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AN ALGORITHM FOR MINIMIZING A CERTAIN CLASS OF QUASIDIFFERENTIABLE FUNCTIONS

V.F. Demyanov
S. Gamidov
T.I. Sivelina

December 1983
WP-83-122

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

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1. INTRODUCTION

One interesting and important class of nondifferentiable functions is that produced by smooth compositions of max-type functions. Such functions are of practical value and have been studied extensively by several researchers [1-3]. We treat them as quasidifferentiable functions and analyze them using quasidifferential calculus.

One special subgroup of this class of functions (namely, the sum of a max-type function and a min-type function) has been studied by T.I. Sivelina [4]. The main feature of the algorithm described in the present paper is that at each step it is necessary to consider a bundle of auxiliary directions and points, of which only one can be chosen for the next step. This requirement seems to arise from the intrinsic nature of nondifferentiable functions.

2. THE UNCONSTRAINED CASE

Let

$$f(x) = F(x, y_1(x), \dots, y_m(x)) \tag{1}$$

where

$$x \in E_n, \qquad y_i(x) \equiv \max_{j \in I_i} \phi_{ij}(x), \qquad I_i \equiv 1:N_i$$

and functions $F(x, y_1, \ldots, y_m)$ and $\phi_{ij}(x)$ are continuously differentiable on E_{n+m} and E_n, respectively.

Take any $g \in E_n$. Then for $\alpha \geq 0$ we have

$$y_i(x+\alpha g) = y_i(x) + \alpha \frac{\partial y_i(x)}{\partial g} + o_i(\alpha, g)$$

where

$$\frac{\partial y_i(x)}{\partial g} \equiv \lim_{\alpha \to +0} \frac{y_i(x+\alpha g) - y_i(x)}{\alpha} = \max_{j \in R_i(x)} (\phi'_{ij}(x), g) \quad ,$$

$$\phi'_{ij}(x) = \frac{\partial \phi_{ij}(x)}{\partial x} \quad ,$$

$$R_i(x) = \{j \in I_i \mid \phi_{ij}(x) = y_i(x)\} \quad ,$$

$$\frac{o_i(\alpha, g)}{\alpha} \xrightarrow[\alpha \to +0]{} 0 \quad . \tag{2}$$

This leads to

$$f(x+\alpha g) = f(x) + \alpha \left[\left(\frac{\partial F(y(x))}{\partial x}, g \right) + \sum_{i \in I} \frac{\partial F(y(x))}{\partial y_i} \frac{\partial y_i(x)}{\partial g} \right] + o(\alpha, g) \tag{3}$$

where

$$I \equiv 1:m, \qquad y(x) \equiv (x, y_1(x), \ldots, y_m(x)) \quad ,$$

and

$$\frac{o(\alpha, g)}{\alpha} \xrightarrow[\alpha \to +0]{} 0 \quad . \tag{4}$$

It is clear that convergence in (2) and (4) is uniform with respect to $g \in S_1 \equiv \{g \in E_n \mid \|g\| = 1\}$. Let

$$I_+(x) = \left\{ i \in I \;\middle|\; \frac{\partial F(y(x))}{\partial y_i} > 0 \right\} ,$$

$$I_-(x) = \left\{ i \in I \;\middle|\; \frac{\partial F(y(x))}{\partial y_i} < 0 \right\} .$$

Then from (3) we have

$$f(x+\alpha g) = f(x) + \alpha \left[\left(\frac{\partial F(y(x))}{\partial x}, g \right) + \sum_{i \in I_+(x)} \max_{j \in R_i(x)} \left(\frac{\partial F(y(x))}{\partial y_i} \phi'_{ij}(x), g \right) + \sum_{i \in I_-(x)} \min_{j \in R_i(x)} \left(\frac{\partial F(y(x))}{\partial y_i} \phi'_{ij}(x), g \right) \right] + o(\alpha, g). \quad (5)$$

It follows from (5) that f is quasidifferentiable and

$$Df(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$$

where

$$\underline{\partial} f(x) = \text{co } A(x), \qquad \overline{\partial} f(x) = \text{co } B(x) ,$$

$$A(x) = \left\{ v \in E_n \;\middle|\; v = \frac{\partial F(y(x))}{\partial x} + \sum_{i \in I_+(x)} \frac{\partial F(y(x))}{\partial y_i} \phi'_{ij}(x), j \in R_i(x) \right\} ,$$

$$B(x) = \left\{ w \in E_n \;\middle|\; w = \sum_{i \in I_-(x)} \frac{\partial F(y(x))}{\partial y_i} \phi'_{ij}(x), j \in R_i(x) \right\} .$$

Recall [5,6] that a necessary condition for $x^* \in E_n$ to be a minimum point of a quasidifferentiable function f on E_n is

$$-\overline{\partial} f(x^*) \subset \underline{\partial} f(x^*) .$$

A point x^* satisfying this inclusion is called an *inf-stationary point* of f on E_n.

