

WHAT ARE WE TALKING ABOUT WHEN WE TALK ABOUT "RISK"?  
A Critical Survey of Risk and Risk Preference Theories

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## PREFACE

Risks have emerged as an important constraint in the evaluation and selection of energy strategies. The work of the Joint IAEA/IIASA Risk Assessment Project (IAEA: International Atomic Energy Agency) is oriented toward providing information on technological risks, and their social aspects, for use in decisions related to the management of risks. The emphasis of this research is upon energy systems.

This Research Memorandum brings together the scattered literature on formal theories of risk and risk preference. The various approaches are presented and critically discussed, especially with reference to expected utility theory, which is the standard theory of decision making under uncertainty. As far as available, the results of experimental tests of the theories are presented, too. Finally, more general aspects of risk are discussed, especially those related to new energy technologies.



## ABSTRACT

The notion of "risk" plays an important role in decisions about modern technologies. We have to learn the discomfoting lesson that modern technologies do not only provide benefits, but also "risks", potential loss of monetary values, of environmental quality, of health, or even life. But what exactly is meant by the term "risk"? The present paper considers more formal aspects of "risk". The concept of risk in mathematical statistics and expected utility theory is discussed in some detail. The major part of the paper describes the existing formal axiomatic (measurement-theoretic) theories of risk and risk preference.

Risk refers to the perception of the riskiness of an option (risk estimation), whereas risk preference refers to decision makers' preferences over a set of risky options. Some people are risk seekers, they like gambling, or mountain climbing etc., others prefer to be on the safe side. These theories are discussed in their relation to each other and to expected utility theory. The position is formulated that expected utility theory does not deal with risk in an adequate and psychologically meaningful way. Some results of empirical tests of the various theories are also presented.

The paper closes with a more general discussion of aspects of risk concerning technological and social decisions. It may very well be the case that aspects of risk have to be considered that do not enter in the formal theories as they are formulated at present.



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## 1. INTRODUCTION

The term "risk" seems to play an important role in much of the current writing on political, social, and technological issues. Especially for the latter, we find references to the risk of the various sources of energy, in particular, of course, atomic energy, power plants, waste disposal, etc. More and more we learn the lesson that technology does not only guarantee benefits of various kinds, but also "risks"--hazards, potential loss of equipment, of health, and eventually even of life. These considerations radiate also into the socio-political sphere, we find phrases like "social risk of the nuclear option", or "risk of an authoritarian development" due to excessive safety requirements of the nuclear options and many others. Almost always the term "risk" is left undefined, is used in an everyday-life meaning. But for scientific purposes, especially when we try to measure risk we have to define it and give an operational definition first. This might seem to be a rather easy task at a first glance, but it is not. There is much fuzziness and disagreement about it, as will be evident later. This paper is addressed to the question of what we mean when we talk about "risk".

It might be helpful to give a short outline of the text to follow. First of all, we shall trace some of the definitions of the term "risk". Next, we will have a look how "risk" is defined and used in various branches of science, in which it plays a role. Third, some empirical psychological work is reported that was devoted to the study of risk, which had an outflow in first attempts to formulate theories of risk and risk preference, which are studied next, together with some experiments that tested their empirical appropriateness. Finally, it is analyzed whether "risk" gains a new dimension when applied to large-scale technological problems.

## 2. SOME HISTORY AND "AD HOC" DEFINITIONS OF RISK

Obviously, the term "risk" is used to denote very different things. Many authors use risk synonymously with uncertainty, e.g., they speak of "decision making under risk" instead of "decision making under uncertainty". Some use it synonymously with probability (of a negative event), such as "the risk that a patient has a certain illness" (Norusis, 1973, p. 10). Fischhoff, et al. (1977) define risk as the perceived probability of dying from various sources. Knight (1921), in his much-quoted early work, uses "risk" for an objective and measurable uncertainty, while the term "uncertainty" is reserved for subjective and non-measurable uncertainty. This differentiation is still made today by some authors, especially in the German economic literature. More recently, this distinction between risk and uncertainty has become less clear since the Bayesian

approach to probability assumes that all probabilities are subjective and that there is hardly ever complete lack of knowledge.

Some authors use "risk" only to indicate the possibility of (financial) losses, e.g., Redlich (1957, p. 35), who says, "To repeat, in my language, 'risk' is equated with the chance of loss and this definition applies to both business and non-business 'risk'".

The notion of risk as used by Redlich and many others is in line with what has been called "pure" risk: Only potential losses are affected. If also potential gains (and their probabilities of occurrence) are to be considered to have an impact on "risk", then this is called "speculative" risk by some authors.

If gains and losses and their respective probabilities are involved, it is very practical to introduce the notion of a lottery (or gamble) which is defined as a probability distribution over outcomes (or consequences, which will be used synonymously). A lottery  $L$  is then defined as a set of probabilities  $p_i$  and a set of consequences  $x_i$  which occur with probabilities  $p_i$ ,  $i = 1, 2, \dots, n$ . Some of the consequences might be losses, others gains, with "win" and "lose" probabilities attached to them. Pure risk, then, refers to lotteries that are defined over the negative part of the real line, whereas there are no such restrictions in the case of "speculative" risk. From a purely formal point of view, there might not be much to it to distinguish between the two, but psychologically there might very well be a difference.

