

MULTIREGIONAL POPULATION PROJECTION

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August 1975

WP-75-106

Paper prepared for presentation at the
7th I.F.I.P. Conference, September 9-12,
1975, University of Nice, France.

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Multiregional Population Projection

Population projections illuminate the impacts of current schedules of births, deaths and migration by drawing out the future consequences of the maintenance of present rates. Methods for developing population projections for single regions are well known, and the mathematics of such exercises have been documented in countless articles, and more recently, in several texts (e.g., Keyfitz, 1968; Pollard, 1973). The mathematics of population projection for multiregional systems that experience internal migration, however, are less known, and it is only recently that concepts such as the multiregional life table have given them a methodological consistency with the conventional mechanics of single-region population projection.

This paper is an exposition of the mathematics of multiregional population projection. We begin by outlining the notion of a multiregional life table. Next, we show how the stationary regional populations of such a life table serve as inputs to numerical calculations carried out with the multiregional versions of the discrete and continuous models of demographic growth [e.g., Leslie, 1945, and Sharpe and Lotka, 1911, respectively]. We then conclude with a brief consideration of some of the spatial consequences of zero population growth.

Although some mathematics is inevitable in an article such as this we have attempted to relegate as much of the mathematical details as possible to the Appendix. Further details also may be found in the text by Rogers (1975).

Finally, an important ingredient of effective strategies to understand and resolve complex problems of a mathematical nature is a powerful notational system. In extending the principal results of single-region population mathematics to multiregional population systems, we generalize conventional

notation as set out, for example, in Keyfitz (1968); although we do not distinguish notationally between continuous and discrete functions. The regional dimension is introduced by means of two subscripts which refer to regions of birth and residence. As in the single-region theory, the argument of a variable usually refers to age, and the right superscript, also enclosed in parentheses, refers to time. The Glossary below brings together most of the variables used in this paper.

GLOSSARY

Observed Population

- ${}_j K_i^{(t)}(x)$ = Number of individuals aged x to $x+4$ years at last birthday in region i at time t who were born in region j .
- $F_i(x)$ = Annual rate of childbearing in region i among individuals aged x to $x+4$ years at last birthday.

Stationary (Life Table) Population

- ${}_j \ell_i(x)$ = Number of individuals at exact age x in region i who were born in region j .
- $q_i(x)$ = Probability of dying within the next 5 years for individuals in region i at exact age x .
- $p_{ij}(x)$ = Probability of residing in region j at exact age $x+5$ for individuals in region i at exact age x .
- ${}_j p_i(x)$ = Probability of residing in region i at exact age x for individuals born in region j .
- $s_{ij}(x)$ = Proportion of x -to- $(x+4)$ -year old residents of region i alive and $x+5$ to $x+9$ years in region j 5 years later.
- ${}_j L_i(x)$ = Number of individuals aged x to $x+4$ years at last birthday in region i who were born in region j . (Also interpretable as the number of person-years lived in region i by j -born individuals between ages x to $x+4$ years.)
- ${}_j T_i(x)$ = Total person-years lived in region i from age x to the end of life by j -born individuals.
- ${}_j e_i(x)$ = Expectation of remaining life in region i for j -born individuals at age x .
- ${}_j R_i(0)$ = Net reproduction rate in region i of j -born individuals.
- B_i = Number of births in the stationary population of region i .

Stable Population

- λ = Stable 5-year growth ratio.
- r = Intrinsic rate of growth.

- b_j = Intrinsic birth rate in region j .
 d_j = Intrinsic death rate in region j .
 o_j = Intrinsic outmigration rate in region j .
 i_j = Intrinsic immigration rate in region j .
 n_j = Intrinsic net migration rate in region j .
 $C_j(x)$ = Proportion of the total stable population in region j that is x -to- $(x+4)$ -years old.
 $(\%)_j$ = Proportion of the total multiregional stable population that is in region j .
 Y_j = Stable equivalent of region j .
 Q_j = Number of births in the stable population of region j .

1. The Multiregional Life Table

A multiregional life table exhibits the mortality and migration history of an artificial population, called a cohort, as it gradually decreases in size until all of its members have died. Normally it is assumed that the age-specific mortality and migration experience to which this cohort is exposed remains constant and that the cohort is undisturbed by emigration and immigration. Consequently, changes in the cohort's membership can only occur in the form of a decrease due to deaths.

The data set out in a multiregional life table originate from a set of probabilities of outmigrating and of dying within each interval of age, $p_{ij}(x)$ and $q_i(x)$, respectively, where $\sum_{j=1}^m p_{ij}(x) + q_i(x) = 1$. Life tables that deal with age intervals of a year are frequently referred to as complete life tables, whereas those using longer intervals are called abridged life tables. We, however, shall ignore this somewhat spurious distinction and for convenience will, without loss of generality, deal only with 5-year age intervals throughout.

Let the regional radix $\ell_i(0)$ denote the number of babies born at a given instant in time in the i^{th} region of an m -region multiregional population system. Subjecting these regional cohorts to the age-specific mortality and mobility of an observed population, we may obtain $\ell_i(x)$, the expected number of individuals who survive to exact age x in region i . However, we need to keep track of where these survivors were born. Consequently, let us introduce an additional subscript on the left-hand side of the variable to designate the region of birth, such that ${}_j\ell_i(x)$ denotes the expected number of survivors alive in region i at age x who were born in region j .

Consider the ${}_j\ell_i(x+5)$ residents of region i at age $x+5$ who were born in region j . They are survivors of the $\sum_{h=1}^m {}_j\ell_h(x)$ j -born individuals who at age

x resided in any one of the m regions ($h = 1, 2, \dots, m$). Denoting by $p_{hi}(x)$ the probability that an individual in region h at age x will survive and be in region i 5 years later, we have the relationship:

$${}_j\ell_i(x+5) = {}_j\ell_1(x)p_{1i}(x) + {}_j\ell_2(x)p_{2i}(x) + \dots = \sum_{h=1}^m {}_j\ell_h(x)p_{hi}(x). \quad (1)$$

It has been estimated that the probability that a female at age 10 and residing in California in 1958 would be living in the rest of the United States 5 years later is 0.058749, and the probability that a female resident of the rest of the United States at the same age in 1958 would still be living there in 1963 is 0.985997. Thus the number of California-born life table survivors at age 15 in the rest of the United States is

$$\begin{aligned} {}_1\ell_2(15) &= {}_1\ell_1(10)p_{12}(10) + {}_1\ell_2(10)p_{22}(10) \\ &= 85,751 (0.058749) + 11,544 (0.985997) \\ &= 5,038 + 11,382 \\ &= 16,420, \end{aligned}$$

a quantity that appears in the second row-first column position in the lower half of the array set out in Table 1. The regional radices for that table were both arbitrarily set to 100,000.

Let ${}_jL_i(x)$ denote the total person-years lived in region i , during the 5-year age interval $(x, x+5)$ by individuals who were born in region j . Assuming a uniform distribution of outmigrations and deaths over the 5-year unit interval of age, we may define the following multiregional generalization of the single-region linear integration formula for deriving $L(x)$:

$${}_jL_i(x) = \frac{5}{2} \left[{}_j\ell_i(x) + {}_j\ell_i(x+5) \right]. \quad (2)$$

TABLE 1 - STATIONARY LIFE TABLE POPULATION, REGIONAL FERTILITY RATES, AND MULTIREGIONAL NET
MATERNITY FUNCTION: UNITED STATES FEMALES, 1958, TWO-REGION MODEL

Region	Age x	${}_1l_1(x)$	${}_2l_1(x)$	${}_1L_1(x)/l_1(0)$	${}_2L_1(x)/l_2(0)$	$F_1(x)$	${}_1\phi_1(x)$	${}_2\phi_1(x)$
California	10	85,751	2,291	4.16220	0.14050	0.00032	0.00134	0.00004
	15	80,737	3,329	3.92220	0.18953	0.04959	0.19451	0.00940
	20	76,151	4,252	3.65597	0.25690	0.12323	0.45052	0.03166
	25	70,088	6,023	3.33854	0.33460	0.08945	0.29862	0.02993
	30	63,454	7,360	3.06113	0.39018	0.05262	0.16109	0.02053
	35	58,991	8,247	2.86151	0.42907	0.02387	0.06831	0.01024
	40	55,469	8,916	2.70519	0.45559	0.00606	0.01640	0.00276
	45	52,738	9,308	2.57330	0.47038	0.00030	0.00078	0.00014
	50	50,194	9,507	2.44204	0.47586	0.00002	0.00004	0.00001
Region	Age x	${}_1l_2(x)$	${}_2l_2(x)$	${}_1L_2(x)/l_1(0)$	${}_2L_2(x)/l_2(0)$	$F_2(x)$	${}_1\phi_2(x)$	${}_2\phi_2(x)$
Rest of U.S.	10	11,544	94,672	0.69909	4.70382	0.00048	0.00034	0.00225
	15	16,420	93,481	0.92960	4.64493	0.04584	0.04261	0.21291
	20	20,764	92,316	1.18150	4.56260	0.12567	0.14848	0.57338
	25	26,496	90,188	1.48124	4.46573	0.09311	0.13792	0.41582
	30	32,754	88,441	1.73468	4.38419	0.05477	0.09502	0.24014
	35	36,634	86,926	1.90037	4.30997	0.02825	0.05369	0.12177
	40	39,381	85,473	2.00486	4.22927	0.00819	0.01642	0.03463
	45	40,813	83,698	2.05530	4.12971	0.00048	0.00100	0.00200
	50	41,399	81,490	2.06703	4.00307	0.00001	0.00003	0.00005

We have estimated the number of California-born life table survivors at age 15 in the rest of the United States to be 16,420. Table 1 gives 11,544 as the corresponding total for those 5 years younger. Hence the total number of person-years lived in the rest of the United States between ages 10 and 15 by the 100,000 California-born females is

$$\begin{aligned} {}_1L_2(10) &= \frac{5}{2} \left[{}_1\ell_2(10) + {}_1\ell_2(15) \right] \\ &= \frac{5}{2} \left[11,544 + 16,420 \right] \\ &= 69,909 , \end{aligned}$$

or 0.69909 years per California-born female.

