WORKING PAPER

TRACKING PROPERTY: A VIABILITY APPROACH

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FOREWORD

This paper is devoted to the characterization of the tracking property connecting solutions to two differential inclusions or control systems through an observation map derived from the viability theorem. The tracking property holds true if and only if the dynamics of the two systems and the contingent derivative of the observation map satisfy a generalized partial differential equation, called the ***contingent differential inclusion***. This contingent differential inclusion is then used in several ways. For instance, knowing the dynamics of the two systems, construct the observation map or, knowing the dynamics of one system and the observation map, derive dynamics of the other system (trackers) which are solutions to the contingent differential inclusion.

It is also shown that the tracking problem provides a natural framework to treat issues such as the zero dynamics, decentralization, and hierarchical decomposition.

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Introduction

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ and the ***system of differential inclusions***

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in G(x(t), y(t)) \end{cases}$$

We further introduce a set-valued map $H : X \rightsquigarrow Y$, regarded as an ***observation map***.

We devote this paper to many issues related to the following ***tracking property***: for every $x_0 \in \mathrm{Dom}(H)$ and every $y_0 \in H(x_0)$, there exist solutions $(x(\cdot), y(\cdot))$ to the system of differential inclusions such that

$$\forall\, t \geq 0,\ \ y(t) \in H(x(t))$$

The answer to this question is a solution to a ***viability problem***, since we actually look for a solution $(x(\cdot), y(\cdot))$ which remains viable in the graph of the observation map H. So, if the set-valued maps F and G are Peano[1] maps and if the graph of H is closed, the Viability Theorem states that the tracking property is equivalent to the fact that the graph of H is a viability domain of $(x, y) \rightsquigarrow F(x, y) \times G(x, y)$.

Recalling that the graph of the contingent derivative $DH(x, y)$ of H at a point (x, y) of its graph is the contingent cone[2] to the graph of H at

[1] A set-valued map is called *Peano* if its graph is nonempty and closed, its values are convex and its growth linear.

[2] The contingent cone $T_K(x)$ to a subset K at $x \in K$ is the closed cone of directions $v \in X$ such that $\lim_{h \to 0+} d_K(x + hv)/h = 0$. It is equal to X when x belongs to the interior of K, coincides with the tangent space when K is smooth and to the tangent cone of convex analysis when K is convex. We say thgat K is *sleek* at x is $y \rightsquigarrow T_K(y)$ is lower semicontinuous at x. In this case, the contingent cone $T_K(x)$ is convex. Convex subsets are sleek.

If (x, y) belongs to the graph of a set-valued map $H : X \rightsquigarrow Y$, the ***contingent derivative*** $DH(x, y)$ of H at (x, y) is the set-valued map from X to Y defined by

$$\mathrm{Graph}(DH(x, y)) := T_{\mathrm{Graph}(H)}(x, y)$$

(x, y), the tracking property is then equivalent to the ***contingent differential inclusion***

$$\forall\ (x, y) \in \mathrm{Graph}(H),\ \ G(x, y) \cap DH(x, y)(F(x, y)) \neq \emptyset$$

We observe that when F and G are single-valued maps f and g and H is a differentiable single-valued map h, the contingent differential inclusion boils down to the more familiar ***system of first-order partial differential equations***[3]

$$\forall\ j = 1, \ldots, m,\ \ \sum_{i=1}^{n} \frac{\partial h_j}{\partial x_i} f_i(x, h(x)) - g_j(x, h(x)) \ = \ 0$$

Since the contingent differential inclusion links the three data F, G and H, we can use it in three different ways:

1. — Knowing F and H, find G or selections g of G such that the tracking property holds (observation problem)

2. — Knowing G (regarded as an ***exosystem***, following Byrnes-Isidori's terminology) and H, find F or selections of f of F such that the tracking property holds (tracking problem)

3. — Knowing F and G, find observation maps H satisfying the tracking property, i.e., solve the above contingent differential inclusion.

Furthermore, we can address other questions such as:

a) — Find the largest solution to the contingent differential inclusion (which then, contains all the other ones if any)

b) — Find single-valued solutions h to the contingent differential inclusion which then becomes

$$\forall\ x \in K,\ \ 0 \in Dh(x)(F(x, h(x))) - G(x, h(x))$$

In this case, the tracking property states that there exists a solution to the ***"reduced" differential inclusion***

$$x'(t) \ \in \ F(x(t), h(x(t)))$$

so that $(x(\cdot), y(\cdot) := h(x(\cdot)))$ is a solution to the initial system of differential inclusions starting at $(x_0, h(x_0))$. Knowing h allows to divide the system by half, so to speak.

[3]For special types of systems of differential equations, the graph of such a map h (satisfying additional properties) is called a *center manifold*. Theorems providing the existence of local center manifolds have been widely used for the study of stability near an equilibrium and in control theory.

The observation and the tracking problems are the two sides of the same coin because the set-valued map H and its inverse play the same roles whenever we regard a single-valued map as a set-valued map characterized by its graph.

Consider then the observation problem: the idea is to observe solutions of a system $x' \in F(x,y)$ by a system $y' \in G(x,y)$ where $G : Y \rightsquigarrow Y$ describes simpler dynamics: equilibria, uniform movement, exponential growth, periodic solutions, etc. This would allow to observe complex systems[4] $x' \in F(x)$ in high dimensional spaces X by simpler systems $y' \in G(y)$ or even better, $y' = g(y)$, in low dimension spaces. We can think of H as an observation map, made of a small number of *sensors* taking into account uncertainty or lack of precision.