There are in principle two ways to characterize probability distributions over outcomes. In the discrete case one could use the set of consequences  $\{x_i\}$  together with their probability of occurrence  $\{p_i\}$ ,  $i = 1, 2, \dots, n$ . A simple lottery could then be given by  $\{\$5, \frac{1}{2}; \$-3, \frac{1}{2}\}$ , i.e., the risky options of winning five dollars with probability of one half, or loosing three dollars with the same probability. A different way to characterize lotteries would be by their moments, that is, expected value ( $E$ ), variance ( $V$ ), skewness ( $S_K$ ), and so on. For two-outcome gambles of the type  $\{a, p; b, 1-p\}$  as used in the example, the moments are as follows

$$\begin{aligned} E &= pa + (1 - p)b \\ V &= p(1 - p)(a - b)^2 \\ S_K &= \frac{1 - 2p}{\sqrt{p(1 - p)}} \end{aligned}$$

with  $a > b$ . In the continuous case, probability distributions over outcomes can only be described by their moments. Which parametrization to use in the discrete case will depend primarily on the risk model to be used.

Alternatively to these parametrizations, lotteries could be characterized by the whole utility function. If expected utility models are used, the distinction to the parametric case is less clear since some of those models lead to (restricted forms) of expected utility models.

A natural extension would be not to consider losses per se, but expected losses, or average loss. This could be set into perspective to the positive side, i.e., expected gains. Domar and Musgrave (1944) define the utility of a lottery as expected gains minus expected loss.

Once the situation is formalized as a choice between lotteries, or a lottery and a sure thing, it seems obvious that risk could depend on numerical characteristics or parameters, of lotteries, such as

- (a) the expected value (mean) of the lottery--the higher the stakes, the more risky the option, and more important,
- (b) some index of dispersion of the lottery--the larger the dispersion, the higher the perceived risk.

It is intuitively evident that the risk of an option depends in part on the expected value--if much is at stake, the whole thing appears riskier. But on the other hand the perceived risk of an option will also depend on "what else could happen". Therefore, some measure of dispersion seems to be the prime candidate for "risk" in the literature. The first one, to my knowledge, to propose a measure of dispersion to quantify risk was Tetens (1786), who proposed one-half of the mean deviation. Markowitz proposed risk to be either the variance or the semi-variance around a chosen value, defined by

$$V_b = E \left[ x - x_b \right]^2 \quad \text{for all } x \leq x_b \quad .$$

The semi-variance is then the mean squared deviation of all  $x$  below  $x_b$ , which is a free parameter and can be set to the decision makers' choice (e.g.,  $x_b$  could be a target rate of return, and only deviations to the left of it could be considered). If  $x_b$  is set equal to zero,  $V_0$  is the "variance over losses". Since lotteries do not have to be symmetric, the semi-variance is not equal to one half of the variance and skewness may very well have an impact on risk perception. Such models were discussed

by Markowitz (1959), Mao (1970a, b), Hogan and Warren (1972, 1974), and by Porter (1974).

In an experimental context, Pollatsek (1966) used the range as an indicator of risk, while Rapoport (1970) or Royden, Suppes and Walsh (1959) used the variance. Measures of dispersion also play an important role in formal risk theories. It seems clear that some notion of dispersion must be an ingredient in any definition or theory of risk.

Definitions or models of risk can be classified according to several points of view. The most important aspects are whether one or more than one parameter is involved, and if several parameters have to be considered, whether the model is compensatory or noncompensatory. Examples of single parameter models are the maximum expected return model (lottery A is preferred to B if and only if A has a higher expectation than B) or the minimum loss probability model (A is preferred to B if it has a smaller loss probability). If more than one parameter is involved, models can be either compensatory or noncompensatory. In compensatory models a bad value of one parameter can be counter-balanced by a good value in some other parameter--the values of the parameters are traded off against each other. The most well-known two parameter trade-off model is the mean-variance trade-off model, as proposed in Markowitz' (1959) portfolio theory. Obvious variants of such models are mean-standard deviation trade-off, mean-semivariance trade-off, mean-probability of loss trade-off, etc. (Libby and Fishburn, 1977). Such models are discussed by many authors, including Borch (1969, 1974), Tsiang (1972, 1974), and Levy (1974). Some references for the mean-semivariance trade-off model were given above.

The most prominent class of noncompensatory risk models are lexicographic models. Lottery A is preferred to lottery B if and only if the risk value of A is smaller than of B (or, if the risk values are the same, if A has a higher expectation than B). The risk value could be equated with the probability of "ruin", for example. Some similar models are discussed in Libby and Fishburn (1977), a general discussion of such models can be found in Fishburn (1974).

Mean and variance are, of course, moments of a probability distribution. If probability distributions are of a normal type, these two characteristics are sufficient to completely determine the distribution. To describe more general types of distributions, more moments are needed. Especially the relation between the third moment, skewness, and the fourth, kurtosis, have to be analyzed in their relation to "risk". But till now there are no formal definitions or models of risk incorporating these higher-order moments.

The only agreement researchers in the area of "risk" could reach up to now seems to be the statement that there is no definition of risk which could be accepted by any larger fraction of the scientific community. Not only might risk perception and risk evaluation be a highly idiosyncratic enterprise, the same may hold true for risk definitions and models.

### 3. THE CONCEPT OF RISK ACROSS SCIENCES

In the following sections the concepts and definitions of risk are analyzed as they appear in various branches of science without trying to achieve completeness. We will then (in Chapter 5) try to find out (1) what the definitions have in common or where they differ, respectively, and (2) whether they agree with, or differ from the psychological, pre-scientific meaning of the term "risk".

#### 3.1. The Concept of Risk in Mathematical Statistics

The following presentation is in the spirit of the decision-theoretic approach to mathematical statistics, since "risk" plays a less important role in "classical" statistics. The definitions follow closely those of Ferguson (1967), see also Raiffa and Schlaifer (1961) and DeGroot (1970) for similar treatments. Only those concepts of decision theory are introduced which are needed for the definition of "risk".