The remainder of the multiregional life table follows directly. First, we complete the survivorship and migration history of the multiregional cohort of babies. Next, we compute the total person-years in prospect beyond age x by region of residence, ${}_jT_i(x)$ say, for each birth cohort ${}_j\ell_j(0)$, where

$${}_jT_i(x) = \sum_{y=x}^z {}_jL_i(y), \quad z \text{ being the last age interval of life. The}$$

expectation of life beyond age x for j -born individuals then follows directly as:

$${}_je(x) = \frac{\sum_{i=1}^m {}_jT_i(x)}{\sum_{i=1}^m {}_j\ell_i(x)} = \frac{{}_jT(x)}{{}_j\ell(x)} ,$$

and

$${}_je_i(x) = \frac{{}_jT_i(x)}{\sum_{i=1}^m {}_j\ell_i(x)} = \frac{{}_jT_i(x)}{{}_j\ell(x)} , \quad (4)$$

Thus, we conclude that a j -born individual currently at age x can expect to live a total of ${}_je(x)$ more years, of which ${}_je_j(x)$ years will be spent in region j and ${}_je_i(x)$ years will be spent in region i , $i=1, 2, \dots, m$, ($i \neq j$).

Summing the various columns of person-years lived in Table 1 and adding to these totals the person-years lived by those under age 10 and over age 55, we may find the total person-years lived beyond each age, by region of residence and birth, and the corresponding expectations of remaining life at each age. From such calculations we conclude, for example, that a California-born baby girl, under the regional mobility and mortality schedules that prevailed in the United States in 1958, has an expectation of life at birth of 73.86 years of which 24.90 years, on the average, will be lived in the rest of the United States. A baby girl born in the rest of the United States, on the other hand, has a life expectancy of 73.11 years, of which only 5.75 years, on the average, will be lived in California.

Table 2 presents the regional expectations of life at birth by region of residence for the same 1958 data that generated Table 1 but disaggregates California into four regions: the San Francisco, Los Angeles, and San Diego Standard Metropolitan Statistical Areas, and the rest of California. For purposes of comparison, the corresponding expectations of life at birth for males are also included.

The data in Table 2 indicate that the migration patterns of males and females are remarkably similar, with males exhibiting slightly higher levels of geographical mobility. (The proportions of expected lifetimes to be lived in the regions of birth are higher for females born in all but the rest of California region.) The heaviest migration level out of California occurs in the San Diego region, where almost a half (45 to 47 percent) of a baby's expected lifetime is expected to be lived outside of California. This no doubt is due to the large number of births that are attributable to Navy and other military personnel stationed in that region.

TABLE 2 - REGIONAL EXPECTATIONS OF LIFE AT BIRTH
 BY REGION OF RESIDENCE: UNITED STATES MALES AND FEMALES, 1958,
 FIVE-REGION MODEL

A. Males

Region of Birth	Region of Residence					Total
	1.	2.	3.	4.	5.	
1. San Francisco S.M.S.A.	32.51	5.50	1.10	5.59	22.92	67.62
2. Los Angeles S.M.S.A.	4.11	36.06	1.56	3.62	22.16	67.50
3. San Diego S.M.S.A.	3.64	7.67	21.72	2.46	31.95	67.44
4. Rest of California	8.81	7.39	1.27	27.09	22.78	67.35
5. Rest of U.S.	1.34	2.69	0.58	0.87	61.26	66.74

B. Females

Region of Birth	Region of Residence					Total
	1.	2.	3.	4.	5.	
1. San Francisco S.M.S.A.	35.96	6.61	1.18	6.02	24.22	73.98
2. Los Angeles S.M.S.A.	4.77	40.81	1.79	3.82	22.97	74.15
3. San Diego S.M.S.A.	4.22	9.05	24.63	2.61	33.26	73.78
4. Rest of California	10.59	9.09	1.37	27.97	24.71	73.73
5. Rest of U.S.	1.42	2.99	0.55	0.83	67.35	73.14

2. The Discrete Model of Multiregional Demographic Growth

Population projections work out the numerical consequences to an initial population of a particular set of assumptions regarding future fertility, mortality, and geographical mobility. The mechanics of such projections typically revolve around three basic steps. The first ascertains the starting age distribution and the age-specific schedules of fertility, mortality, and migration to which this population has been subject during a past period. The second adopts a set of assumptions regarding the future behavior of such schedules. And the third derives the consequences of applying these schedules to the initial population.

The discrete model of multiregional demographic growth expresses the population projection process by means of a matrix operation in which a multiregional population, set out as a vector, is multiplied by a projection matrix that survives that population forward through time (see Appendix-A.2). The projection calculates the region and age-specific survivors of a multiregional population of a given sex and adds to this total the new births that survive to the end of the unit time interval. This process may be described by the following system of equations:

$$K_i^{(t+1)}(0) = \sum_{x=\alpha-5}^{\beta-5} \sum_{j=1}^m b_{ji}(x) K_j^{(t)}(x) \quad \begin{matrix} \alpha-5 \leq x \leq \beta-5, \\ i = 1, 2, \dots, m \end{matrix} \quad (5)$$

$$K_i^{(t+1)}(x+5) = \sum_{j=1}^m s_{ji}(x) K_j^{(t)}(x) \quad \begin{matrix} x=0, 5, 10, \dots, z \\ i = 1, 2, \dots, m \end{matrix} \quad (6)$$

where we continue to assume a time and age interval of 5 years, and where

$b_{ji}(x)$ = the average number of (female) babies born during a 5-year unit interval and alive in region i at the end of that interval, per (female) person in region j aged x to $x+4$ years at the beginning of the interval;

- $s_{ji}(x)$ = the probability that a (female) resident of region j aged x to $x+4$ years will be alive and in region i 5 years later;
 $K_j^{(t)}(x)$ = the (female) population in region j aged x to $x+4$ years at time t ;
 α = the first age of childbearing;
 β = the last age of childbearing;
 z = the last age interval of life (e.g., 85 years and over).

As in the single-region model, survival of individuals from one moment in time to another, 5 years later, is calculated by diminishing a regional population to take into account the decrement due to mortality. In the multi-regional model, however, we also need to include the decrement due to outmigration and the increment contributed by immigration. An analogous problem is presented by surviving children born during the 5-year interval. Some of these migrate with their parents; others are born after their parents have migrated but before the unit time interval has elapsed.

In the United States an estimated 446,634 [$=K_1^{(t)}(20)$] and 5,149,902 [$=K_2^{(t)}(20)$] women, aged 20 to 24 years at last birthday, were living in California and in the rest of the United States, respectively, at mid-year of 1958. Recalling the two-region life table population for California and the rest of the United States that appears in Table 1 and using the formula for $s_{ii}(x)$ that is developed in Equation (A.12) of the Appendix we may compute, for example,

$$s_{11}(20) = \frac{\frac{1L_1(25)}{1L_2(20)} - \frac{2L_1(25)}{2L_2(20)}}{\frac{1L_1(20)}{1L_2(20)} - \frac{2L_1(20)}{2L_2(20)}} = \frac{\frac{3.33854}{1.18150} - \frac{0.33460}{4.56260}}{\frac{3.65597}{1.18150} - \frac{0.25690}{4.56260}} = 0.90596.$$

Analogous calculations yield the outmigration proportion $s_{21}(20) = 0.02232$.