For $x^* \in E_n$ to be a local minimum point of f it is sufficient[+] that

$$-\overline{\partial} f(x^*) \subset \text{int } \underline{\partial} f(x^*) \quad .$$

The following lemmas can be derived from the above necessary and sufficient conditions:

Lemma 1. For any set of coefficients

$$\left\{\lambda_{ij} \middle| i \in I_-(x^*), j \in R_i(x^*), \lambda_{ij} \geq 0, \sum_{j \in R_i(x^*)} \lambda_{ij} = 1\right\}$$

there exists another set of coefficients

$$\left\{\lambda_{ij} \middle| i \in I_+(x^*), j \in R_i(x^*), \lambda_{ij} \geq 0, \sum_{j \in R_i(x^*)} \lambda_{ij} = 1\right\}$$

such that

$$\frac{\partial F(y(x^*))}{\partial x_i} + \sum_{i \in I} \frac{\partial F(y(x^*))}{\partial y_i} \sum_{j \in R_i(x^*)} \lambda_{ij} \phi'_{ij}(x^*) = 0 \quad . \qquad (6)$$

(If $\frac{\partial F(y(x^*))}{\partial y_i} = 0$ put $\lambda_{ij} = 0 \quad \forall j \in R_i(x^*)$.)

Condition (6) is a multipliers rule - note the difference between it and the Lagrange multipliers rule for mathematical programming.

It follows from (6) that x^* is a stationary point of the smooth function

$$F_\lambda(x) = F\left(x, \sum_{j \in R_1(x^*)} \lambda_{1j}\phi_{1j}(x), \ldots, \sum_{j \in R_m(x^*)} \lambda_{mj}\phi_{mj}(x)\right) \quad ,$$

and if $\underline{\partial} f(x^*)$ consists of more than one point then the set $\{\lambda_{ij}\}$ is not unique. (Of course, it may not be unique even if $\underline{\partial} f(x^*)$ is a singleton.)

[+]For an arbitrary quasidifferentiable function condition (7) is sufficient for a minimum only with certain additional assumptions. However condition (7) is sufficient for functions described by (1).

Lemma 2. If for any $w \in B(x^*)$ there exist sets

$$\{v_i \mid i \in 1:(n+1)\} \quad \text{and} \quad \left\{\alpha_i \mid \alpha_i > 0, \sum_{i=1}^{n+1} \alpha_i = 1\right\}$$

such that the vectors $\{v_i\}$ form a simplex (i.e., vectors $\{v_i - v_{n+1} \mid i \in 1:n\}$ are linearly independent) and $w = \sum_{i=1}^{n+1} \alpha_i v_i$, then x^* is a local minimum point of f on E_n.

We shall now introduce the following sets, where $\varepsilon \geq 0$, $\mu \geq 0$:

$$R_{i\varepsilon}(x) = \{j \in I_i \mid \phi_{ij}(x) \geq y_i(x) - \varepsilon\} \quad ,$$

$$\underline{\partial}_\varepsilon f(x) = \text{co}\left\{v \in E_n \mid v = \frac{\partial F(y(x))}{\partial x} + \sum_{i \in I_+(x)} \frac{\partial F(y(x))}{\partial y_i} \phi'_{ij}(x), j \in R_{i\varepsilon}(x)\right\},$$

$$B_\mu(x) = \left\{w \in E_n \mid w = \sum_{i \in I_-(x)} \frac{\partial F(y(x))}{\partial y_i} \phi'_{ij}(x), j \in R_{i\mu}(x)\right\} \quad .$$

Let f be defined by (1). A point $x^* \in E_n$ will be called an ε-*inf-stationary point* of f on E_n if

$$-\overline{\partial} f(x^*) \subset \underline{\partial}_\varepsilon f(x^*) \quad .$$

We shall now describe an algorithm for finding an ε-inf-stationary point, with $\varepsilon > 0$ and $\mu > 0$ fixed.