For instance, when $G \equiv 0$, we obtain constant observations. Observation maps H such that $F(x) \cap DH(x,y)^{-1}(0) \neq \emptyset$ for all $y \in H(x)$ provide solutions satisfying

$$\forall\, t \geq 0,\;\; x(t) \in H^{-1}(y_0) \;\text{ where }\; y_0 \in H(x_0)$$

In other words, inverse images $H^{-1}(y_0)$ are closed viability domains[5] of F. *Viewed through such an observation map, the system appears in equilibrium.*

More generally, if there exists a linear operator $A \in \mathcal{L}(Y,Y)$ such that

$$\forall\, y \in \mathrm{Im}(H),\, \forall\, x \in H^{-1}(y),\;\; F(x) \cap DH(x,y)^{-1}(Ay) \neq \emptyset$$

then we obtain solutions $x(\cdot)$ satisfying the time-dependent viability condition

$$\forall\, t \geq 0,\;\; x(t) \in H^{-1}(e^{At}y_0) \;\text{ where }\; y_0 \in H(x_0)$$

so that we can use the exhaustive knowledge of linear differential equations to derive behavioral properties of the solutions to the original system.

[4]We can use this tracking property as a *mathematical metaphor* to model the concept of .... metaphors in epistemology. The simpler system (the model) $y' \in G(y)$ is designed to provide *explanations* of the evolution of the unknown system $x' \in F(x)$ and the tracking property means that the *metaphor* H is valid (*non falsifiable*). Evolution of knowledge amounts to "increase" the observation space Y and to *modify* the system G (replace the model) and/or the observation map H (obtain more experimental data), checking that the tracking property (the validity or the consistency of the metaphor) is maintained.

[5]When $Y := \mathbf{R}$, such maps can be called "prime integrals" (or "energy functions") of F, because when both $F := f$ and $H := h$ are single-valued, we find the usual condition $h'(x) \cdot f(x) = 0$.

But instead of checking whether such or such dynamics G satisfy the tracking property, we can look for systematic ways of finding them. For that purpose, it is natural to appeal to the selection procedures studied in [6, Chapter 6]. For instance, the most attractive idea is to choose the minimal selection $(x,y) \mapsto g^\circ(x,y)$ of the set-valued map

$$(x,y) \rightsquigarrow DH(x,y)(F(x,y))$$

which, by construction, satisfies the contingent differential inclusion. We shall prove that under adequate assumptions, the system

$$\begin{cases} i) & x'(t) \in F(x(t),y(t)) \\ ii) & y'(t) = g^\circ(x(t),y(t)) \end{cases}$$

has solutions (satisfying automatically the tracking property) even though the minimal selection g° is not necessarily continuous (see [13,3,?] for the use of minimal selections).

The drawback of the minimal selection and the other ones of the same family is that g° depends upon x. We would like to obtain single-valued dynamics g independent of x. They are selections of the set-valued map G_H defined by

$$G_H(y) := \bigcap_{x \in H^{-1}(y)} DH(x,y)(F(x,y))$$

We must appeal to Michael's Continuous Selection Theorem to find continuous selections g of this map, so that the system

$$\begin{cases} i) & x'(t) \in F(x(t),y(t)) \\ ii) & y'(t) = g(y(t)) \end{cases}$$

has solutions satisfying the tracking property.

The size of the set-valued map G_H *measures in some sense a degree of inadequacy of the observation of the system* $x' \in F(x)$ through H, because the larger the images of G_H, the more dynamics g tracking an evolution of the differential inclusion.

Tracking problems are intimately related to the observation problem: Here, the system $y' \in G(y)$, called the *exosystem*, is given, and so are their solutions when the initial states are fixed. The problem is to *regulate the system* $x'(t) \in F(x(t),y(t))$ for finding solutions $x(\cdot)$ *that match the solutions to the exosystem* $y'(t) \in G(y(t))$ in the sense that $y(t) \in H(x(t))$, or, more to the point, $x(t) \in H^{-1}(y(t))$.

Decentralization of control systems, as well as *decoupling properties*, are instances of this problem.

An instance of decentralization can be described as follows: We take $X := Y^n$, $F(x) := \prod_{i=1}^n F_i(x_i)$, and the viability subset is given in the form

$$K := \{(x_1, \ldots, x_n) \mid \sum_{i=1}^n x_i \in M\}$$

so that we observe the individual evolutions $x_i'(t) \in F_i(x_i(t))$ through their sum $y(t) := \sum_{i=1}^n x_i(t)$. Decentralizing the system means solving

— first a differential inclusion $y'(t) \in G(y(t))$ providing a viable solution $y(\cdot)$ in the viability subset $M \subset Y$, and

— second, find solutions to the differential inclusions $x_i'(t) \in F_i(x_i(t))$ satisfying the (time-dependent) viability condition

$$\sum_{i=1}^n x_i(t) = y(t)$$

condition which does not depend anymore on M.

Hierarchical decomposition happens whenever the observation map is a composition product of several maps determining the *successive levels of the hierarchy*. The evolution at each level is linked to the state of the lower level and is regulated by controls depending upon the evolution of the state-control of the lower level.