##### Definitions

1.  $H$  a nonempty set, called alternatively states of nature, hypotheses, or parameter space, depending on the context in which they are used; generic element  $\theta$
2.  $A$  a nonempty set, called action space, actions available to the decision maker (DM); generic element  $a$
3.  $X$  a random variable, whose distribution depends on  $\theta$ ;  $x$  indicates an observation of  $X$
4.  $S$  sample space (taken here as finite dimensional Euclidean space)
5.  $P$  probability measure, defined on (Borel) subsets of  $S$
6.  $P_\theta$  probability measure, defined on  $\theta \in H$  for subsets of  $S$
7.  $L(\theta, a)$  real-valued function, defined on the Cartesian product of  $H$  and  $A$ .

A statistical decision problem (or "game" can be characterized by the triple  $(H, A, L)$ ; "nature" chooses a  $\theta$  in  $H$  and an actor (the decision maker; DM) chooses an action  $a$  in  $A$ . Before choosing  $a$ , the actor does not know the "true statue of nature" nor has he any influence on nature. Depending jointly on his action  $a$  and nature's  $\theta$ , the DM will suffer a loss  $L(\theta, a)$ .

The loss is zero if the DM chooses the best action in the situation.

To be able to choose the best action, the DM will want to get some information. He obtains information by performing an experiment in which he observes the realization (outcome)  $x$  of a random variable  $X$ , the density  $P_\theta$  of which depends on the  $\theta$  chosen by nature. The action the DM will take depends, of course, on  $x$ . A decision function is a function  $d: X \rightarrow A$  which preassigns a decision  $a = d(X)$  to each observation.

The loss the decision maker will suffer depends on the random variable  $X$ . Therefore, the loss itself can be considered as a random variable,

$$L(\theta, d(X))$$

The expected loss of  $L(\theta, d(X))$  when  $\theta$  is the true state of nature is called the risk function,

$$R(\theta, d) = E_\theta [L(\theta, d(X))]$$

which gives the average loss when  $\theta$  is the true state of nature and the DM chooses  $d$ .

For many purposes it is sufficient to define the expectation as the Riemann integral

$$R(\theta, d) = \int L(\theta, d(x)) dF_X(x/\theta)$$

where  $F_X(x/\theta)$  indicates the distribution function of the random variable  $X$ , given the true state of nature is  $\theta$ .

The definition of a risk function may be interpreted by a quote from Ferguson (1967, p. 9): "It is a custom, ..., that the choice of a decision function should depend only on the risk function  $R(\theta, d)$ , (the smaller the value the better) and not otherwise on the distribution of the random variable  $L(\theta, d(X))$ . (For example, this would entail the supposition that a poor man would be indifferent when choosing between the offer of \$10,000 as an outright gift, and the offer of a gamble that would give him \$20,000 with probability one half and \$0 with probability one half.)" Ferguson then continues in stating that there is good reason for the statistician or DM to behave this way, provided the loss is measured in terms of utility. This point is not further elaborated in the book. Other statisticians do not care at all how the loss is measured, e.g., Mood and Graybill (1963).

If the statistician is willing to assume a prior distribution over the parameter space  $H$ , as it is done in Bayesian statistics, the Bayes risk of a decision rule is defined as the

expectation of the risk function. (The formal statement is somewhat more involved and will not be presented here.)

To summarize:

1. The risk function is the expected value of the loss function (no prior distribution assumed)
2. The Bayes risk is the expectation of the risk function (prior distribution is assumed).

If the DM chooses a decision rule as to minimize risk, he will minimize an expectation, i.e., the first moment of the loss (or risk) function. This is formally equivalent to the expected value or expected utility principle to be discussed in Section 3.3. It must be kept in mind that the expectation is just a real number. The whole situation is reflected in this number in such a way that the DM has a preference ordering over his actions with respect to that number: minimize loss (or risk). No specific psychological meaning is attached to the term "loss". The fact that only loss is considered but not gains is only induced by the problem formulation, for methodological convenience.

### 3.2. Risk in "Modern" Utility Theory

"Modern" utility theory begins with the pioneering work of one of the greatest mathematicians of our century, John von Neumann, and was laid down in 1944 in a book entitled "Theory of Games and Economic Behavior", by von Neumann and Morgenstern. Mostly the second edition, published in 1947, is referenced, because it contains the proofs of the theorems. Later work by Savage (1954), Luce and Raiffa (1957), Fishburn (1964, 1970), Keeney and Raiffa (1976), and many others, has substantially enriched and refined the theory. This kind of utility refers to conditions of decision making under uncertainty, as opposed to decision making under certainty of the (neo)classical economic school.

I will not try to describe what "modern utility" is, but will right away describe how risk is handled within this framework. To be able to do so, some formal machinery is required. This will be introduced first. For a systematic introduction into unidimensional utility theory, the readers may wish to consult Chapter 4 of Keeney and Raiffa. We follow their presentation.

A lottery,  $L$ , is defined as a probability distribution over consequences (outcomes), i.e., the lottery yields outcome  $x_i$  with probability  $p_i$ ,  $i = 1, 2, \dots, n$ . The (uncertain) consequences of a lottery are conceived as a random variable  $\tilde{x}$ . By definition of the expectation operation, the expected consequence is given by

$$\bar{x} = E [\tilde{x}] = \sum_{i=1}^n p_i x_i, \quad (1)$$

and the expected utility by

$$E [u(\tilde{x})] = \sum_{i=1}^n p_i u(x_i), \quad (2)$$

where  $u(x_i)$  denotes the utility attached to outcome  $x_i$ . It is assumed that the decision maker wishes to choose the lottery which maximizes expected utility ("EU-model").