Choose an arbitrary $x_0 \in E_n$. Suppose that x_k has been found. If

$$-\overline{\partial} f(x_k) \subset \underline{\partial}_\varepsilon f(x_k) \tag{7}$$

then x_k is an ε-inf-stationary point and the process terminates. If, on the other hand, (7) is not satisfied then for every $w \in B_\mu(x_k)$ we find

$$\min_{v \in \underline{\partial}_\varepsilon f(x_k)} \|w+v\| = \|w+v_k(w)\| \quad .$$

If $w + v_k(w) \neq 0$ then let $g_k(w) = -\frac{w+v_k(w)}{\|w+v_k(w)\|}$ and compute

$$\min_{\alpha \geq 0} f(x_k+\alpha g_k(w)) = f(x_k+\alpha_k(w)g_k(w)) \quad . \tag{8}$$

If $w + v_k(w) = 0$ then take $\alpha_k(w) = 0$ and find

$$\min_{w\in B_\mu(x_k)} f(x_k+\alpha_k(w)g_k(w)) = f(x_k+\alpha_k(w_k)g_k(w_k)) \quad .$$

We then set

$$x_{k+1} = x_k + \alpha_k(w_k)g_k(w_k) \quad .$$

It is clear that

$$f(x_{k+1}) < f(x_k) \quad . \tag{9}$$

By repeating this procedure we obtain a sequence of points $\{x_k\}$. If it is a finite sequence (i.e., consists of a finite number of points) then its final element is an ε-inf-stationary point by construction. Otherwise the following result holds.

Theorem 1. If the set $D(x_0) = \{x \in E_n | f(x) \leq f(x_0)\}$ is bounded then any limit point of the sequence $\{x_k\}$ is an ε-inf-stationary point of f on E_n.

Proof. The existence of limit points follows from the boundedness of $D(x_0)$. Let x^* be a limit point of $\{x_k\}$, i.e., $x^* = \lim_{k_s\to\infty} x_{k_s}$. It is clear that

$$x^* \in D(x_0) \quad .$$

Assume that x^* is not an ε-inf stationary point. Then there exists a $w^* \in B_o(x^*)$ such that

$$\min_{v\in\underline{\partial}_\varepsilon f(x^*)} \|w^*+v\| = a > 0 \quad . \tag{10}$$

We shall denote by w_k^* the point in $B_\mu(x_k)$ which is nearest to w^* and by $\rho(w_k^*)$ the distance of w_k^* from w^*. It is obvious that $\rho(w_{k_s}^*) \xrightarrow[k_s\to\infty]{} 0$. It may also be seen that the mapping $\underline{\partial}_\varepsilon f(x)$ is upper-semicontinuous. From (10) and the above statements it follows that there exists a $K<\infty$ such that

$$\min_{v\in\underline{\partial}_\varepsilon f(x_{k_s})} \|w_{k_s}^*+v\| \equiv \|w_{k_s}^*+v(w_{k_s}^*)\| = a_{k_s} \geq \frac{a}{2} \quad \forall k_s > K . \tag{11}$$

Now we have

$$f(x_{k_s}+\alpha g_{k_s}) = f(x^*+(x_{k_s}-x^*+\alpha g_{k_s})) =$$

$$f(x^*) + \frac{\partial f(x^*)}{\partial[x_{k_s}-x^*+\alpha g_{k_s}]} + o(\|x_{k_s}-x^*+\alpha g_{k_s}\|) \tag{12}$$

where

$$g_{k_s} \equiv g_{k_s}(w_{k_s}^*) = -\frac{w_{k_s}^*+v_{k_s}(w_{k_s}^*)}{\|w_{k_s}^*+v_{k_s}(w_{k_s}^*)\|} ,$$

$$\frac{\partial f(x^*)}{\partial[x_{k_s}-x^*+\alpha g_{k_s}]} = \sum_{i\in I_+(x^*)} \max_{j\in R_i(x^*)} \left(\frac{\partial F(y(x^*))}{\partial x} + \frac{\partial F(y(x^*))}{\partial y_i}\phi'_{ij}(x^*) ,\right.$$

$$\left. x_{k_s}-x^* + \alpha g_{k_s}\right) + \sum_{i\in I_-(x^*)} \min_{j\in R_i(x^*)} \left(\frac{\partial F(y(x^*))}{\partial y_i}\phi'_{ij}(x^*) , x_{k_s}-x^* + \alpha g_{k_s}\right). \tag{13}$$