1 The Tracking Property

1.1 Characterization of the Tracking Property

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ and a set-valued map $H : X \rightsquigarrow Y$, called the *observation map*:

Definition 1.1 *We shall say that F, G and H satisfy the* tracking property *if for any initial state $(x_0, y_0) \in \mathrm{Graph}(H)$, there exists at least one solution $(x(\cdot), y(\cdot))$ to the* system of differential inclusions

$$\begin{cases} x'(t) & \in & F(x(t), y(t)) \\ y'(t) & \in & G(x(t), y(t)) \end{cases} \tag{1}$$

satisfying

$$\forall\, t \geq 0, \;\; y(t) \;\in\; H(x(t))$$

We shall say that a set-valued map $H : X \rightsquigarrow Y$ *is a solution to the* contingent differential inclusion *if its graph is a* closed *subset of* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$ *and if*

$$\forall\, (x,y) \in \mathrm{Graph}(H), \;\; G(x,y) \cap DH(x,y)(F(x,y)) \tag{2}$$

We deduce at once from the viability theorems of [6, Chapter 3] the following:

Theorem 1.2 *Let us assume that* $F : X \times Y \rightsquigarrow X$*,* $G : X \times Y \rightsquigarrow Y$ *are Peano maps and that the graph of the set-valued map* H *is a closed subset of* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$.

1. — The triple (F, G, H) *enjoys the tracking property if and only if* H *is a solution to the* contingent differential inclusion *(2).*

2. — There exists a largest *solution* $H_\star$ *to the contingent differential inclusion (2) contained in* H*. It enjoys the following property: whenever an initial state* $y_0 \in H(x_0)$ *does not belong to* $H_\star(x_0)$*, then all solutions* $(x(\cdot), y(\cdot))$ *to the system of differential inclusions (1) satisfy*

$$\begin{cases} i) \quad \forall\, t \geq 0, \quad y(t) \;\notin\; H_\star(x(t)) \\ ii) \quad \exists\, T > 0 \quad \text{such that } y(T) \;\notin\; H(x(T)) \end{cases} \tag{3}$$

3. — If the set-valued maps $H_n \subset H$ *are solutions to the contingent differential inclusion (2), so is their graphical upper limit*[6]*.*

We shall be interested in particular by single-valued solutions h to the partial contingent differential inclusion

$$\forall\, x \in K, \;\; 0 \in Dh(x)(F(x,h(x))) - G(x,h(x))$$

In this case, the stability property implies the following statement: *Let us consider an equicontinuous sequence of continuous solutions* h_n *to the contingent differential inclusion converging pointwise to a function* h*. Then* h *is still a solution to the contingent differential inclusion.*

[6] The *graphical upper limit* of a sequence of set-valued maps H_n is the set-valued map whose graph is the (Kuratowski) upper limit of the graphs of the H_n's.

First, a pointwise limit h of single-valued maps h_n is a selection of the graphical upper limit of the h_n. The latter is equal to h when h_n remain in an equicontinuous subset: Indeed, in this case, any limit of elements $(x_n, h_n(x_n))$ being of the form $(x, h(x))$ belongs to the graph of h.

Remark — We could also introduce two other kinds of ***contingent differential inclusions***:

$$\forall\, (x,y) \in \mathrm{Graph}(H),\ \ DH(x,y)(F(x,y)) \ \subset\ G(x,y)$$

and

$$\forall\, (x,y) \in \mathrm{Graph}(H),\ \ G(x,y) \ \subset \bigcap_{u \in F(x,y)} DH(x,y)(u)$$

The first inclusion implies obviously that any solution $(x(\cdot), y(\cdot))$ to the viability problem

$$x'(t) \ \in\ F(x(t), y(t)) \ \ \&\ \ x(t) \ \in\ H^{-1}(y(t))$$

parametrized by the absolutely continuous function $y(\cdot)$ is a solution to the differential inclusion

$$y'(t) \ \in\ G(x(t), y(t))$$

The second inclusion states the the graph of H is an invariance domain of the set-valued map $F \times G$. Assume that F and G are Lipschitz with compact values on a neighborhood of the graph of F. By the Invariance Theorem of [6, Theorem 5.4.5], the second inclusion is equivalent to the following strong tracking property:

For any initial state $(x_0, y_0) \in \mathrm{Graph}(H)$, *every solution* $(x(\cdot), y(\cdot))$ to the system of differential inclusions (1) starting at (x_0, y_0) satisfies $y(t) \in H(x(t))$ for all $t \geq 0$. □

We shall address now the problem of constructing ***trackers***, which are selections of the set-valued map Φ

$$(x,y) \ \rightsquigarrow\ \Phi(x,y) \ :=\ DH(x,y)(F(x,y))$$

For that purpose, we recall what we mean by selection procedure of a set-valued map F from a metric space X to a normed space Y.

1.2 Selection Procedures

Definition 1.3 (Selection Procedure) *Let X be a metric space, Y be a normed space and F be a set-valued map from X to Y. A* selection procedure *of a set-valued map $F : X \rightsquigarrow Y$ is a set-valued map $S_F : X \rightsquigarrow Y$ satisfying*

$$\begin{cases} i) & \forall x \in \mathrm{Dom}(F), \;\; S(F(x)) := S_F(x) \cap F(x) \neq \emptyset \\ ii) & \textit{the graph of } S_F \textit{ is closed} \end{cases}$$

The set-valued map $S(F) : x \rightsquigarrow S(F(x))$ is called the selection *of F.*

The set-valued map defined by

$$S_F^\circ(x,y) \;:=\; \{v \in Y \mid \|v\| \leq d(0, F(x,y))\} \tag{4}$$

is naturally a selection procedure of a set-valued map with closed convex values which provides the minimal selection.

We can easily provide more examples of selection procedures through optimization thanks to the Maximum Theorem.