The sum of the two products $s_{11}(20) K_1^{(t)}(20)$ and $s_{21}(20) K_2^{(t)}(20)$ gives the expected number of women aged 25 to 29 in California in 1963 [Equation (6)]:

$$\begin{aligned} K_1^{(t+1)}(25) &= (0.90596)(466,634) + (0.02232)(5,149,902) \\ &= 537,719. \end{aligned}$$

Utilizing the formula for the fertility rates $b_{ij}(x)$ set out in Equation (A.13) of the Appendix we may calculate the contribution made to the first age group in the rest of the United States in 1963 by surviving female children of 20-to 24-year-old women residents of California in 1958:

$$\begin{aligned} b_{12}(20) &= \frac{1}{2}[(0.11260)F_1(20) + (0.90596)(0.11260)F_1(25) \\ &\quad + (0.09050)(4.90700)F_2(25)] \quad , \end{aligned}$$

into which we may substitute $F_1(20) = 0.12323$, $F_1(25) = 0.08945$, and $F_2(25) = 0.09311$ to find

$$b_{12}(20) = 0.03217.$$

Applying this rate to the estimated 446,634 females in California aged 20 to 24 years in 1958, we find their contribution to the first age group in the rest of the United States in 1963 to be $446,634 (0.03217) = 14,368$ girls. Adding this total to the corresponding contribution made by 20-to 24-year-old females in the rest of the United States in 1958 we obtain the total contribution to $K_2^{(t+1)}(0)$ made by U.S. women aged 20 to 24 in 1958, and aggregating all such totals across the childbearing ages [Equation (5)], we find $K_2^{(t+1)}(0) = 9,638,313$, the resident population aged 0 to 4 at last birthday in the United States at mid-year 1963. Adding this total to the resident population at all other ages gives the rest of the United States a projected grand total of 86,612,665 females for 1963. California's projected total female population for the same year is 8,646,045. In Table 3B we find that the corresponding totals in 1958 were 80,844,419 and 7,395,438, respectively.

It is well known that a regional population which is closed to (i.e., undisturbed by) migration will, if subjected to an unchanging regime of mortality and fertility, ultimately achieve a stable constant age composition that increases at a constant stable growth ratio, λ say. In Rogers (1975) it is shown that this same property obtains region-by-region in the case of a multiregional population system the totality of which is closed to migration and subjected to an unchanging multiregional schedule of mortality, fertility, and internal migration. Knowledge of the asymptotic properties of such a population projection helps us understand the meaning of observed age-specific birth, death, and migration rates. In particular, the quantity $r = 0.2 \ln \lambda$ gives the intrinsic rate of growth that is implied by the indefinite continuation of observed schedules of mortality, fertility, and migration. Table 3 shows that this rate is 0.02064 in the 2-region projection and 0.02065 in the 5-region projection. Both rates are below the 0.02070 yielded by the single-region model. The differences are a consequence of aggregation bias.

A related but equally useful demographic measure is the stable equivalent Y (Keyfitz, 1969) of each region and its proportional allocation across age groups in that region, $C_i(x)$, which is the region's stable age composition. The former may be obtained by projecting the observed multiregional population forward until it becomes stable and dividing the resulting age-region-specific totals by the stable growth ratio λ raised to the n^{th} power, where n is the number of iterations that were needed to achieve stability. Summing across all age groups in a region gives the regional stable equivalent Y_i , whilst dividing the number in each age group in region i by Y_i gives $C_i(x)$, region i 's age composition at stability. Finally, dividing each region's stable equivalent by the sum total of all regional stable equivalents gives $(\%)_i$, the stable regional share of the total multiregional population in region i at stability.

Table 3 presents the above described demographic measures for our California-rest of the United States data of 1958. Also included are intrinsic rates of birth, death, and migration. (These are defined in Appendix A.2.) Note that if the 1958 schedules of growth were to continue unchanged, California's population would ultimately stabilize at about 18 percent of the national total (doubling its 1958 share) and would increase at an annual rate of approximately 20.6 per 1000. Three-fourths of California's stable population would reside in the San Francisco and Los Angeles SMSAs and about one-third of the population would be under 15 years of age. Net migration into the state would be positive, but both the San Diego SMSA and the rest of California each would experience a slight net outmigration of about 0.4 per 1000. Both the highest birth rate and the highest proportion of the aged would be found in the San Diego SMSA, a reflection of San Diego's dual roles as military base and retirement haven.

TABLE 3 - MULTIREGIONAL PROJECTIONS TO STABILITY AND ASSOCIATED
PARAMETERS: UNITED STATES FEMALES, 1958

A. FIVE-REGION PROJECTION

Projections and Stable Growth Parameters	REGION OF RESIDENCE				
	1. San Francisco SMSA.	2. Los Angeles SMSA.	3. San Diego SMSA.	4. Rest of Cal.	5. Rest of U.S.
K(1958)	1,941,994	3,723,919	446,390	1,283,135	80,844,419
% (1958)	0.0220	0.0422	0.0051	0.0145	0.9162
K(2008)	7,561,538	14,488,817	2,334,043	4,634,969	180,567,030
% (2008)	0.0361	0.0691	0.0112	0.0221	0.8615
Y	3,620,347	6,612,727	1,023,696	2,210,093	61,171,949
%	0.0485	0.0886	0.0137	0.0296	0.8196
C(0-14)	0.3275	0.3297	0.3404	0.3520	0.3456
C(15-64)	0.5953	0.5904	0.5570	0.5717	0.5828
C(65+)	0.0773	0.0799	0.1027	0.0763	0.0716
λ	1.10878				
r	0.02065				
b	0.02593	0.02612	0.02826	0.02780	0.02744
$\Delta=b-r$	0.00528	0.00547	0.00760	0.00714	0.00679
d	0.00652	0.00628	0.00721	0.00676	0.00665
i	0.02242	0.01832	0.03163	0.02920	0.00245
o	0.02117	0.01751	0.03202	0.02958	0.00259
n	0.00125	0.00081	-0.00039	-0.00039	-0.00014

B. TWO-REGION AND AGGREGATED MULTIREGIONAL PROJECTIONS

Projections and Stable Growth Parameters	TWO-REGION MODEL		AGGREGATIONS OF MULTIREGIONAL MODELS		SINGLE- REGION MODEL
	1. California	2. Rest of U.S.	TWO-REGION	FIVE-REGION	
K(1958)	7,395,438	80,844,419	88,239,857	88,239,857	88,239,857
K(2008)	28,704,425	180,787,223	209,491,647	209,586,397	209,416,093
Y	13,182,724	61,427,080	74,609,804	74,638,813	74,172,787
%	0.1767	0.8233	1.0000	1.0000	1.0000
C(0-14)	0.3337	0.3456	0.3435	0.3435	0.3443
C(15-64)	0.5865	0.5828	0.5835	0.5834	0.5835
C(65+)	0.0798	0.0716	0.0730	0.0732	0.0722
λ	1.10874		1.10874	1.10878	1.10905
r	0.02064		0.02064	0.02065	0.02070
b	0.02651	0.02744	0.02728	0.02727	0.02734
$\Delta=b-r=d$	0.00587	0.00680	0.00663	0.00662	0.00664

3. The Continuous Model of Multiregional Demographic Growth

The principal contribution of the continuous model of demographic growth lies in its ability to trace through the ultimate consequences of applying a given ~~schedule~~ of fixed age-specific rates of fertility, mortality, and migration to a population of a single sex. It is, therefore, a natural generalization of the multiregional life table's stationary population whose total births are equal to total deaths. When births are not forced to equal deaths, but instead are assumed to occur according to rates that are forever fixed, we obtain the more interesting model of a stable multiregional population. By associating the births of a current generation with those of a preceding generation, one can develop several important constants that describe the ultimate growth and regional age distributions of such a population.

A continuous model of single-sex population growth may be defined for a multiregional population system by means of a straightforward generalization of the corresponding single-region model. Beginning with the number of female births at time t in each region, $B_i(t)$, say, we note that the number of women aged x to $x+dx$ in region i at time t , were born since time zero and are survivors of those born x years ago anywhere in the multiregional system and now living in region i at age x , that is $\sum_{j=1}^m B_j(t-x) {}_j p_i(x) dx$, where $x \leq t$. At time t , these women give birth to

$$\left[\sum_{j=1}^m B_j(t-x) {}_j p_i(x) \right] m_i(x) dx$$

children in region i per year. Here ${}_j p_i(x)$ denotes the probability that a baby girl born in region j will survive to age x in region i , and $m_i(x) dx$ is the annual rate of female childbearing among women aged x to $x+dx$ in region i .