Since

$$\max_{i\in I} a_i + \min_{i\in I} b_i \leq \max_{i\in I} [a_i+b_i] \leq \max_{i\in I} a_i + \max_{i\in I} b_i ,$$

$$\min_{i\in I} a_i + \min_{i\in I} b_i \leq \min_{i\in I} [a_i+b_i] \leq \min_{i\in I} a_i + \max_{i\in I} b_i ,$$

it follows from (13) that

$$\frac{\partial f(x^*)}{\partial [x_{k_s} - x^* + \alpha g_{k_s}]} = \alpha\Bigg[\left(\frac{\partial F(y(x^*))}{\partial x}, g_{k_s}\right) + \sum_{i\in I_+(x^*)} \max_{j\in R_i(x^*)} \left(\frac{\partial F(y(x^*))}{\partial y_i}\right.$$

$$\left.\phi'_{ij}(x^*), g_{k_s}\right) + \sum_{i\in I_-(x^*)} \min_{j\in R_i(x^*)} \left(\frac{\partial F(y(x^*))}{\partial y_i}\phi'_{ij}(x^*), g_{k_s}\right)\Bigg] +$$

$$+ \sum_{i\in I} \beta_i(\alpha, x_{k_s} - x^*) = \alpha \frac{\partial f(x^*)}{\partial g_{k_s}} + \sum_{i\in I} \beta_i(\alpha, x_{k_s} - x^*) \quad , \qquad (14)$$

where

$$\beta_i(\alpha, x_{k_s} - x^*) \in \Bigg[\min_{j\in R_i(x^*)} \left(\frac{\partial F(y(x^*))}{\partial x} + \frac{\partial F(y(x^*))}{\partial y_i} \phi'_{ij}(x^*), x_{k_s} - x^*\right),$$

$$\max_{j\in R_i(x^*)} \left(\frac{\partial F(y(x^*))}{\partial x} + \frac{\partial F(y(x^*))}{\partial y_i} \phi'_{ij}(x^*), x_{k_s} - x^*\right)\Bigg] .$$

It is clear that $\beta_i(\alpha, x_{k_s} - x^*) \xrightarrow[k_s\to\infty]{} 0$ uniformly with respect to α.

From (11) it also follows that for k_s sufficiently large,

$$\frac{\partial f(x^*)}{\partial g_{k_s}} \le -\frac{a}{4} \quad .$$

From (14) and (12) we conclude that there exist values of $\alpha_0 > 0$ and k_s such that

$$f(x_{k_s} + \alpha_0 g_{k_s}) < f(x^*) \quad .$$

But this is impossible since

$$f(x_{k_s+1}) = \min_{w\in B_\mu(x_{k_s})} f(x_{k_s} + \alpha_{k_s}(w) g_{k_s}(w)) \le f(x_{k_s} + \alpha_{k_s}(w^*_{k_s}) g_{k_s}(w^*_{k_s}))$$

$$= \min_{\alpha>0} f(x_{k_s} + \alpha g_{k_s}(w^*_{k_s})) \le f(x_{k_s} + \alpha_0 g_{k_s}) < f(x^*) \quad .$$

This contradicts (9) and the fact that $f(x_k) \xrightarrow[k\to\infty]{} f(x^*)$.

3. THE CONSTRAINED CASE

Let us consider the set

$$\Omega = \{x \in E_n \mid h(x) \leq 0\} \tag{15}$$

where

$$h(x) = H(x, y_{m+1}(x), \ldots, y_p(x)) \quad ,$$

$$y_i(x) = \max_{j \in I_i} \phi_{ij}(x) \quad , \quad I_i \equiv 1:N_i \quad , \quad i \in (m+1):p \quad ,$$

and the functions $H(x, y_{m+1}, \ldots, y_p)$ and $\phi_{ij}(x)$ are continuously differentiable on E_{n-m+p} and E_n, respectively. Let the function f be of the form (1). The function h is quasidifferentiable and its quasidifferential can be described analogously to that of f in Section 2. The set Ω defined by (15) is called quasidifferentiable.