Proposition 1.4 *Let us assume that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with compact values. Let $V : \mathrm{Graph}(F) \mapsto \mathbf{R}$ be continuous. Then the set-valued map S_F defined by:*

$$S_F(x) := \{y \in Y \mid V(x,y) \leq \inf_{y' \in F(x)} V(x,y')\}$$

is a selection procedure *of F which yields selection $S(F)$ equal to:*

$$S(F(x)) = \{y \in F(x) \mid V(x,y) \leq \inf_{y' \in F(x)} V(x,y'))\}$$

Proof — Since F is lower semicontinuous, the function

$$(x,y) \mapsto V(x,y) + \sup_{y' \in F(x)} (-V(x,y'))$$

is lower semicontinuous thanks to the Maximum Theorem. Our proposition follows from :

$$\begin{array}{l} \mathrm{Graph}(S_F) = \\ \{(x,y) \mid V(x,y) + \sup_{y' \in F(x)}(-V(x,y')) \leq 0\} \;\; \Box \end{array}$$

Most selection procedures through game theoretical models or equilibria are instances of this general selection procedure based on Ky Fan's Inequality (see [2, Theorem 6.3.5] for instance).

Proposition 1.5 *Let us assume that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with convex compact values. Let $\varphi : X \times Y \times Y \mapsto \mathbf{R}$ satisfy*

$$\begin{cases} i) & \varphi(x, y, y') \text{ is lower semicontinuous} \\ ii) & \forall (x, y) \in X \times Y, \;\; y' \mapsto \varphi(x, y, y') \text{is concave} \\ iii) & \forall (x, y) \in X \times Y, \;\; \varphi(x, y, y) \leq 0 \end{cases}$$

Then the map S_F associated with φ by the relation

$$S_F(x) := \{y \in Y \mid \sup_{y' \in F(x)} \varphi(x, y, y') \leq 0\}$$

is a selection procedure of F yielding the selection map $x \mapsto S(F(x))$ defined by

$$S_F(x) := \{y \in F(x) \mid \sup_{y' \in F(x)} \varphi(x, y, y') \leq 0\}$$

Proof — Ky Fan's inequality states that the subsets $S_F(x)$ are not empty since the subsets $F(x)$ are convex and compact. The graph of S_F is closed thanks to the assumptions and the Maximum Theorem because it is equal to the lower section of a lower semicontinuous function:

$$\mathrm{Graph}(S_F) = \{(x, y) \mid \sup_{y' \in F(x)} \varphi(x, y, y') \leq 0\} \;\; \square$$

Proposition 1.6 *Assume that $Y = Y_1 \times Y_2$, that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with convex compact values and that $a : X \times Y_1 \times Y_2 \to \mathbf{R}$ satisfies*

$$\begin{cases} i) & a \textit{ is continuous} \\ ii) & \forall (x, y_2) \in X \times Y_2, \;\; y_1 \mapsto a(x, y_1, y_2) \textit{ is convex} \\ iii) & \forall (x, y_1) \in X \times Y_1, \;\; y_2 \mapsto a(x, y_1, y_2) \textit{ is concave} \end{cases}$$

Then the set-valued map S_F associating to any $x \in X$ the subset

$$\begin{array}{rcl} S_F(x) & := & \{(y_1, y_2) \in Y_1 \times Y_2 \textit{ such that} \\ \multicolumn{3}{l}{\forall (z_1, z_2) \in F(x), \;\; a(x, y_1, z_2) \leq a(x, z_1, y_2)\}} \end{array}$$

is a selection procedure of F (with convex values). The selection map $S(F(\cdot))$ associates with any $x \in X$ the subset

$$\begin{array}{rcl} S(F)(x) & := & \{(y_1, y_2) \in F(x) \textit{ such that} \\ \multicolumn{3}{l}{\forall (z_1, z_2) \in F(x), \;\; a(x, y_1, z_2) \leq a(x, y_1, y_2) \leq a(x, z_1, y_2)\}} \end{array}$$

of saddle-points *of $a(x, \cdot, \cdot)$ in $F(x)$.*

Proof — We take

$$\varphi(x, (y_1, y_2), (y_1', y_2')) \;\; := a(x, y_1, y_2') - a(x, y_1', y_2)$$

and we apply the above theorem. $\square$

1.3 Construction of trackers

Any selection of the map Φ defined by

$$\forall\, (x,y) \in \mathrm{Graph}(H),\;\; \Phi(x,y) \;:=\; DH(x,y)(F(x,y))$$

provides dynamics which satisfy the tracking property, provided that the system has solutions.

Naturally, we can obtain such selections by using selections procedures $G := S_\Phi$ of Φ (see Definition 1.3) which have convex values and linear growth, since the solutions to the system

$$\begin{cases} i) & x'(t) \;\in\; F(x(t),y(t)) \\ ii) & y'(t) \;\in\; S_\Phi(x(t),y(t)) \end{cases}$$

satisfying the tracking (which exist by Theorem 1.2) are solutions to the system

$$\begin{cases} i) & x'(t) \;\in\; F(x(t),y(t)) \\ ii) & y'(t) \;\in\; S(\Phi)(x(t),y(t)) := \Phi(x(t),y(t)) \cap S_\Phi(x(t),y(t)) \end{cases}$$

Let us mention only the case of the minimal selection g° of Φ defined by

$$\begin{cases} i) & g^\circ(x,y) \;\in\; DH(x,y)(F(x,y)) \\ ii) & \|g^\circ(x,y)\| \;=\; \inf_{v \in DH(x,y)(F(x,y))} \|v\| \end{cases}$$

Theorem 1.7 *Assume that the Peano map F is continuous and that H is a sleek closed set-valued map satisfying, for some constant $c > 0$,*

$$\forall\, (x,y) \in \mathrm{Graph}(H),\;\; \|DH(x,y)\| \;\leq c$$

where $\|DH(x,y)\| := \sup_{\|u\| \leq 1} \inf_{v \in DH(x,y)(u)} \|v\|$ denotes the norm of the closed convex process $DH(x,y)$. Then the system observed by the minimal selection g° of $DH(\cdot,\cdot)(F(\cdot,\cdot))$

$$\begin{cases} i) & x'(t) \;\in\; F(x(t),y(t)) \\ ii) & y'(t) \;=\; g^\circ(x(t),y(t)) \end{cases}$$

has solutions enjoying the tracking property.