Integrating the above expression over all x and adding $W_i(t)$ to include births to women already alive at time zero gives the fundamental integral

equation system

$$B_i(t) = W_i(t) + \int_0^{t-\beta} \left[\sum_{j=1}^m B_j(t-x) {}_j p_i(x) \right] m_i(x) dx, \quad i = 1, 2, \dots, m \quad (7)$$

For all t beyond the last age of childbearing, those surviving from time zero will no longer contribute to current births, i.e., $W_i(t) = 0$ for $t > \beta$, and (7) then reduces to the homogeneous equation system

$$B_i(t) = \int_0^{t-\beta} \sum_{j=1}^m B_j(t-x) {}_j p_i(x) m_i(x) dx = \int_0^{t-\beta} \sum_{j=1}^m B_j(t-x) {}_j \phi_i(x) dx \quad (8)$$

$$t > \beta \quad i = 1, 2, \dots, m,$$

where ${}_j \phi_i(x) = {}_j p_i(x) m_i(x)$ is the multiregional generalization of the net maternity function of the single-region model (Keyfitz, 1968, Ch.6). With this multiregional net maternity function we may associate the moments

$${}_j R_i(n) = \int_{\alpha}^{\beta} x^n {}_j \phi_i(x) dx \quad i, j = 1, 2, \dots, m \quad (9)$$

among which ${}_j R_i(0)$ is of particular interest inasmuch as it defines the number of (girl) children expected to be born in region i to a (girl) baby now born in region j . Summing this measure over regions of residence we find the region-of-birth reproduction rate for region j :

$${}_j R(0) = {}_j R_1(0) + {}_j R_2(0) + \dots = \sum_{i=1}^m {}_j R_i(0) \quad (10)$$

Alternatively, summing the same measure over regions of birth we obtain the region-of-residence reproduction rate for region i :

$$R_i(0) = \frac{B_1}{B_i} {}_1 R_i(0) + \frac{B_2}{B_i} {}_2 R_i(0) + \dots = \sum_{j=1}^m \frac{B_j}{B_i} {}_j R_i(0) \quad (11)$$

where the weights introduced into the summation reflect the total number of births in each region.

As in the single-region model, the solution of (7) can be found by first obtaining a solution of (8) and then choosing values for the arbitrary constants in that solution so that in addition to satisfying (8), $B_i(t)$ also satisfies (7). Following the procedure used in the single-region model (e.g., Keyfitz, 1968, Ch.5) we adopt the trial solution $B_i(t) = Q_i e^{rt}$ and rewrite (8) as

$$Q_i = \int_{\alpha}^{\beta} \sum_{j=1}^m Q_j e^{-rx} {}_j p_i(x) m_i(x) dx, \quad i = 1, 2, \dots, m,$$

where the range of integration has been narrowed to take into account that $m_i(x) \neq 0$ only for $\alpha \leq x \leq \beta$. Finally, dividing both sides of the equation by Q_i gives the multiregional characteristic equation system

$$1 = \sum_{j=1}^m \frac{Q_j}{Q_i} {}_j \psi_i(r), \quad i = 1, 2, \dots, m, \quad (12)$$

$$\text{where } {}_j \psi_i(r) = \int_{\alpha}^{\beta} e^{-rx} {}_j p_i(x) m_i(x) dx = \int_{\alpha}^{\beta} e^{-rx} {}_j \phi_i(x) dx.$$

Single-region arguments may be used to show that the system of equations in (12) can have only one real root and that any complex roots which satisfy (12), must occur in complex conjugate pairs (Keyfitz, 1968, Ch.5). Furthermore, the real root r is greater than the real part of any complex root. Consequently, the birth sequence $B_i(t) = \sum_{h=1}^{\infty} Q_{hi} e^{r_h t}$ is increasingly dominated by the first term $Q_{1i} e^{r_1 t}$ as t becomes large. Thus, ultimately

$$B_i(t) \doteq Q_{1i} e^{r_1 t} = Q_i e^{rt}.$$

Exponential births lead to an exponentially growing population with a stable distribution in which each age-by-region subpopulation maintains a constant proportional relationship to the total population and increases at the same intrinsic rate of growth, r . The influence of the initial population distribution is forgotten as time goes by, a condition known as ergodicity.

In the single-region model one normally evaluates $\Psi(r)$ with the numerical approximation

$$\Psi(r) = \sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} \frac{L(x)F(x)}{\ell(0)} ,$$

in which the integral $\int_0^5 e^{-r(x+t)} p(x+t)m(x+t)dt$ is replaced by the product of $e^{-r(x+2.5)}$, $\frac{L(x)}{\ell(0)}$, and $F(x)$, the observed fertility rate. The summation is over ages x which are multiples of 5.

An analogous approach may be followed in the multiregional model. We evaluate the integral

$$\int_0^5 e^{-r(x+t)} {}_j p_i(x+t) m_i(x+t) dt$$

as the product of $e^{-r(x+2.5)}$, $\frac{{}_j L_i(x)}{\ell_j(0)}$, and $F_i(x)$.

Thus we have

$${}_j \Psi_i(r) = \sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} \frac{{}_j L_i(x) F_i(x)}{\ell_j(0)}$$

Using the data set out in Table 1, we may compute, for example,

$${}_2 \Psi_1(0) = {}_2 R_1(0) = \sum_{x=10}^{50} \frac{{}_2 L_1(x)}{\ell_2(0)} F_1(x) = 0.105 ,$$

which defines the number of baby girls by which a woman born in the rest of the United States will be replaced in California. Analogous computations give

$${}_1 R_1(0) = 1.192 \quad {}_1 R_2(0) = 0.495 \quad {}_2 R_2(0) = 1.603 .$$

We conclude therefore, that under the 1958 schedule of growth, a girl born in the rest of the United States will be replaced, in the subsequent generation, by ${}_2 R(0) = 0.105 + 1.603 = 1.708$ baby girls of whom 0.105 will be born in California. Corresponding measures for a California-born girl are 1.687 and 0.495, respectively.

But $r = 0$ is clearly not the solution of Equation (12). This can be readily established by substituting the values of ${}_j\Psi_i(0)$ into (12) and solving for Q_2/Q_1 and Q_1/Q_2 , respectively. Solving the first equation we obtain $Q_2/Q_1 = -1.829$; the solution to the second is $Q_1/Q_2 = -1.218$ or, equivalently, $Q_2/Q_1 = -0.821$. Since we have two different estimates of the same quantity it is clear that we have not yet found the correct value for r . By a process of iteration we ultimately converge to $r = 0.02059$, for which

$${}_1\Psi_1(r) = 0.711 \quad {}_1\Psi_2(r) = 0.282 \quad {}_2\Psi_1(r) = 0.060 \quad {}_2\Psi_2(r) = 0.941 .$$

Substituting these into Equation (12) gives $Q_2/Q_1 = 4.823$ in both cases, i.e.,

$$\begin{aligned} 0.711 + (4.823)0.060 &= 1 \\ (1/4.823)0.282 + 0.941 &= 1 \end{aligned}$$

4. The Spatial Consequences of Zero Population Growth

During the past decade, several White House task forces, countless congressional committees, and scores of public interest groups have attempted to define the outlines of a desirable national population growth policy, taking as their starting point the widespread conviction that such growth is not taking place the way it should. Even though these committees and task forces span more than a decade and several administrations, their respective products have been remarkably similar in coverage, major themes, and proposals. Most begin by projecting the nation's population growth to a net increase of anywhere from 80 to 145 million Americans by the turn of the century. Almost all of these study groups then assert that without public intervention, a majority of citizens will inherit steadily growing, already overcrowded and poorly planned metropolitan areas. (See, for example, the various reports of the U.S. Commission on Population Growth and the American Future, 1972.)

The contention that America's population crisis stems from a propensity to overbreed overlooks the evident fact that any demographic imbalance in the U.S. today is less one of absolute numbers than of their maldistribution. The notion of a population distribution policy therefore has wide appeal but, unfortunately, insufficient substance. An important contributing factor to this lack of substance is our poor understanding of the dynamics of multiregional demographic growth and distribution.

Demographers agree that because of the large number of young people in America's population today, immediate zero population growth is not a practical objective. Consequently, most projected paths toward a stationary population assume an average of approximately 2.11 births per woman from now on and hold mortality fixed. On the assumption of zero or negligibly small net immigration, such a projection leads in about 70 years to a stationary population that is

approximately 40 to 50 percent larger than the current population. Much has been made of the social and economic consequences of such a population and particularly important have been the analyses of its stationary age composition--an age composition that would have a higher median age and virtually constant numbers from age zero to 50. (See, for example, Coale, 1972.)

But what of the spatial distribution of such a stationary national population? What are the alternative paths in a geographic context? Will we, for example have as Alonso (1973, p. 191) puts it "a nationally stable population ... composed of many localities declining in population, many localities growing, and only some remaining stable"?

A nationally stationary population may arise out of a growth process which exhibits a zero growth rate in each short interval of time or it may develop out of a long-run average zero growth rate which occurs as a consequence of a combination of sequences of positive growth, of zero growth, and of decline. Since no obvious advantages arise from the latter case, demographers quite naturally have viewed the attainment of a stationary population as arising from a continuation of zero growth in the short-run. Thus the normal assumption involves a fixed mortality schedule and fertility set at replacement level.

An analogous situation arises in the case of a multiregional population. By augmenting the assumptions of fixed mortality and replacement level fertility with the assumption of fixed migration we may obtain a stationary multiregional population. In such a case, each region in the system will grow at a zero rate of growth. (Alternatively, we may assume that zero growth for the multiregional system is a consequence of an aggregation of zero and nonzero growth rates in its constituent regions. The dynamics of this situation are more complex and will not be considered in this paper.)