The certainty equivalent of a lottery is the amount  $\hat{x}$  such that the DM is indifferent between  $L$  and the sure-thing  $\hat{x}$  (i.e., the prospect of getting amount  $x$  for certain), i.e.,  $\hat{x} \sim L^{1/}$ . The utility of  $\hat{x}$  is given by  $u(\hat{x})$ , the utility of the lottery is defined by its expected utility in Equ. (2), so we have:

$$u(\hat{x}) = E [u(\tilde{x})]. \quad (3)$$

The certainty equivalent is also called cash equivalent and selling price of  $L$ .

Now we come to the definition of risk aversion. Speaking intuitively, a person is risk averse if he or she is conservative, does not like to gamble, etc. In the framework of utility theory, risk aversion is defined as preferring the expected consequence of any nondegenerate lottery  $L$  to  $L$  itself. (A lottery is called nondegenerate if no single consequence has the probability of one of occurring, a sure-thing is therefore a degenerate lottery.) From this it follows directly that the utility of the expected consequence must be greater than the expected utility of the lottery, again valid for all nondegenerate lotteries,

$$u [E(\tilde{x})] > E [u(\tilde{x})]. \quad (4)$$

An immediate consequence of that definition is the fact that a decision maker is risk averse if and only if his utility function is concave; this theorem is very simple to prove

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<sup>1/</sup>The sign  $\sim$  indicates indifference,  $\succ$  (strict) preference, and  $\succsim$  weak preference (preference-indifference). Numerical relations are denoted by  $>$  or  $\geq$ , as usual. "Iff" is used here to denote "if and only if".



(Keeney and Raiffa, 1976, p. 149). The DM is risk prone if he prefers any nondegenerate lottery to the expected consequence of that lottery. The utility function of such a DM is convex.

Figure 1 provides an example of a utility function of a risk averse decision maker. The certainty equivalent  $\hat{x}$  is smaller than the expected consequence  $\bar{x}$  of  $L$ ; this is generally true for all risk averse DMs who have increasing utility functions over nondegenerate lotteries. Obviously, the DM is cautious in the sense that he is willing to give up some amount as compared with the expected consequence in order to get a smaller amount  $\hat{x}$  for sure. The difference between  $\bar{x}$  and  $\hat{x}$  is called risk premium (RP)<sup>2/</sup>,

$$RP(\tilde{x}) = \bar{x} - \hat{x} \quad , \quad (5)$$

the risk premium equals the difference between the expected consequence and certainty equivalent. The RP is positive for a risk averse DM, given the utility function is increasing.

Until now it was implicitly assumed that the  $x_i$  were positive, i.e., potential gains. Now the DM could be faced with the frustrating situation that all the  $x_i$ 's were negative, the lottery  $L^-$  is a probability distribution over losses (negative consequences). In such a situation a DM who is risk averse would try to get rid of that lottery, he would be willing to pay a certain amount for achieving that goal. The amount he would be willing to pay to get rid of the lottery is called insurance premium (IP),

$$IP = -\hat{x} \quad , \quad (6)$$

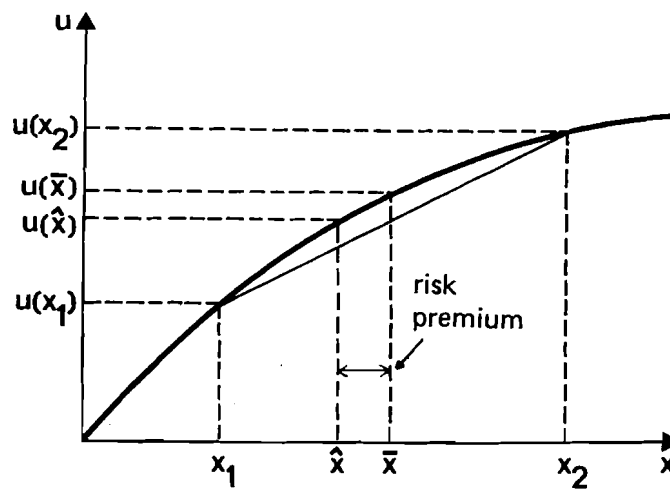
that is, the insurance premium is the negative of the certainty equivalent. The DM would be willing to pay an "insurance premium" of \$  $K$  if his certainty equivalent for lottery  $L^-$  is  $-K$ . Grayson (1960), for example, analyzing the utility functions (for money) for oil wildcatters found several persons exhibiting convex utility functions.

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<sup>2/</sup>This definition of "risk premium" deviates from that of insurance mathematics, which uses the term for the whole premium to be paid for getting the insurance contract.

FIGURE 1

Example of a utility function of a risk averse DM



The risk premium of a lottery  $\ell(x_1, p, x_2)$  equals the the expected value  $\bar{x}$  of that lottery, minus the certainty equivalent  $\hat{x}$ . The risk premium is equal to the amount the decision maker is willing to give up from the expected value to avoid the risk inherent in the lottery.

Let me pause for a moment to point out some differences between the classical economist's "utility" function and the one derived above. The "utility" function with decreasing marginal utility (e.g., with a concave shape) has no measurement-theoretic defined units. As Keeney and Raiffa point out correctly (1976, p. 150), such "utility" functions have no valid interpretation in terms of expected utilities--any such assertions are meaningless. Furthermore, in my opinion, it has to be pointed out that the interpretation of a concave increasing utility function as representing a risk averse DM is valid if and only if the utility function was established by a procedure involving the establishment of certainty equivalents for lotteries (or similar procedures), i.e., for "risky" utility measurement procedures (see, e.g., Fischer, 1977). Or, in other words, the characterization of utility functions representing risk averse or risk prone DMs is valid only for von Neumann-Morgenstern utility functions, but not for the classical economist's "utility" function. The next few sections deal with measures of risk aversion.