Proof— By [5, Theorem 3.1.1] ,the set-valued map $(x,y,u) \rightsquigarrow DH(x,y)(u)$ is lower semicontinuous. We deduce then from the lower semicontinuity of

F that the set-valued map Φ is also lower semicontinuous. Since $DH(x,y)$ is a convex process, it maps the convex subset $F(x,y)$ to the convex subset $\Phi(x,y)$. Therefore, Φ being lower semicontinuous with closed convex images, its minimal selection S_Φ° defined by (4) is closed with convex values. Furthermore,

$$\|g^\circ(x,y)\| \leq c\|F(x,y)\| \leq c'(\|x\|+\|y\|+1)$$

since $\|DH(x,y)\| \leq c$ and the growth of F is linear. Then the system

$$\begin{cases} i) & x'(t) \in F(x(t),y(t)) \\ ii) & y'(t) \in S_\Phi^\circ(x(t),y(t)) \cap c'(\|x(t)\|+\|y(t)\|+1)B \end{cases}$$

has solutions enjoying the tracking property by Theorem 1.2. Therefore for almost all $t \geq 0$,

$$y'(t) \in \Phi(x(t),y(t)) \cap S_\Phi^\circ(x(t),y(t)) = g^\circ(x(t),y(t)) \quad \square$$

1.4 The Observation Problem

We consider the important case when $G : Y \rightsquigarrow Y$ does not depend upon x. Hence the contingent differential inclusion becomes

$$\forall\, x \in \mathrm{Dom}(H), \forall\; y \in H(x), \quad G(y) \cap DH(x,y)(F(x,y)) \neq \emptyset$$

Example Let us consider the case of *descriptor systems*

$$Ex'(t) = Ax(t) + Bu(t)$$

which we want to observe through $H \in \mathcal{L}(X,Y)$ by the linear equation

$$y'(t) = Gy(t)$$

where $G \in \mathcal{L}(Y,Y)$. We introduce the matrices (A, GH) from X to $X \times Y$ and

$$\begin{pmatrix} E & B \\ H & 0 \end{pmatrix} \text{ from } X \times Z \text{ to } X \times Y$$

We observe that the system enjoys the tracking property if and only if

$$\mathrm{Im}(A, GH) \subset \mathrm{Im}\begin{pmatrix} E & B \\ H & 0 \end{pmatrix}$$

In this case, the velocities $x'(t)$ and the controls $u(t)$ are supplied by the linear system

$$\begin{cases} Ex'(t) - Bu(t) & = Ax(t) \\ Hx'(t) & = GHx(t) \end{cases}$$

which can be solved by linear algebraic formulas. □

Example: Energy Maps (or Zero Dynamics) The simplest dynamics is obtained when $G \equiv 0$: in this case, each subset $H^{-1}(y)$ is a viability domain of $F(\cdot, y)$, because, for any $y \in \mathrm{Im}(H)$ and $x_0 \in H^{-1}(y)$, there exists a solution $x(\cdot)$ such that $x(t) \in H^{-1}(y_0)$ for all $t \geq 0$.

This viability property becomes:

$$\forall\, y \in \mathrm{Im}(H),\ \forall\, x \in H^{-1}(y),\ \ F(x,y) \cap DH(x,y)^{-1}(0) \neq \emptyset$$

When F is a Peano map, we deduce that it is also equivalent to condition

$$\forall\, y \in \mathrm{Im}(H),\ \forall\, x \in H^{-1}(y),\ \ F(x,y) \cap T_{H^{-1}(y)}(x) \ \neq\ \emptyset$$

We shall say that such a set-valued map H is an *energy map* of F.

In the general case, the evolution with respect to a parameter y of the viability kernels of the closed subsets $H^{-1}(y)$ under the set-valued map $F(\cdot, y)$ is described by the inverse of the largest solution $H_\star$:

Corollary 1.8 *Let $F : X \rightsquigarrow X$ be a Peano map. Then for any finite dimensional vector-space Y, there exists a largest closed energy map $H_\star : X \rightsquigarrow Y$ of F, a solution to the inclusion*

$$\forall\, x \in \mathrm{Dom}(H),\ \forall\, y \in H(x),\ DH(x,y)(F(x,y)) \ni 0$$

The inverse images $H_\star^{-1}(y)$ are the viability kernels of the subsets $H^{-1}(y)$ under the maps $F(\cdot, y)$:

$$\mathrm{Viab}_{F(\cdot,y)}(H^{-1}(y)) \ = \ H_\star^{-1}(y)$$

The graphical upper limit of energy maps is still an energy map.

Then *the graph of the map $y \rightsquigarrow \mathrm{Viab}_{F(\cdot,y)}(H^{-1}(y))$ is closed, and thus upper semicontinuous whenever the domain of H is bounded.*

When the observation map H is a single-valued map h, the contingent differential inclusion becomes

$$\forall\, x,\ \exists\, u \in F(x,y) \ \text{ such that } 0 \ \in\ Dh(x)(u)$$

When h is differentiable and $F := f$ is single-valued, we find the classical characterization

$$< h'(x), f(x) > = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x) f_i(x) = 0$$

of *energy functions* or *prime integrals*[7] of the differential equation $x' = f(x)$.