If mortality is fixed and one thousand baby girls born at each moment replace themselves, on the average, with a thousand baby girls as they move past their childbearing years, we will ultimately obtain a stationary zero growth population. But the women who survive to the childbearing ages must have enough daughters to replace not only themselves but also those women who have not survived to become mothers. Thus we specify that the net (and not the gross) reproduction rate of the female population be unity, i.e., $R(0)=1$. Reducing observed age-specific fertility rates proportionally to obtain a net reproduction rate of unity then is one way of achieving a stationary population. The last column in the lower half of Table 4 shows that had the U.S. female population in 1958 immediately moved to replacement levels, the 88 million female population of that year would have grown to 113 million (the stationary equivalent, Y) before attaining zero population growth.

The multiregional analog of the above calculation is straightforward. We simply reduce the observed age-specific regional fertility rates proportionally until region-of-birth net reproduction rates are all equal to unity, i.e., ${}_jR(0) = 1, j = 1, 2, \dots, m$. (To avoid interrupting the flow of the argument, we have relegated the discussion of such a calculation to the Appendix.) The mechanics of the population projection process itself, however, remain unchanged.

Table 4 sets out some of the more interesting consequences of an immediate movement to replacement levels of fertility by the 1958 U.S. female population . that has served as our numerical example throughout this paper. (Note that Table 4 is the zero growth counterpart of Table 3, which illuminated the long-run consequences of an unchanging continuance of present rates.) Several findings are of some interest and merit elaboration.

First, observe that the spatial allocations or shares of the stationary multiregional population in Table 4 do not differ significantly from those of

TABLE 4 - MULTIREGIONAL PROJECTIONS TO ZERO GROWTH AND ASSOCIATED
PARAMETERS: UNITED STATES FEMALES, 1958

A. FIVE-REGION PROJECTION

Projections and Stable Growth Parameters	REGION OF RESIDENCE				
	1. San Francisco S.M.S.A.	2. Los Angeles S.M.S.A.	3. San Diego S.M.S.A.	4. Rest of Cal.	5. Rest of U.S.
K(1958)	1,941,994	3,723,919	446,390	1,283,135	80,844,419
% (1958)	0.0220	0.0422	0.0051	0.0145	0.9162
K(2008)	4,132,157	7,869,750	1,216,643	2,398,295	95,274,261
% (2008)	0.0373	0.0710	0.0110	0.0216	0.8592
Y	5,887,834	10,558,059	1,546,379	3,337,595	92,180,796
%	0.0519	0.0930	0.0136	0.0294	0.8121
C(0-14)	0.1997	0.1960	0.1705	0.1979	0.2014
C(15-64)	0.6296	0.6223	0.5874	0.6212	0.6309
C(65+)	0.1707	0.1818	0.2422	0.1810	0.1677
λ	1.00000				
r	0.00000				
b	0.01359	0.01317	0.01178	0.01315	0.01375
$\Delta=b-r$	0.01359	0.01317	0.01178	0.01315	0.01375
d	0.01334	0.01305	0.01599	0.01438	0.01366
i	0.01906	0.01624	0.03167	0.02775	0.00224
o	0.01931	0.01636	0.02746	0.02651	0.00233
n	-0.00025	-0.00012	0.00421	0.00124	-0.00009

B. TWO-REGION AND AGGREGATED MULTIREGIONAL PROJECTIONS

Projections and Stable Growth Parameters	TWO-REGION MODEL		AGGREGATIONS OF MULTIREGIONAL MODELS		SINGLE- REGION MODEL
	1. California	2. Rest of U.S.	TWO-REGION	FIVE-REGION	
K(1958)	7,395,438	80,844,419	88,239,857	88,239,857	88,239,857
K(2008)	15,442,904	95,404,038	110,846,942	110,891,106	110,653,183
Y	20,765,005	92,670,056	113,435,061	113,512,666	112,988,412
%	0.1831	0.8169	1.0000	1.0000	1.0000
C(0-14)	0.1927	0.2014	0.1998	0.1997	0.2003
C(15-64)	0.6249	0.6310	0.6299	0.6296	0.6308
C(65+)	0.1824	0.1676	0.1703	0.1707	0.1689
λ	1.00000		1.00000	1.00000	1.00000
r	0.00000		0.00000	0.00000	0.00000
b	0.01322	0.01375	0.01365	0.01364	0.01369
$\Delta=b-r=d$	0.01322	0.01375	0.01365	0.01364	0.01369

the stable multiregional population in Table 3. In both cases, California receives approximately 18 percent of the national population with the San Francisco, Los Angeles and San Diego SMSA's receiving 5, 9, and $1\frac{1}{2}$ percent, respectively. Thus it appears that the spatial allocation effects of proportionally reduced fertility are negligible.

Although the redistributational effects of proportionally reduced fertility are negligible, the age compositional effects are not. As in the single-region model, reduced fertility produces an older population which has a much higher percentage of its members in the 65 years and over age group. However, the interaction of reduced fertility and fixed migration schedules produces an uneven regional allocation of the aged. Thus although California under zero growth would have about 18 percent of its population in the 65-year and over age group, San Diego would have more (24 percent) while San Francisco would have less (17 percent). The spatial population dynamics leading up to this result are clear. San Diego, because it is a retirement haven, receives relatively "older" inmigrants than does San Francisco. As the proportion of the aged increases nationally, San Diego will receive a heavier than average net inflow of migrants. This is why its net migration rate changes from a negative 0.4 per 1000 in Table 3 to a positive 4.2 per 1000 in Table 4. San Francisco's corresponding rates, on the other hand, exhibit a reverse shift, decreasing from a positive net migration rate of 1.2 per 1000 to a negative rate of 0.2 per 1000.

Finally, Table 4 shows that regions which exhibit higher than average birth rates prior to zero growth will have lower than average birth rates during zero growth. Once again San Diego offers an interesting case study. According to Table 3, it has the highest intrinsic birth rate of all 5 regions in the system (28 per 1000). Yet in Table 4 its intrinsic birth rate is the lowest (12 per 1000). The population dynamics producing this reversal are the same as those outlined

earlier and result from the relatively older population that San Diego would have under zero growth. (Note that San Diego's intrinsic death rate is the highest both before and during zero growth).

In one of his contributions to the final reports produced by the Commission on Population Growth and the American Future, Peter Morrison (1972, p. 547) observed:

"...demographic processes interact in subtle and often complex ways, and the mechanisms by which declining fertility would influence population redistribution are only partially understood."

It is hoped that this paper has identified and illuminated some of these mechanisms.

REFERENCES

- Alonso, W. (1973). "Urban Zero Population Growth," Daedalus, CII, 191-206.
- Coale, A.J. (1972). "Alternative Paths to a Stationary Population," in Demographic and Social Aspects of Population Growth, Ed., C.F. Westoff and R. Parke, Jr., U.S. Commission on Population Growth and the American Future (Washington, D.C.: U.S. Government Printing Office), pp.589-603.
- Feeney, G. (1970). "Stable Age by Region Distributions," Demography, VI, 341-348.
- Gantmacher, F.R. (1959). The Theory of Matrices, trans, K.A. Hirsch, Vol. 2 (New York: Chelsea Publishing Co.).
- Keyfitz, N. (1968). Introduction to the Mathematics of Population, (Reading, Mass.: Addison-Wesley).
- _____. (1969). "Age Distribution and the Stable Equivalent," Demography, VI, 261-269.
- LeBras, H. (1971). "Equilibre et Croissance de Populations Soumises a des Migration," Theoretical Population Biology, II, 100-121.
- Leslie, P.H. (1945). "On the Use of Matrices in Certain Population Mathematics," Biometrika, XXXIII, 183-212.
- Morrison, P.A. (1972). "The Impact of Population Stabilization on Migration and Redistribution," in Population, Distribution, and Policy, Ed., S.M. Mazie, U.S. Commission on Population Growth and the American Future (Washington, D.C.: U.S. Government Printing Office), pp. 543-560.
- Parlett, B. (1970). "Ergodic Properties of Population, I: The One Sex Model," Theoretical Population Biology, I, 191-207.
- Pollard, J.H. (1973). Mathematical Models for the Growth of Human Populations (London: Cambridge University Press).
- Rogers, A. (1966). "The Multiregional Matrix Growth Operator and the Stable Interregional Age Structure," Demography, III, 537-544.
- _____. (1971). Matrix Methods in Urban and Regional Analysis, (San Francisco, Cal.: Holden-Day).
- _____. (1974). "The Multiregional Net Maternity Function and Multiregional Stable Growth," Demography, XI, 473-481.
- _____. (1975). Introduction to Multiregional Mathematical Demography (New York: John Wiley).
- Sharpe, F.R. and A.J. Lotka (1911). "A Problem in Age-Distribution," Philosophical Magazine, Ser. 6, XXI, 435-438.
- Stone, L.O. (1968). "Stable Migration Rates from the Multiregional Growth Matrix Operator," Demography, V, 439-442.
- Sykes, Z.M. (1969). "On Discrete Stable Population Theory," Biometrics, XXV, 285-293.