Measures of Risk Aversion. As we just saw, von Neumann-Morgenstern utility functions with a concave shape indicate risk aversion. Now it would be nice to be able to express the degree of risk aversion. Is the degree of concaveness, i.e., the bend of the curve, a valid indicator of risk aversion? If this were true, the second derivative  $u''$  of  $u$  with respect to  $x$  should give us the information needed. But, as shown by an example in Keeney and Raiffa (1976, p. 159), this is not the case. Two utility functions may have different second derivations, but do have the same RP associated with them. But the sign of  $u''$  gives some information, if  $u''$  is negative, the slope of the curve is concave and the DM exhibits risk aversion, if  $u''$  is positive, the slope is convex and we conclude that the DM holds a risk prone attitude toward choices between lotteries (it will be assumed throughout that  $u$  is twice continuously differentiable).

Now, following Pratt (1964), local risk aversion at  $x$  is defined by

$$r(x) \equiv u''(x)/u'(x) \quad , \quad (7)$$

i.e., the curvature of the utility curve at a point  $x$  is set into perspective to the slope.

Multivariate risk aversion will not be treated here, see, e.g., Stiglitz (1969) or Keeney (1973).

### 3.3. Risk in Early Social Science Experimentation

It is but recently that "risk" is a research topic in the social sciences. Some of the early experimental work will be briefly reported now. The findings of some of the experiments led to the development of formal theories of risk, to be reported in the next chapter.

Most of the more recent empirical work that has any relation to "risk" was done in the framework of behavioral decision theory (Edwards, 1954a, b, c, 1961; Becker and McClintock, 1967; Rapoport and Wallsten, 1972; Slovic, Fischhoff and Lichtenstein, 1977). All of these studies use very simple lotteries, to be called "gambles" in the sequel. The "riskiness" was always derived from preferential choice data, direct estimates of perceived riskiness were never given directly (as, for example, in Huang's (1971) experiments). Results from the two different response modes need not be identical, of course. This whole issue refers to the old debate of "revealed" vs. directly assessed quantities, see, e.g., Fischhoff, et al. (1977) discussion of Starr (1969). As long as the expected value of a lottery is fixed and there is no skewness involved, variance remains the main candidate for preferences. Then, variance is identified with risk, so preferences are based on risk only. If more than this one parameter varies, results are not so easy to interpret.

The most elementary gamble  $g$  is of the following kind:

$$g : \{r_W, p_W, r_L\}$$

to be read as : Win amount  $r_W$  with probability  $p_W$ , otherwise lose  $r_L$  (with probability  $p_L = 1 - p_W$ ). Often  $p_W = p_L = \frac{1}{2}$ . But sometimes the lotteries have a more complicated form.

When people have to choose among such lotteries, to what aspect of the situation do they react?

- (1) People exhibit preferences for certain probabilities, e.g., Edwards (1953, 1954a, b, c);
- (2) People exhibit preferences for certain levels of variance, e.g., Coombs and Pruitt (1960).

Which is more important? According to Edwards (1954c), probability preferences are the more important of the two. But in this experiment probability and variance preferences were totally confounded. Results of Davidson and Marschak (1959) and Lichtenstein (1965) indicate that variance preferences are very important in determining choices between gambles. For their preferred level of variance  $\underline{S}$ s give up a considerable amount of expected value.

- (3) People exhibit preferences for skewness levels, e.g., Coombs and Pruitt (1960) and Lichtenstein (1965), while
- (4) Kurtosis preferences could not be established (Lichtenstein, 1965).

Problems arise in the interpretation of the results of some of the studies mentioned since various factors do not vary independently of each other--depending on the design of the experiment.

Furthermore, it must be stressed that there is (almost) no theoretical background behind these studies, a fact that has been criticized by Coombs (1972), for example. All of them have rather a kind of explanatory character.

Points (2) to (4) above were formulated in terms of moments of a probability distribution. There are also some experiments which were conducted on the components of the lotteries, i.e., the probabilities to win and lose ( $p_W$  and  $p_L$ ) and the amounts to win and lose ( $r_W$  and  $r_L$ ).

According to results of Slovic (1967), Slovic and Lichtenstein (1968) and Andriessen (1971), people seem to place more weight on the lose components  $p_L$  and  $r_L$  than on the win components. These results were obtained by simple regression techniques, where the worth of a gamble is described as a linear combination of the components

$$W = \beta_0 + \beta_1 p_L + \beta_2 r_L + \beta_3 p_W + \beta_4 r_W .$$

The  $\beta$ 's are just parameters fitted by least squares, and the model is assumed, or superimposed. It is not shown that  $Ss$  actually use this information aggregation rule which obviously does not follow SEU theory.

Anderson and Shanteau (1970) took a perhaps more promising approach to the same problem. Instead of a regression paradigm, these authors used a ANOVA design with a functional measurement analysis (see Anderson, 1974). This makes it possible to test more general composition principles. The components were taken as subjective representations of the probabilities and values involved.

Anderson and Shanteau found that a multiplying model did quite a good job across different experimental situations, while the adding model exhibited some more serious inadequacies, thus casting some doubts as to the appropriateness of a regression formulation of the problem.