The problem is to find $\min_{x \in \Omega} f(x)$. As in (3) we have

$$h(x+\alpha g) = h(x) + \alpha\left[\frac{\partial H(\tilde{y}(x))}{\partial x} + \sum_{i \in I'} \frac{\partial H(\tilde{y}(x))}{\partial y_i} \frac{\partial y_i(x)}{\partial g} + o'(\alpha, g)\right]$$

where

$$I' = (m+1):p \; ; \; \frac{o'(\alpha,g)}{\alpha} \xrightarrow[\alpha \to +0]{} 0 \quad ,$$

$$\tilde{y}(x) = (x, y_{m+1}(x), \ldots, y_p(x)) \quad .$$

Let

$$I'_+(x) = \left\{ i \in I' \,\middle|\, \frac{\partial H(\tilde{y}(x))}{\partial y_i} > 0 \right\}$$

$$I'_-(x) = \left\{ i \in I' \,\middle|\, \frac{\partial H(\tilde{y}(x))}{\partial y_i} < 0 \right\} .$$

Now we have

$$h(x+\alpha g) = h(x) + \alpha \sum_{i\in I_+'(x)} \max_{j\in R_i(x)} \left(\frac{\partial H(\tilde{y}(x))}{\partial x} + \frac{\partial H(\tilde{y}(x))}{\partial y_i} \phi_{ij}'(x), g \right) +$$

$$+ \alpha \sum_{i\in I_-'(x)} \min_{j\in R_i(x)} \left(\frac{\partial H(\tilde{y}(x))}{\partial y_i} \phi_{ij}'(x), g) \right) + o'(\alpha, g)$$

where

$$R_i(x) = \{j \in I_i \mid \phi_{ij}(x) = y_i(x)\} \quad .$$

We now introduce the sets

$$R_{i\varepsilon}(x) = \{j \in I_i \mid \phi_{ij}(x) \geq y_i(x) - \varepsilon\} \quad ,$$

$$\underline{\partial}_{\varepsilon} h(x) = \text{co}\left\{ v\in E_n \,\middle|\, v = \frac{\partial H(\tilde{y}(x))}{\partial x} + \sum_{i\in I_+'(x)} \frac{\partial H(\tilde{y}(x))}{\partial y_i} \phi_{ij}'(x), j\in R_{i\varepsilon}(x) \right\}$$

$$B_{\mu}'(x) = \left\{ w \in E_n \,\middle|\, w = \sum_{i\in I_-'(x)} \frac{\partial H(\tilde{y}(x))}{\partial y_i} \phi_{ij}'(x), j \in R_{i\mu}(x) \right\}$$

where $\varepsilon \geq 0$, $\mu \geq 0$.

Several equivalent necessary conditions for a minimum have been obtained [6,7,8]. Here we take the necessary condition in the form proposed by A. Shapiro [8]:

In order that $x^* \in \Omega$ be a minimum point of a quasidifferentiable function f defined on a quasidifferentiable set Ω, it is necessary that

$$- \overline{\partial} f(x^*) \subset \underline{\partial} f(x^*) \qquad \text{for } h(x^*) < 0 \tag{16}$$

$$-[\overline{\partial} f(x^*) + \overline{\partial} h(x^*)] \subset \text{co}\{\underline{\partial} f(x^*) - \overline{\partial} h(x^*), \underline{\partial} h(x^*) - \overline{\partial} f(x^*)\} \quad \text{for } h(x^*) = 0 \quad . \tag{17}$$

Take $\varepsilon \geq 0$, $\tau \geq 0$. We shall call $x^* \in \Omega$ an (ε,τ)-*inf-stationary point* of f on Ω if

$$-\ \overline{\partial} f(x^*) \subset \underline{\partial}_\varepsilon f(x^*) \qquad \text{for } h(x^*) < -\tau \quad ,$$

$$-[\overline{\partial} f(x^*) + \overline{\partial} h(x^*)] \subset \mathrm{co}\{\underline{\partial}_\varepsilon f(x^*) - \overline{\partial}_\varepsilon h(x^*), \underline{\partial}_\varepsilon h(x^*) - \overline{\partial}_\varepsilon f(x^*)\}$$
$$\text{for } -\tau \leq h(x^*) \leq 0 \quad .$$

We shall now describe an algorithm for finding an (ε,τ)-inf-stationary point with $\varepsilon > 0$, $\mu > 0$ and $\tau > 0$ fixed.