APPENDIX

THE MATHEMATICS OF MULTIREGIONAL POPULATION GROWTH

A.1 The Multiregional Life Table

The components of a multiregional life table originate from a set of survivorship-outmigration matrices $\underline{P}(x)$, $x = 0, 5, 10, \dots, z-5$, which are applied in the appropriate sequence to a collection of survival matrices $\underline{\ell}(x)$ which describe the number of individuals attaining age x by region of birth and residence. Starting with the first survival matrix $\underline{\ell}(0)$, a diagonal matrix, we obtain

$$\underline{\ell}(5) = \underline{P}(0)\underline{\ell}(0)$$

where, for example, in a two-region system

$$\underline{\ell}(5) = \begin{bmatrix} {}_1\ell_1(5) & {}_2\ell_1(5) \\ {}_1\ell_2(5) & {}_2\ell_2(5) \end{bmatrix} \quad \underline{P}(0) = \begin{bmatrix} p_{11}(0) & p_{21}(0) \\ p_{12}(0) & p_{22}(0) \end{bmatrix} \quad \underline{\ell}(0) = \begin{bmatrix} \ell_1(0) & 0 \\ 0 & \ell_2(0) \end{bmatrix}$$

More generally, we may define the following matrix counterpart to Equation (1):

$$\underline{\ell}(x+5) = \underline{P}(x) \underline{\ell}(x) \quad , \quad (A.1)$$

where the elements in the i^{th} row and j^{th} column of $\underline{\ell}(x)$ and $\underline{P}(x)$ are ${}_j\ell_i(x)$ and $p_{ji}(x)$, respectively.

Observe the transposed subscripting in the matrices, and note the Markovian assumption that is implicit in (A.1), i.e., the same probabilities are applied to all residents of a region irrespective of their previous life-residence history.

We have elected to defy traditional matrix notation and use transposed subscripting in order to preserve a left-to-right ordering of places of residence while retaining the usual "matrix-times-a-vector" multiplication process used in the single-region model (e.g., in Keyfitz, 1968 , Ch. 3).

Integrating (A.1) over 5-year intervals to find

$$\underline{L}(x) = \int_0^5 \underline{\ell}(x+t) dt \quad ,$$

we use the linear approximation

$$\tilde{L}(x) = \frac{5}{2} \left[\tilde{\ell}(x) + \tilde{\ell}(x+5) \right] , \quad (\text{A.2})$$

where the element in the i^{th} row and j^{th} column of $\tilde{L}(x)$ is ${}_j\tilde{L}_i(x)$.

Summing the various person-years lived to obtain

$$\tilde{T}(0) = \sum_{x=0}^z \tilde{L}(x) , \quad (\text{A.3})$$

we may calculate the matrix of expectations of life at birth by place of birth and residence:

$$\tilde{e}(0) = \tilde{T}(0) \tilde{\ell}(0)^{-1} , \quad (\text{A.4})$$

where the element in the i^{th} row and j^{th} column of $\tilde{e}(0)$ is ${}_j\tilde{e}_i(0)$.

A.2 The Discrete Model of Multiregional Demographic Growth

The multiregional growth and distribution process defined in Section 2 of this paper enables us to express the relation between the population at time t and time $t+1$ as a set of linear, first-order, homogeneous difference equations with constant coefficients. These can be expressed compactly in matrix form. Two of the most logical formulations use alternative generalizations of the so-called Leslie matrix for the single-region case. The arrangement of elements defined in Rogers (1966) was of the form

$$\tilde{G} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{21} & \cdots & \tilde{G}_{m1} \\ \tilde{G}_{12} & \tilde{G}_{22} & \cdots & \tilde{G}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{G}_{1m} & \tilde{G}_{2m} & \cdots & \tilde{G}_{mm} \end{bmatrix}$$

where

$$\tilde{G}_{ij} = \begin{bmatrix} 0 & 0 & b_{ij}(\alpha-5) & \cdots & b_{ij}(\beta-5) & 0 & \cdots & 0 \\ s_{ij}(0) & & & & & & & \vdots \\ & s_{ij}(5) & & & & & & \vdots \\ & & \cdot & & & & & \vdots \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & s_{ij}(z-5) & 0 \end{bmatrix} \quad (\text{A.5})$$

for $i, j = 1, 2, \dots, m$.

An alternative arrangement, suggested by Feeney (1970), is

$$\tilde{H} = \begin{bmatrix} \tilde{0} & \tilde{0} & \tilde{B}(\alpha-5) & \cdots & \tilde{B}(\beta-5) & \tilde{0} & \cdots & \tilde{0} \\ \tilde{S}(0) & & & & & & & \vdots \\ & \tilde{S}(5) & & & & & & \vdots \\ & & \cdot & & & & & \vdots \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \tilde{S}(z-5) & \tilde{0} \end{bmatrix}$$

We shall follow Feeney's designation of \tilde{G} as the multiregional matrix growth operator and of \tilde{H} as the generalized Leslie matrix. The matrix expression of the multiregional growth process using the multiregional matrix growth operator is

$$\{\tilde{K}^{(t+1)}\} = \tilde{G} \{\tilde{K}^{(t)}\} \quad , \quad (A.6)$$

where

$$\{\tilde{K}^{(t)}\} = \begin{Bmatrix} \tilde{K}_1^{(t)} \\ \tilde{K}_2^{(t)} \\ \vdots \\ \tilde{K}_m^{(t)} \end{Bmatrix} \quad \text{and} \quad \{\tilde{K}_i^{(t)}\} = \begin{Bmatrix} K_i^{(t)}(0) \\ K_i^{(t)}(5) \\ \vdots \\ K_i^{(t)}(z) \end{Bmatrix}$$

The same growth process expressed with the generalized Leslie matrix is

$$\{\tilde{\bar{K}}^{(t+1)}\} = \tilde{H} \{\tilde{\bar{K}}^{(t)}\} \quad (A.7)$$

where

$$\{\tilde{\bar{K}}^{(t)}\} = \begin{Bmatrix} \tilde{\bar{K}}^{(t)}(0) \\ \tilde{\bar{K}}^{(t)}(5) \\ \vdots \\ \tilde{\bar{K}}^{(t)}(z) \end{Bmatrix} \quad \text{and} \quad \{\tilde{\bar{K}}^{(t)}(x)\} = \begin{Bmatrix} K_1^{(t)}(x) \\ K_2^{(t)}(x) \\ \vdots \\ K_m^{(t)}(x) \end{Bmatrix}$$

Survivorship and Outmigration

Consider an m-region population system with $k_i(x)$ individuals at age x in region i . Then, as in the life table population, we have that

$$k_i(x) = {}_1k_i(x) + {}_2k_i(x) + \dots = \sum_{j=1}^m {}_jk_i(x) \quad , \quad i = 1, 2, \dots, m, \quad (A.8)$$

where ${}_jk_i(x)$ denotes the number of j -born individuals who are in region i at age x . The expected survivors of the multiregional population after five years are

$$k_j(x+5) = \sum_{i=1}^m k_i(x) p_{ij}(x) \quad , \quad j = 1, 2, \dots, m,$$

where $p_{ij}(x)$ is the probability that an individual in region i at age x will be in region j at age $x+5$. Substituting the definitional relationship of (A.8) into the above equation and utilizing the fact that i -born individuals can never become members of a j -born population, and vice-versa, we have that

$${}_jk_i(x+5) = \sum_{h=1}^m {}_jk_h(x) p_{hi}(x) \quad , \quad i, j = 1, 2, \dots, m$$

or, in matrix form,

$$\underline{\tilde{k}}(x+5) = \underline{P}(x) \underline{\tilde{k}}(x) ;$$

whence

$$\underline{P}(x) = \underline{\tilde{k}}(x+5) \underline{\tilde{k}}(x)^{-1} .$$

The expected survivors of this multiregional population aged x to $x + dx$, $\underline{\tilde{k}}(x)dx$, say, after 5 years are

$$\underline{\tilde{k}}(x+5)dx = \underline{P}(x)\underline{\tilde{k}}(x)dx$$

and those between ages x and $x+5$ expected to survive 5 years are

$$\underline{\tilde{K}}(x+5) = \int_0^5 \underline{\tilde{k}}(x+5+t)dt = \int_0^5 \underline{P}(x+t) \underline{\tilde{k}}(x+t)dt \quad (A.9)$$

As in the single-region model, we assume that the survivorship and out-migration behavior of the stationary life table population adequately represents that of the empirical population. Thus, substituting the life table matrix product $\underline{\tilde{L}}(x+5) \underline{\tilde{L}}(x)^{-1}$ for $\underline{P}(x)$ in (A.9), we obtain

$$\underline{\tilde{K}}(x+5) = \int_0^5 \underline{\tilde{L}}(x+5+t) \underline{\tilde{L}}(x+t)^{-1} \underline{\tilde{K}}(x+t) dt \quad .$$