The largest closed energy map contained in h is necessarily the restriction of h to a closed subset of the domain of h, *which is the viability kernel of* $h^{-1}(0)$. The restriction of the differential inclusion to the viability kernel of $h^{-1}(0)$ is (almost) what Byrnes and Isidori call the *zero dynamics of* F (in the framework of smooth nonlinear control systems).

Remark — The Equilibrium Map. We associate with each parameter y the set

$$E(y) := \{x \in H^{-1}(y) \mid 0 \in F(x,y)\}$$

of equilibria of $F(\cdot, y)$ viable in $H^{-1}(y)$. We say that $E : Y \rightsquigarrow X$ is the *equilibrium map*.

We can derive some information on this equilibrium map from its derivative, that we can compute easily:

Theorem 1.9 *Assume that both* $H : X \rightsquigarrow Y$ *and* $F : X \times Y \rightsquigarrow X$ *are closed and sleek and that*

$$\begin{cases} \forall\, (x,y) \in \mathrm{Graph}(H), \ \forall\, (u,v,w) \in X \times Y \times X, \\ \exists\, v_1 \in DH(x,y)(u_1) \ \textit{ such that } \ w \in DF(x,y,0)(u+u_1, v+v_1) \end{cases}$$

Then the contingent derivative of the equilibrium map is the equilibrium map of the derivative:

$$u \in DE(y,x)(v) \Longleftrightarrow u \in DH(x,y)^{-1}(v) \ \ \& \ \ 0 \in DF(x,y,0)(u,v)$$

Proof — We observe that by setting $\pi(x,y) := (x,y,0)$, the graph of E^{-1} can be written:

$$\mathrm{Graph}(E^{-1}) := \mathrm{Graph}(H) \cap \pi^{-1}(\mathrm{Graph}(F))$$

[7] When f is real-valued, thei is the "contingent version" of the Hamilton-Jacobi equation. See the the papers and the forthcoming monograph of Frankowska [14] for its exhaustive study and the connections with the *viscosity solutions*.

and we apply [5, Theorem 4.3.3], which states that if the transversality condition: for all $(x, y) \in \mathrm{Graph}(E^{-1})$,

$$\pi\left(T_{\mathrm{Graph}(H)}(x,y)\right) - T_{\mathrm{Graph}(F)}(\pi(x,y)) \;=\; X \times Y \times X$$

holds true, then

$$T_{\mathrm{Graph}(E^{-1})}(x,y) \;:=\; T_{\mathrm{Graph}(H)}(x,y) \cap \pi^{-1}\left(T_{\mathrm{Graph}(F)}(\pi(x,y))\right)$$

Recalling that the contingent cone to the graph of a set-valued map is the graph of its contingent derivative, the assumption of our proposition implies the transversality condition. We then observe that the latter equality yields the conclusion of the proposition. □

Using the inverse function and the localization theorems presented in [5, section 5.4], we can derive the same kind of informations as the ones provided by [5, Proposition 5.4.7.].

For instance, set

$$Q(y,x) \;:=\; u \in DH(x,y)^{-1}(0) \mid 0 \;\in\; DF(x,y,0)(u,0)$$

Then, for any equilibrium $x \in E(y)$ and any closed cone P satisfying $P \cap Q(y,x) = \{0\}$, there exists $\varepsilon > 0$ such that

$$E(y) \cap (x + \varepsilon(P \cap B)) \;=\; \{x\}$$

where B denotes the ubit ball. In particular, *an equilibrium* $x \in E(y)$ *is locally unique whenever* $0 \in DH(x,y)^{-1}(0)$ *is the unique equilibrium of* $DF(x,y,0)(\cdot,0)$.

Furthermore, if the set $E(y)$ of equilibria is convex, then

$$E(y) \;\subset\; x + Q(y,x) \;\square$$

More generally, the behavior of observations of some solutions to the differential inclusion $x' \in F(x,y)$ may be given as the prescribed behavior of solutions to differential equations $y' = g(y)$, where g is a selection of

$$g(y) \in \bigcap_{x \in H^{-1}(y)} DH(x,y)(DF(x,y))$$

In the case when the differential equation $y' = g(y)$ has a unique solution $r(t)y_0$ staring from y_0, the solution $x(\cdot)$ satisfies the condition

$$\forall\, t \geq 0,\;\; x(t) \in H^{-1}(r(t)y(0)),\;\; x(0) \in H^{-1}(y(0))$$

When g is a linear operator $G \in \mathcal{L}(Y,Y)$, it can be written

$$\forall\, t \geq 0,\;\; x(t) \in H^{-1}(e^{Gt}y(0)),\;\; x(0) \in H^{-1}(y(0))$$

When $H \equiv h$ is a single-valued differentiable map, then the map G_H can be written

$$G_H(y) \;:=\; \bigcap_{h(x)=y} h'(x)F(x,y)$$

and a single-valued map g is a selection of G_H if and only if

$$\forall\, x \in \mathrm{Dom}(H),\;\; 0 \in h'(x)F(x,y) - g(h(x))$$

The problem arises to construct such maps g.

1.5 Construction of Observers

These maps g are selections of the map $G_H : Y \rightsquigarrow Y$ defined by

$$G_H(y) \;:=\; \bigcap_{x \in H^{-1}(y)} (DH(x,y)(F(x,y)))$$

(The set-valued map G_H *measures so to speak a degree of disorder of the system* $x' \in F(x,y)$, because the larger the images of G_H, the more observed dynamics g tracking an evolution of the differential inclusion.)