Choose an arbitrary $x_0 \in \Omega$. Suppose that $x_k \in \Omega$ has been found. If condition (16) or (17) is satisfied at x_k then x_k is an (ε,τ)-inf-stationary point and the process terminates. There are two other possibilities:

(a) $h(x_k) < -\tau$,

(b) $-\tau \leq h(x_k) \leq 0$.

In case (a) we perform one step in the minimization of the function f, using the same algorithm as in Section 2 except that

$$\min_{\alpha \geq 0} f(x_k + \alpha g_k(w))$$

must be replaced by

$$\min_{\substack{\alpha \geq 0 \\ x_k + \alpha g_k(w) \in \Omega}} f(x_k + \alpha g_k(w))$$

in (8).

In case (b) we have to find

$$\min\{\|w_1+w_2+v\| \mid v \in \mathrm{co}\{\underline{\partial}_\varepsilon f(x_k) - \overline{\partial}_\varepsilon h(x_k), \underline{\partial}_\varepsilon h(x_k) - \overline{\partial}_\varepsilon f(x_k)\}$$

$$= \|w_1+w_2+v_k(w_1+w_2)\|$$

for every $w_1 \in B_\mu(x_k)$ and $w_2 \in B'_\mu(x_k)$.

Compute

$$\min_{\substack{\alpha \geq 0 \\ h(x_k(\alpha)) \leq 0}} f(x_k(\alpha)) = f(x_k(w_1,w_2)) \tag{18}$$

where

$$x_k(\alpha) = x_k - \alpha(w_1+w_2+v_k(w_1+w_2)) \quad .$$

We then find

$$\min\{f(x_k(w_1,w_2)) \mid w_1 \in B_\mu(x_k), w_2 \in B_\mu'(x_k)\} = f(x_k(w_{k1},w_{k2})) \quad .$$

Setting $x_{k+1} = x_k(w_{k1},w_{k2})$, it is clear that

$$x_{k+1} \in \Omega \quad , \qquad f(x_{k+1}) < f(x_k) \quad .$$

Repeating this procedure, we construct a sequence of points $\{x_k\}$. If it is a finite sequence then the final element is an (ε,τ)-inf-stationary point of f on Ω; otherwise it can be shown that the following theorem holds.

Theorem 2. If the set $\mathcal{D}(x_0) = \{x \in \Omega \mid f(x) \leq f(x_0)\}$ is bounded then any limit point of the sequence $\{x_k\}$ is an (ε,τ)-inf-stationary point of f on Ω.

Proof. Theorem 2 can be proved in the same way as Theorem 1.

Remark 1. If the initial point x_0 does not belong to Ω it is necessary to take a few preliminary steps in the minimization of function h until a point belonging to Ω is obtained.

Remark 2. To find an inf-stationary point (i.e., an (ε,τ)-inf-stationary point where $\varepsilon = \tau = 0$) it is necessary for ε to tend to zero (this can be achieved using the standard mathematical programming techniques).

Remark 3. It is possible to extend the proposed approach to the case where

$$f(x) = \max_{i \in I} F_i(x, y_{i1}(x), \ldots, y_{im_i}(x))$$

$$h(x) = \max_{j \in J} H_j(x, z_{j1}(x), \ldots, z_{jm_j}(x)) \quad , \quad y_{ik}(x) = \max_{\ell \in I_{ik}} \phi_{ik\ell}(x)$$

and the functions $F_i(x, y_{i1}, \ldots, y_{im_i})$, $H_j(x, z_{j1}, \ldots, z_{jm_j})$, $\phi_{ik\ell}(x)$ are continuously differentiable.

Remark 4. Instead of the one-dimensional minimization proposed in (18) it is possible to take

$$x_k(w_1, w_2) = x_k - \lambda_k(w_1 + w_2 + v_k(w_1 + w_2))$$

where

$$\lambda_k \xrightarrow[v \to \infty]{} +0 \quad , \quad \sum_{k=0}^{\infty} \lambda_k = +\infty \quad .$$

ACKNOWLEDGMENT

The authors wish to thank cordially Professor A.M. Rubinov for his valuable advice and Helen Gasking for her careful editing.

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