In social psychology, the so-called "risky shift" phenomenon has gained great attention--several hundred papers were published on it. Risky shift refers to the observation first reported by Stoner (1961) that groups have the tendency to prefer to accept riskier options than the average of the group members. A detailed analysis of that literature from a decision-theoretic viewpoint is given in Schaefer (1978). The main conclusion is

that risk is not inherent in the tasks subjects have to do. There may be shifts, but not on a dimension to be called "risk".

Some very insightful experiments on risk in relation to insurance buying behavior in a laboratory setting were performed by Slovic (1976). He offered subjects "fair" insurance premiums (premium equal to the expected loss) for various  $(p_L, r_L)$  combinations, ranging from (.001; 1,000) to (.5; 2);  $r_L$  in dollars. Typically, subjects bought much more insurance for the high probability-low loss event, the maximum being about 70% for all offers for (.25; 4). Slovic then tested several parameters. So, for example, if the premium is subsidized, people buy a little more insurance, while for commercial insurances they buy less. But the general pattern is left unchanged. When the situation was transformed into a more realistic setting of a "farm game", the same pattern emerged again, although generally more insurance was bought.

These findings are highly interesting and contrary to what one might have predicted. While in the public discussion "low probability-high consequence" events play a dominant role today--people organize into committees, protest on the streets against options with such characteristics, e.g., nuclear fueled power stations, etc., the same (?) people are obviously not willing to insure themselves against events which effect them perhaps even more directly, such as floods or earthquakes.

Slovic proposes two possible explanations for his findings. First, the utility function may be convex over losses, instead of concave as is normally assumed. Such a functional form would indicate diminishing marginal disutility over losses. Convex utility curves were actually found in some studies, e.g., by Galanter (1975, Galanter and Pliner (1974), Swalm (1966), and Tversky and Kahnemann (1975), and the above mentioned work by Grayson. A convex utility function would imply, taking SEU theory for granted, that nobody buys insurance.

The second explanation is a threshold model for the probabilities involved: If they are too small, i.e., below a certain threshold, they are ignored. People just act as if "that could not happen" to them. Anyway, both of Slovic's explanations are in contrast to SEU theory.

Slovic also thought about ways of how to sell insurances, especially to people in areas menaced by natural hazards, such as floods and earthquakes. One possibility would be to sell insurances in form of a "package", insuring against the preferred high probability-low consequence events, but also to some extent against the ruin combination of low probability-high consequence events. This could be coupled with other measures, such as reimbursement of a part of the premium if "nothing happened". Furthermore, the time horizon for the

insurance plan might be important: If the ruin probability is computed for, say, a time period of 20 years in some residential area, the probability is significantly higher as compared to a reference time of a year. With this little trick, the threshold may be overcome.

#### 4. FORMAL THEORIES OF RISK AND RISK PREFERENCE

##### 4.1. Why Formal Theories? What Are Formal Theories?

Formal theories offer some advantages over purely verbally formulated theories. One obvious advantage is precision: Formal theories are based on a set of assumptions (axioms) from which consequences are deduced in a mathematical way. If the real data fulfill the axioms, often rather strong statements can be made. The axioms should be stated such that they make sense and are testable, though it will not always be possible to test all of them (Adams, Fagot, and Robinson, 1970). Since certain structural relations are involved in most of the axioms, the failure to fulfill them will cast light on what may be wrong. It might then be possible to weaken the axioms (at the expense of less informative conclusions) or to re-formulate them.

The theories we will have to consider belong to the domain of the axiomatic theory of measurement (Krantz, Luce, Suppes and Tversky, 1971). Very generally, axiomatic theories proceed as follows. A person has to make a set of judgments or decisions. Usually the judgments are ordinal ones, e.g., the judgment "rod a is longer than rod b". The goal of measurement consists in mapping some of the features that hold true in the empirical world into numbers, such that the relationships which govern the empirical world (the length of a set of rods, for example) are faithfully mapped into numbers.

More formally, a measurement procedure consists of the following two steps:

(1) Representation theorem: A mapping from an empirical relational system into a numerical one is constructed which is at least a homomorphism. A relational system is a set together with one or more relations defined on the elements of the set. Let  $A$  be a set of elements of value and a binary relation " $\preceq$ " interpreted as "is not preferred to", the relational system is then given by  $\langle A, \preceq \rangle$ . Thus  $a \preceq b$ ,  $b \in A$ , means that  $a$  is not preferred to  $b$ . The corresponding numerical relational system might be  $\mathbb{R}$ , the reals, together with the relation " $\leq$ ", "not greater than". Again, very loosely, the homomorphism states that if  $a \preceq b$  in the empirical relational system holds true, then also  $a \leq b$  in the reals. A homomorphism is a mapping which preserves the structure.

(2) Uniqueness theorem. The uniqueness theorem defines what other homomorphisms are also permitted, by which the scale quality of the measurement is defined, e.g., semiorder, interval scale, ratio scale.

The whole system has the advantage that the responses of a person are not given on a scale the quality of which is stated by the experimenter (e.g., interval scale for ratings), rather, the scale quality is derived and can be taken for granted, once the conditions are met. Furthermore, it is important to note that in many such procedures that have a measurement-theoretic foundation, only responses on an ordinal scale are required, such as "apple a is not heavier than apple b". This task should be very easy for judges. Only procedures with a measurement-theoretic foundation can guarantee that one really knows the scale quality, which is one of the essentials of measurement. A short glance in the above-mentioned book by Krantz, et al., will convince that the actual construction of theories with an axiomatic measurement-theoretic foundation is by no means a triviality.