By using Michael's Continuous Selection Theorem, we obtain the following

Theorem 1.10 *Assume that the set-valued map F is continuous with convex compact images and linear growth, that H is a sleek closed set-valued map the domain of which is bounded and that there exists a constant $c > 0$ such that*

$$\forall\, (x,y) \in \mathrm{Graph}(H),\;\; \|DH(x,y)\| \;\leq c$$

Assume also that there exist constants $\delta > 0$ and $\gamma > 0$ such that, for any map $x \mapsto e(x) \in \gamma B$,

$$\delta B \cap \bigcap_{x \in H^{-1}(y)} (DH(x,y)(F(x,y)) - e(x)) \;\neq\; \emptyset$$

Then there exists a continuous map g such that the solutions of

$$\begin{cases} i) & x'(t) \;\in\; F(x(t),y(t)) \\ ii) & y'(t) \;=\; g(y(t)) \end{cases}$$

enjoy the tracking property for any initial state $(x_0,y_0) \in \mathrm{Graph}(H)$.

Proof — The proof of the above theorem showed that the set-valued map Φ is lower semicontinuous with compact convex images. Furthermore, the set-valued map H^{-1} is upper semicontinuous with compact images since we assumed the domain of H bounded. Then the lower semicontinuity criterion [5, Theorem 1.5.3] implies that the set-valued map G_H is also lower semicontinuous with compact convex images. Then there exists a continuous selection g of G_H, so that the above system does have solutions viable in the graph of H. □

2 The Tracking Problem

2.1 Tracking Control Systems

Let $H : X \rightsquigarrow Y$ be an observation map. We consider two control systems

$$\left\{ \begin{array}{lll} i) & \text{for almost all } t \geq 0, \; x'(t) & = \; f(x(t), u(t)) \\ ii) & \text{where } u(t) \; \in \; U(x(t)) & \end{array} \right. \tag{5}$$

and

$$\left\{ \begin{array}{ll} i) & \text{for almost all } t \geq 0, \; y'(t) \; = g(y(t), v(t)) \\ ii) & \text{where } v(t) \; \in \; V(y(t)) \end{array} \right. \tag{6}$$

on the state and observation spaces respectively, where $U : X \rightsquigarrow Z_X$ and $V : Y \rightsquigarrow Z_X$ map X and Y to the control spaces Z_X and Z_Y and where $f : \text{Graph}(U) \mapsto X$ and $g : \text{Graph}(V) \mapsto Y$.

We introduce the set-valued maps $R_H(x,y) : Z_Y \rightsquigarrow Z_X$ defined by

$$R_H(x,y;v) = \left\{ \begin{array}{ll} \{u \in U(x) | f(x,u) \in DH(x,y)^{-1}(g(y,v))\} & \text{if } v \in V(y) \\ \emptyset & \text{if } v \notin V(y) \end{array} \right.$$

Corollary 2.1 *Assume that the set-valued maps U and V are Peano maps and that the maps f and g are continuous, affine with respect to the controls and with linear growth. The two control systems enjoy the tracking property if and only if*

$$\forall \, (x,y) \in \text{Graph}(H), \;\; \text{Graph}(R_H(x,y)) \neq \emptyset$$

Then the system is regulated by the regulation law

$$\text{for almost all } t \geq 0, \;\; u(t) \in R_H(x(t), y(t); v(t))$$

When $H \equiv h$ is single-valued and differentiable and when we set $f(x,u) := c(x) + g(x)u$ and $g(y,v) := d(y) + e(y)v$ where $g(x)\cdot$ and $e(y)\cdot$ are linear operators, we obtain the formula

$$R_h(x;v) := U(x) \cap (h'(x)g(x))^{-1}(d(h(x)) - h'(x)c(x) + e(h(x)v))$$

2.2 Decentralization of a control system

We assume that the viability set of the control system (5) is defined by constraints of the form $K := L \cap h^{-1}(M)$ where

$$\left\{ \begin{array}{ll} i) & L \subset X \text{ and } M \subset Y \text{ are sleek} \\ ii) & h \text{ is a } C^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K := L \cap h^{-1}(M),\ Y = h'(x)T_L(x) - T_M(h(x)) \end{array} \right. \tag{7}$$

We associate with the two systems (5), (6) the ***decoupled viability constraints***

$$\left\{ \begin{array}{lll} i) & \forall t \geq 0, & x(t) \in L \\ ii) & \forall t \geq 0, & h(x(t)) = y(t) \\ iii) & \forall t \geq 0, & y(t) \in M \end{array} \right. \tag{8}$$

It is obvious that the ***state component*** $x(\cdot)$ of any solution $(x(\cdot), y(\cdot))$ to the system ((5),(6)) satisfying viability constraints (8) is a solution to the initial control system (5) viable in the set K defined by (7).

On the other hand, solutions to the system (5) viable in K can be obtained in two steps:

— first, find a solution $y(\cdot)$ to the control system (6) ***viable in*** M and then,

— second, find a solution $x(\cdot)$ the control system (5) satisfying the viability constraints

$$\left\{ \begin{array}{lll} i) & \forall t \geq 0, & x(t) \in L \\ ii) & \forall t \geq 0, & h(x(t)) = y(t) \end{array} \right. \tag{9}$$

which do not involve anymore the subset $M \subset Y$ of constraints.