And adopting the multiregional analog of the numerical approximation to this integral that is normally used in the single-region model [e.g., Keyfitz, 1968 , p. 247] we find

$$\begin{aligned} \underline{\tilde{K}}(x+5) &= \int_0^5 \underline{\tilde{L}}(x+5+t) dt \left[\int_0^5 \underline{\tilde{L}}(x+t) dt \right]^{-1} \int_0^5 \underline{\tilde{K}}(x+t) dt = \underline{\tilde{L}}(x+5) \underline{\tilde{L}}(x)^{-1} \underline{\tilde{K}}(x) \\ &= \underline{\tilde{S}}(x) \underline{\tilde{K}}(x) \quad , \end{aligned} \quad (\text{A.10})$$

where

$$\underline{\tilde{S}}(x) = \underline{\tilde{L}}(x+5) \underline{\tilde{L}}(x)^{-1} \quad . \quad (\text{A.11})$$

In the case of a two-region model, Equation (A.11) yields, for example,

$$s_{ii}(x) = \frac{\frac{iL_i(x+5)}{iL_j(x)} - \frac{jL_i(x+5)}{jL_j(x)}}{\frac{iL_i(x)}{iL_j(x)} - \frac{jL_i(x)}{jL_j(x)}} \quad (\text{A.12})$$

Generally population projections do not call for population totals disaggregated by place of birth. Hence, we may consolidate the $\underline{\tilde{K}}(x)$ matrix into the vector

$$\left\{ \overline{\underline{\tilde{K}}}^{(t)}(x) \right\} = \underline{\tilde{K}}^{(t)}(x) \left\{ \underline{\tilde{1}} \right\} \quad , \text{ and post-multiply both sides of (A.10)}$$

$$\text{by } \left\{ \underline{\tilde{1}} \right\} \text{ to find } \left\{ \overline{\underline{\tilde{K}}}^{(t+1)}(x+5) \right\} = \underline{\tilde{S}}(x) \left\{ \overline{\underline{\tilde{K}}}^{(t)}(x) \right\} \quad .$$

We have added a t superscript to denote time and have placed a bar over the vector in order to distinguish it from the vector $\left\{ \underline{\tilde{K}}^{(t)} \right\}$, to which we ascribed a somewhat different structure in (A.6).

Fertility

To complete the multiregional population projection, we must include an estimate of the total number of births during the unit time interval. Denoting the age-region-specific annual birth rate by $F_i(x)$, we multiply it by the arithmetic mean of the initial and final populations of ages x to $x+4$ in region i :

$$\frac{K_i^{(t)}(x) + K_i^{(t+1)}(x)}{2} = \frac{1}{2} \left(K_i^{(t)}(x) + \sum_{j=1}^m s_{ji}(x-5) K_j^{(t)}(x-5) \right),$$

and since this number is exposed for 5 years, we multiply it by 5. The women in region i aged x to $x+4$ together with those aged $x+5$ to $x+9$ at last birthday will contribute, during the 5-year time interval, a total of

$$\frac{5}{2} \left\{ K_i^{(t)}(x) + K_i^{(t+1)}(x) \right\} F_i(x) + \frac{5}{2} \left\{ K_i^{(t)}(x+5) + K_i^{(t+1)}(x+5) \right\} F_i(x+5)$$

births. Of these, a proportion $\frac{i L_j(0)}{5 \ell_i(0)}$ will be surviving residents of region

j at the end of the time interval. Adding through all ages of childbearing, α through β , rearranging, and adopting the more compact notation of matrix algebra, we have

$$\begin{aligned} \left\{ \bar{K}^{(t+1)}(0) \right\} &= \frac{1}{2} \sum_{x=\alpha-5}^{\beta-5} \underline{L}(0) \underline{\ell}(0)^{-1} \left[\underline{F}(x) + \underline{F}(x+5) \underline{S}(x) \right] \left\{ \bar{K}^{(t)}(x) \right\} \\ &= \sum_{x=\alpha-5}^{\beta-5} \underline{B}(x) \left\{ \bar{K}^{(t)}(x) \right\}, \quad \text{say,} \end{aligned}$$

where $\underline{\ell}(0)$ and $\underline{F}(x)$ are diagonal matrices with regional radices and regional fertility rates, respectively, set out along their principal diagonals.

We have therefore the age-region-specific fertility elements

$$b_{ij}(x) = \frac{1}{2} \left[\frac{i L_j(0)}{\ell_i(0)} F_i(x) + \sum_{h=1}^m s_{ih}(x) \frac{h L_j(0)}{\ell_h(0)} F_h(x+5) \right] \quad (\text{A.13})$$

Stable Growth

Having expressed population growth in matrix form, as in (A.6), we now may examine, by means of repeated multiplication, the long-run implications of maintaining current age-specific birth, death and outmigration rates.

For example, observe that

$$\{\tilde{K}^{(t+2)}\} = \tilde{G} \{\tilde{K}^{(t+1)}\} = \tilde{G}^2 \{\tilde{K}^{(t)}\} ,$$

and, in general,

$$\{\tilde{K}^{(t+n)}\} = \tilde{G}^n \{\tilde{K}^{(t)}\} . \quad (A.14)$$

The properties of such a projection as n increases indefinitely have been studied by Leslie (1945) and Keyfitz (1968), among others, and more recently by Sykes (1969), Parlett (1970), and Rogers (1975). This body of theory, commonly referred to as stable growth theory, draws on the properties of matrices with nonnegative elements, and, in particular, on what is commonly referred to as the Perron-Frobenius theorem.

The Perron-Frobenius theorem establishes that any nonnegative, indecomposable, primitive square matrix has a unique, real, positive characteristic root, λ say, that is larger in absolute value than any other characteristic root of that matrix. Moreover, one can associate with this dominant characteristic root a characteristic vector that has only positive elements, which for convenience we shall assume are scaled so as to sum to unity.

To establish the asymptotic properties of the projection process in (A.14), we first partition each submatrix \tilde{G}_{ij} of \tilde{G} at the highest age of reproduction [i.e., after the column headed by $b_{ij}(\beta-5)$ in (A.5)]. Then we permute the rows and columns of \tilde{G} to form the similar matrix $\hat{\tilde{G}}$ in which the upper partitioned parts of each \tilde{G}_{ij} are brought together as the submatrix \tilde{V} to form the upper left corner of $\hat{\tilde{G}}$. Since \tilde{G} and $\hat{\tilde{G}}$ are similar matrices, they have the same characteristic roots. Because

only the submatrix \underline{V} of \hat{G} can affect the growth of the population below the highest age of reproduction, it can be shown that the characteristic roots of the submatrix \underline{V} completely determine those of \hat{G} , and therefore of G .

From here on we may follow the usual sequence of arguments used in the single-region theory. First, we may establish the primitivity of \underline{V} , under certain conditions regarding the positioning of positive fertility elements in that matrix. We then may draw on the Perron-Frobenius theorem to establish the existence of a unique, real, positive, dominant characteristic root and associated positive characteristic vector. Finally, we may show that successive powers of \underline{V} converge to a limiting form with proportional columns, leading to the observation that the effects of the initial population distribution on those at future points in time diminish as time increases, and ultimately disappear entirely as the population assumes the stable distribution. This feature of the projection process usually is established by diagonalizing \underline{V} under the assumption that its characteristic roots are distinct (Rogers, 1971, pp. 422-423).

The sequence of arguments outlined above hinges on the establishment of conditions under which \underline{V} is primitive. One such condition in the single-region theory is that at least two adjacent fertility elements in the first row be positive (Pollard, 1973, pp. 46-47). A natural generalization of this condition to the multiregional case is, of course, that two adjacent fertility elements be positive in each and every submatrix \underline{V}_{ij} of \underline{V} . However, as in the single-region case this condition can be shown to be unnecessarily restrictive (see Rogers, 1975, Ch. 5).

Intrinsic Rates

As Stone (1968) points out, classical work in stable population theory has ignored migration and therefore has not dealt with the notion of stable migration rates. His proposed formulas for stable age-specific migration rates are our starting point for developing the notion of intrinsic migration rates for the multiregional theory.

Stone defines

$${}_5i_j(x) = \frac{\sum_{\substack{h=1 \\ h \neq j}}^m K_h^{(t)}(x) s_{hj}(x)}{K_j^{(t)}(x)}$$

and

$${}_5o_j(x) = \sum_{\substack{h=1 \\ h \neq j}}^m s_{jh}(x)$$

to be the stable age-specific immigration and outmigration rates, respectively, of a regional population experiencing stable growth. Analogously,

$${}_5d_j(x) = 1 - \sum_{h=1}^m s_{jh}(x)$$

may be defined to be the stable age-specific death rate of the same population. Thus among people aged x in region i at stability, a proportion ${}_5o_j(x)$ leave the region during a 5 year interval of time while, simultaneously, a fraction ${}_5d_j(x)$ die and a total of

$$\sum_{\substack{h=1 \\ h \neq j}}^m K_h^{(t)}(x) s_{hj}(x)$$

individuals migrate into the region to give it an immigration rate of ${}_5i_j(x)$.