Perhaps one should add some sceptical remarks on the value of axiomatic theories of measurement. First, not all assumptions (axioms) are really testable. This is generally true for structural assumptions, like the Archimedean, but also, though to a lesser extent, for independence assumptions (see, e.g., von Winterfeldt, 1976). Second, if there is only a single violation, there exists, strictly speaking, no representation. Or, in other words, there is no developed theory of measurement errors. Elements of such a theory are promised for the second volume of the Krantz, et al., book. Alternatively, one could consider to construct probabilistic axiomatic theories of measurement, as proposed by Domotor (1969). But the development of such theories is not too satisfactory yet. Third, a practical disadvantage is the fact that often very many judgments are required, some of which may be rather artificial or hypothetical. Nevertheless, the development of axiomatic theories for the social sciences is a very important contribution, because of the theoretical status of a science can be best assessed from the status of the development of its measurement theory, or theories (and techniques, of course).

#### 4.2. Risk Perception, Risk Preference, and Risk Management

The distinction mentioned in the heading is obvious, though often overlooked or at least surrounded with some "fuzziness". Credit must be given to Clyde Coombs who, in all his writings, was always very clear on that issue. Risk refers to the riskiness of an option. It is a matter of perception, or estimation. The measurement of risk, therefore, induces an order relation on risk estimates. These dimensions are independent of each other. Risk preference, on the other hand, simply refers to the amount of risk you like; do you prefer to gamble or to be on the safe side? More specifically: What



amount of risk do you like best? This is a question of risk evaluation; risk preference induces an order relation on risk evaluation. Two persons may very well agree on the riskiness of a set of gambles, but may nevertheless prefer different gambles, rank-order them differently according to their personal preference. This is not to say that people should agree on riskiness of options. After this distinction was made, it seems evident that there are different theories for both aspects, risk preference and risk (estimation). Otway (1977) introduces a third aspect, risk management, which refers to the organizational and political handling of risky options. This latter aspect is especially relevant for risks that affect society, such as the risks of various transportation models, of energy systems, and large-scale technology in general.

#### 4.3. Theories of Risk

##### 4.3.1. Polynomial Psychophysics of Risk (Coombs and Huang)

The first formal theory of a structure of perceived risk was proposed and tested by Coombs and Huang (1970). The theory deals with three transformations of simple gambles of the kind  $(W, p, L)$  with  $p = \frac{1}{2}$  and  $W > L$ , that is, gambles with positive expected values.

The transformations used were of the following kind:

$$(a) \quad a(g) \equiv (W + a, p, L - a) ; \quad E[a(g)] = E[g]. \quad (8)$$

This transformation has the property that all possible determinants of "risk", such as variance, maximal loss, expected loss, expected regret, increase with  $a$ . One could expect, therefore, that perceived risk of  $a(g)$  is a monotonically increasing function of  $a$ .

$$(b) \quad b(g) \equiv (W + b, p, L + b) ; \quad E[b(g)] = E[g] + b. \quad (9)$$

Transformation  $b(g)$  has no effect on the variance or on expected regret, but changes the maximal loss and the expected loss, which covary inversely with  $b$ . The expected value increases by amount  $b$ .

$$(c) \quad c(g) \equiv (W, p, L)^c, \text{ i.e., } g \text{ is played } c \text{ times independently} \\ E[c(g)] = c \cdot E[g]. \quad (10)$$

This transformation  $c(g)$  is a convolution of  $g$  with itself.

It was postulated that perceived risk  $R$  would follow a distributive conjoint measurement model,

$$R(g) = \left[ \alpha(g) + \beta(g) \right] \gamma(g) , \quad (11)$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are the subjective correspondants to  $a$ ,  $b$  and  $c$ . The model was tested by a conjoint measurement analysis.

#### 4.3.2. Pollatsek and Tversky's Risk Systems

Let us next consider the Pollatsek and Tversky (1970) risk theory in some more detail. On the basis of some assumption on the orderings of perceived risk a ratio scale of risk is derived. First, a formal statement of the theory will be given, to be followed by some interpretation and cross references.

##### Definitions

- $S = \{A, B, C, \dots\}$  probability distribution over  $\mathbb{R}$
- $A \circ B$  convolution of probability distributions, i.e., if two lotteries are given by  
 $A = (a_1, p_1; a_2, p_2)$  and  
 $B = (b_1, q_1; b_2, q_2)$ , then  
 $A \circ B = (a_1+b_1, p_1q_1; a_1+b_2, p_1q_2; a_2+b_1, p_2q_1; a_2+b_2, p_2q_2)$
- $\approx$  binary relation of comparative risk  
" $A \approx B$ ", is to be read as "A is at least as risky as B"
- $\emptyset$  the null gamble: a value of zero is obtained with probability one

The relational system  $\langle S, \circ, \approx \rangle$  is a risk system if the axioms of extensive measurement (weak order, cancellation (monotonicity), solvability, Archimedean) are fulfilled. The convolution operation as defined above is rather similar to the convolution operation in length measurement.

What do the axioms impose on the judgments? The first axiom states the comparability of all lotteries, and their transitive order; the second the compatability with the convolution operation. The third and fourth are technical axioms as they are typically used in the axiomatic theory of measurement; they are not testable, whereas the first two are testable.

As stated in 4.1., the representation theorem is the core of any measurement theory. It reads as follows: If A1 - A4 are fulfilled, then it is possible to construct an additive ratio scale which preserves the risk order. If  $\langle S, \circ, \approx \rangle$  is a risk system, then there exists a real-valued function  $R$ , defined on  $S$ , such that for arbitrary  $A, B, \in S$ :

- i)  $A \succsim B$  iff  $R(A) \geq R(B)$
- ii)  $R(A \circ B) = R(A) + R(B)$
- iii) If  $R'$  is another function fulfilling i) and ii), then  $R'(A) = \alpha R(A)$  for certain  $\alpha > 0$ ; (iii) gives the uniqueness condition.