This decentralization problem is a particular case of the observation problem for the set-valued map H defined by

$$H(x) := \left\{ \begin{array}{ll} h(x) & \text{if } x \in L \ \& \ h(x) \in M \\ \emptyset & \text{if not} \end{array} \right.$$

whose contingent derivative is equal under assumptions (7) to

$$DH(x)(u) := \begin{cases} h'(x)u & \text{if } u \in T_L(x) \ \& \ h'(x)u \in T_M(h(x)) \\ \emptyset & \text{if not} \end{cases}$$

We know that the regulation map of the initial system (5), (6) on the subset K defined by (7) is equal to

$$R_K(x) = \{u \in U(x) \cap T_L(x) \mid h'(x)f(x,u) \in T_M(h(x))\}$$

The regulation map of the projected control system (6) on the subset M is defined by

$$R_M(y) = \{v \in V(y) \mid g(y,v) \in T_M(y)\}$$

We introduce now the set-valued map R_H which is equal to

$$R_H(x,y;v) := \{u \in U(x) \cap T_L(x) \mid h'(x)f(x,u) = g(y,v)\}$$

We observe that

$$\forall x \in K, \ R_H(x,h(x);R_M(h(x))) \subset R_K(x)$$

The regulation map regulating solutions to the system ((5),(6)) satisfying viability conditions (8) is equal to $x \rightsquigarrow R_H(x,h(x);R_M(h(x)))$. Therefore, the regulation law feeding back the controls from the solutions are given by: for almost all $t \geq 0$

$$\begin{cases} i) & v(t) \in R_M(y(t)) \\ ii) & u(t) \in R_H(x(t);v(t)) \end{cases}$$

The first law regulates the solutions to the control system (6) viable in M and the second regulates the solutions to the control system (5) satisfying the viability constraints (9).

Remark — The reason why this property is called decentralization lies in the particular case when $X := Y^n$, when $h(x) := \sum_{i=1}^n x_i$ and when the control system (5) is

$$\forall i = 1,\dots,n, \ x_i'(t) = f_i(x_i(t),u_i(t)) \text{ where } u_i(t) \in U_i(x_i(t))$$

constrained by

$$\forall i = 1,\dots,n, \ x_i(t) \in L_i \ \& \ \sum_{i=1}^n x_i(t) \in M$$

We introduce the regulation map R_H defined by

$$R_H(x_1, \dots, x_n, y; v)$$

$$:= \{u \in \bigcap_{i=1}^n (U_i(x_i) \cap T_{L_i}(x_i)) \mid \sum_{i=1}^n f_i(x_i, u) = g(y, v)\}$$

This system can be decentralized first by solving the viability problem for system (6) in the viability set M through the regulation law $v(t) \in R_M(y(t))$.

This being done, the state-control $(y(\cdot), v(\cdot))$ being known, it remains in a second step to study the evolution of the n control systems

$$\forall i = 1, \dots, n, \quad x_i'(t) \;=\; f_i(x_i, u(t))$$

through the regulation law

$$u(t) \;\in\; R_H(x_1(t), \dots, x_n(t), y(t); v(t)) \;\square$$

Economic Interpretation — We can illustrate this problem with an economic interpretation: the state $x := (x_1, \dots, x_n)$ describes an allocation of a commodity $y \in M$ among n consumers. The subsets L_i represent the consumptions sets of each consumer and the subset M the set of available commodities. The control u plays the role of the price system of the consumptions goods and v the price of the production goods. Differential equations $x_i' = f_i(x_i, u)$ represent the behavior of each consumer in terms of the consumption price and $y' = g(y, v)$ the evolution of the production process.

The decentralization process allows us to decouple the production problem and the consumption problem. See more details in [6, Chapter 15] on dynamical economic models. □

2.3 Hierarchical Decomposition Property

For simplicity, we restrict ourself here to the case when *the observation map* $H \equiv h := h_2 \circ h_1$ *is the product of two differentiable single-valued maps* $h_1 : X \mapsto Y_1$ and $h_2 : Y_1 \mapsto Y_2$.

We address the following issue: Can we observe the evolution of a solution to a control problem (5) through $h_2 \circ h_1$ by observing it

— first through h_1 by a control system

$$\begin{cases} i) & \text{for almost all } t \geq 0, \; y_1'(t) \;= g_1(y_1(t), v_1(t)) \\ ii) & \text{where } v_1(t) \;\in\; V_1(y_1(t)) \end{cases} \tag{10}$$

and then,

— second, observing this system through h_2.

We introduce the maps R_h, R_{h_1} and R_{h_2} defined respectively by

$$\begin{cases} R_h(x;v) & := \{u \in U(x) \mid h'(x)f(x,u) = g(h(x),v) \\ & \text{if } v \in V(h(x))\} \\ \\ R_{h_1}(x;v_1) & = \{u \in U(x) \mid h_1'(x)f(x,u) = g_1(h_1(x),v_1) \\ & \text{if } v_1 \in V(h_1(x))\} \\ \\ R_{h_2}(x_1;v) & = \{v_1 \in V_1(x_1) \mid h_2'(x_1)g_1(x_1,v_1) = g(h_2(x_1),v) \\ & \text{if } v \in V(h_2(x_1))\} \end{cases}$$

and we see at once that

$$R_{h_1}(x;R_{h_2}(h_1(x);v)) \subset R_h(x;v)$$

Therefore, if the graph of $v \rightsquigarrow R_{h_1}(x;R_{h_2}(h_1(x);v))$ is not empty, we can recover from the evolution of a solution $y(\cdot)$ to the control system (6) a solution $y_1(\cdot)$ to the control system (10) by the tracking law

$$\text{for almost all } t, \;\; v_1(t) \in R_{h_2}(y_1(t),v(t))$$

and then, a solution $x(\cdot)$ to the control system (5) by the tracking law

$$\text{for almost all } t, \;\; u(t) \in R_{h_1}(x(t),v_1(t))$$

This can illustrate hierarchical organization *which is found in the evolution of so many macrosystems. The decomposition of the observation map as a product of several maps determines the successive levels of the hierarchy. The evolution at each level obeys the constraint binding its state to the state of the lower level. It is regulated by controls determined (in a set-valued way) by the evolution of the state-control of the lower level.*

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