But outmigrations, deaths, and immigrations also occur among babies during the

same time interval. For example,

$$\sum_{\substack{h=1 \\ h \neq j}}^m j L_h(0) / {}^5\ell_j(0)$$

is the proportion of births in region j who migrate and survive to be members of the first age group outside of region j at the start of the next unit time interval. Hence, the consolidated "crude" 5-year stable outmigration rate of region j is

$${}^5o_j = \frac{\sum_{x=0}^z K_j^{(t)}(x) {}^5o_j(x)}{\sum_{x=0}^z K_j^{(t)}(x)} + {}^5b_j \sum_{\substack{h=1 \\ h \neq j}}^m \frac{j L_h(0)}{{}^5\ell_j(0)},$$

where 5b_j is the corresponding 5-year stable birth rate:

$${}^5b_j = \frac{\frac{5}{2} \sum_{x=\alpha}^{\beta-5} [K_j^{(t)}(x) + K_j^{(t+1)}(x)] F_j(x)}{\sum_{x=0}^z K_j^{(t)}(x)}.$$

By an analogous argument

$${}^5d_j = \frac{\sum_{x=0}^z K_j^{(t)}(x) {}^5d_j(x)}{\sum_{x=0}^z K_j^{(t)}(x)} + {}^5b_j \left(1 - \frac{\sum_{h=1}^m j L_h(0)}{{}^5\ell_j(0)} \right).$$

and 5i_j can be found as a residual since, by definition

$$1 = {}^5b_j + {}^5d_j + {}^5o_j + {}^5i_j = e^{5r}, \quad (\text{A.15})$$

where λ is the dominant characteristic root of the projection matrix \underline{G} , and $r = 0.2 \ln \lambda$. To transform (A.15) into the corresponding equation for intrinsic

rates:

$$r = b_j - d_j - o_j + i_j = b_j - d_j + n_j = b_j - \Delta_j, \quad (\text{A.16})$$

we rewrite (A.15) as

$$e^{5r} - 1 = {}^5b_j - {}^5d_j - {}^5o_j + {}^5i_j$$

and multiply both sides of the equation by $\frac{r}{e^{5r}-1}$ to find (A.16) where

$$\left. \begin{aligned} b_j &= \left(\frac{r}{e^{5r}-1} \right) {}^5b_j & d_j &= \left(\frac{r}{e^{5r}-1} \right) {}^5d_j \\ o_j &= \left(\frac{r}{e^{5r}-1} \right) {}^5o_j & i_j &= \left(\frac{r}{e^{5r}-1} \right) {}^5i_j \\ n_j &= i_j - o_j & \Delta_j &= d_j + o_j - i_j = d_j - n_j \end{aligned} \right\} \quad (\text{A.17})$$

A.3 The Continuous Model of Multiregional Demographic Growth

The set of fundamental integral equations in Equation (7) may be collected together in matrix form as the multiregional renewal equation

$$\{\underline{B}(t)\} = \{\underline{W}(t)\} + \int_0^t \underline{M}(x) \underline{P}(x) \{\underline{B}(t-x)\} dx, \quad (\text{A.18})$$

where $\underline{M}(x)$ is a diagonal matrix with diagonal elements $m_i(x)$ and where the element in the i^{th} row and j^{th} column of $\underline{P}(x)$ is ${}_j p_i(x)$. Note that $\{\underline{B}(t)\}$ and $\{\underline{W}(t)\}$ are vectors, and observe that $\underline{P}(x)$ denotes a matrix that is different from $\underline{P}(x)$.

For the case where t exceeds β , the last age of childbearing, births among those surviving from time zero will be zero, that is, $\{\underline{W}(t)\} = \{0\}$, and (A.18) reduces to the homogeneous equation first set out by LeBras (1971):

$$\{\underline{B}(t)\} = \int_0^\beta \underline{M}(x) \underline{P}(x) \{\underline{B}(t-x)\} dx \quad (\text{A.19})$$

The matrix product $\underline{M}(x) \underline{P}(x)$, which we shall denote by the matrix $\underline{\Phi}(x)$, is the multiregional net maternity function [Rogers, 1974] with which we may associate the moment matrices

$$\underline{R}(n) = \int_0^\beta x^n \underline{\Phi}(x) dx$$

To solve (A.19) we adopt the trial solution $\{\underline{B}(t)\} = \{\underline{Q}\} e^{rt}$ which when substituted into (A.19)

$$\text{gives } \{\underline{Q}\} = \int_0^\beta e^{-rx} \underline{M}(x) \underline{P}(x) \{\underline{Q}\} dx$$

$$= \left[\int_0^\beta e^{-rx} \underline{\Phi}(x) dx \right] \{\underline{Q}\}$$

$$= \underline{\Psi}(r) \{\underline{Q}\}, \quad (\text{A.20})$$

where $\underline{\Phi}(x) = \underline{M}(x) \underline{P}(x)$ is the multiregional net maternity function and

$$\underline{\Psi}(r) = \int_0^\beta e^{-rx} \underline{\Phi}(x) dx \text{ is the } \underline{\text{multiregional characteristic matrix}}.$$

We now have reduced our problem from one of solving the integral equation in (A.19) to that of solving (A.20) which, unlike (A.19), is a function of only a single variable, r . To solve for r in (A.20), we rewrite that equation as

$$[\Psi(r) - I] \{Q\} = \{0\},$$

from which we conclude that $\{Q\}$ is the characteristic vector that corresponds to the characteristic root of unity of the matrix $\Psi(r)$, and r is the number for which that matrix has a characteristic root of unity or, equivalently,

$$|\Psi(r) - I| = 0. \quad (A.21)$$

The matrix $\Psi(r)$ normally has only positive elements. Consequently by the well-known Perron-Frobenius properties of positive matrices [Gantmacher, Vol. II, 1959, pp. 53-66] it has a real and positive dominant characteristic root, $\lambda_1(r)$, say, which is a simple root and is greater than the absolute value of any other characteristic root. Moreover, we may associate with $\lambda_1(r)$ a characteristic vector $\{Q_1\}$, say, that has only positive elements. Finally, the dominant characteristic root $\lambda_1(r)$ decreases in value as r is increased.

The dominant characteristic root $\lambda_1(r)$ is in fact a function that assigns to any value of r the dominant characteristic root of the matrix $\Psi(r)$. This function is continuous, concave upward throughout, and its values decrease monotonically from $+\infty$ to 0 as its argument increases from $-\infty$ to $+\infty$. Consequently, $\lambda_1(r) = 1$ can occur only once. And since $\lambda_1(r)$ is the dominant characteristic root of $\Psi(r)$, it will always assume the value of unity at the highest value of those real roots r that satisfy (A.21). That is, $\lambda_1(r_1) = 1$, $\lambda_k(r_k) = 1$, $\lambda_1(r_k) > 1$, whence $r_1 > r_k$.

A.4 The Spatial Consequences of Zero Population Growth

A baby girl born at a given moment in region j will, according to Equation (10), have a net reproduction rate of

$${}_jR(0) = \sum_{i=1}^m {}_jR_i(0) = \sum_{i=1}^m \sum_{x=\alpha}^{\beta-5} {}_jL_i(x) F_i(x) \quad j = 1, 2, \dots, m. \quad (A.22)$$

Suppose that we wish to reduce each of these m observed regional rates to unity by proportionally reducing each region's fertility rates. That is, imagine a set of m fractions $\gamma_1, \gamma_2, \dots, \gamma_m$ such that

$$\sum_{i=1}^m \gamma_i {}_jR_i(0) = \sum_{i=1}^m \sum_{x=\alpha}^{\beta-5} {}_jL_i(x) \gamma_i F_i(x) = 1, \quad j = 1, 2, \dots, m. \quad (A.23)$$

or, more compactly, in matrix form

$$\tilde{R}'(0)\{\gamma\} = \{1\}, \quad (A.24)$$

where the apostrophe denotes transposition. The requisite fractions then may be obtained by premultiplying both sides of (A.24) by the inverse of the transpose of the net reproduction matrix, whence

$$\{\gamma\} = [\tilde{R}'(0)]^{-1} \{1\} \quad (A.25)$$

By way of illustration, recall the net reproduction matrix for our 2-region population example of California and the rest of the United States set out in Section 3. The inverse of its transpose is

$$[\tilde{R}'(0)]^{-1} = \begin{bmatrix} 0.86263 & -0.26665 \\ -0.05635 & 0.64126 \end{bmatrix}.$$

Thus

$$\{\gamma\} = \begin{Bmatrix} 0.59598 \\ 0.58491 \end{Bmatrix},$$

and multiplying the two sets of regional age-specific rates in Table 1 by these fractions gives the replacement levels of fertility that lead to the zero-growth results presented in the lower half of Table 4. (It is of interest to note that in summing the revised regional age-specific rates

and multiplying by 5 to obtain the regional gross reproduction rates, GRR_i say, we find that they are not identical. The GRR for California is 1.0295, while that for the rest of the United States is 1.0435.)
