad Univ. Press, 1982 (in Russian).
6. V.F. Demyanov and L.V. Vasiliev. Nondifferentiable Optimization. Nauka, Moscow, 1981 (in Russian).
7. S. Kakutani. A generalization of Brower's fixed point theorem. Duke Math. J., Vol. 8 (3), 1941, pp. 457-459
8. L.V. Kantorovich and G.P. Akilov. Functional Analysis. Nauka, Moscow, 1977.