The approach taken by Pollatsek and Tversky is essentially an extensive measurement structure. But in contrast to other such structures, the risk scale can take on negative values. This is not as unreasonable as one might first think, since a convolution of two lotteries A and B may be perceived to be less risky than A and B taken alone. Intuitively, however, a negative risk is not a very appealing notion.

A further consequence of the assumption is the fact that a sure-thing need not have a risk value of zero associated with it. A sure-thing of amount zero has a risk of zero, a positive amount has a negative risk, and vice versa. This may not be a very appealing property of a risk measure, too. It might be instructive to go through a little example: Lottery  $A = (200, \frac{1}{2}, -200)$  is judged to be riskier than  $B = (300, \frac{1}{2}, -100)$  by most people. But B was generated by the addition of a sure-thing of 100 to A. Therefore, there must be a negative risk associated with the option to get \$100 for sure. This consideration motivates the following axiom:

A5: Positivity, if K is a degenerated lottery with  $k > 0$ ,  $k \in K$ , then:  $A \succsim A \circ k$  for all A in S.

To come to even stronger consequences, some more axioms have to be introduced:

A6: Monotonicity, for all  $A, B \in S$  with  $E[A] = E[B] = 0$  and for arbitrary  $t \in \mathbb{R}^1$ ,  $t > 1$ ,

- i)  $tA \succ A$
- ii)  $A \succsim B$  iff  $tA \succsim tB$ .

Axiom 6 tells us that risk increases for lotteries with zero expected value, if they are multiplied by a positive constant, and that the risk order is preserved under multiplication with a positive constant. (i) is similar to an ad hoc assumption on risk used by Coombs: The throw of a coin with a higher denomination is more risky than the throw of a coin with lower denomination (Coombs used only a weak relationship), and (ii) says that the risk order is independent of the denomination of the coin. A6 imposes a quite strong behavioral assumption, which can be tested, of course.

A7: Continuity, a purely technical axiom which is not behaviorally relevant.

A relational system fulfilling the axioms of extensive measurement and in addition axioms A5 to A7 is called a regular risk system. The analysis of such a system leads to an important theorem: If  $\langle S, o, \approx \rangle$  is a regular risk system, then there is one and only one  $\theta$ ,  $0 < \theta \leq 1$ , such that for all  $A, B$  in  $S$  (with finite expected value and variance):

$$A \approx B \text{ iff } R(A) \geq R(B) , \text{ with } R(A) = \theta V[A] - (1-\theta) E[A].$$

This very astonishing result tells us that in a regular risk system a risk ordering is generated by a linear combination of expected value and variance, i.e., the perceived riskiness of a lottery depends only on the expected value and the variance and on a single parameter,  $\theta$ , which is person specific, of course. The value of  $\theta$  can easily be derived from some judgments on risk-equivalence between lotteries. The value of  $\theta$  defines the relative contribution of the expected value and of the variance to the perceived riskiness of a lottery, i.e., it captures a variance-expected value trade-off ("VE-theory"). From the early definitions (see 2.) it was evident that many authors viewed the variance as the single most important factor in determining the risk of an option. But it is also evident that the expected value has to play a role. Now, both of these contributing factors are captured in a very simple formula. As will be noted, a further possible candidate, skewness, is not incorporated into the theory.

From a set of axioms, none of which seems to be really counter-intuitive, some very strong consequences have been deduced. The most important is theorem 2, giving risk as being generated by a linear combination of expected value and variance. Some other aspects are also worth being restated, such as the possibility for a negative risk and a non-zero risk associated with a sure-thing. These aspects might prove very difficult to empirically fulfill the theory.

A further property related to this risk theory can be stated as follows: Define a preference order that depends only on expectation and variance as VE-dependent, if for any two lotteries  $\ell$  and  $g$

$$V(\ell) = V(g) \quad \text{and} \quad E(\ell) = E(g) \quad \rightarrow \quad \ell \sim g \quad .$$

A preference order is dependent on a (Pollatsek-Tversky) risk measure, or R-dependent, whenever

$$R(\ell) = R(g) \quad \Rightarrow \quad \ell \sim g \quad ,$$

and

$$R(\ell) = \theta V(\ell) - (1-\theta) E(\ell) \quad , \quad 0 < \theta \leq 1 \quad .$$

Any preference order that is R-dependent is also VE-dependent, but not conversely. If utility can be expressed as a power series in money, then there is no utility function which is compatible with a preference order that depends only on the risk measure, and only the quadratic utility function is compatible with a VE-dependent preference order. As Krantz, et al. (1971) point out further, this negative result can be taken as evidence against Pollatsek and Tversky's risk system or against expected utility theory. As a consequence, as it appears to me a theory of risk must be defined such that more satisfactory preference orders are compatible with it.

#### 4.3.3. Huang's Theory of Expected Risk

Huang (1971a) proposed a theory of risk in which "risk" is defined as expected risk (ER):

$$ER = \sum_{x \in X} P(x) f(x) \quad ,$$

in which

$f$  denotes a real valued function on  $X$ , the risk function,  
 $x \in X$  is an outcome, or consequence,  
 $P$  is a probability measure on  $X$ ,  
 $X$  is the set of all possible outcomes.

A lottery or gamble  $g_1$  is subjectively not more risky than  $g_2$ ,  $g_1 \preceq g_2$ , if the following conditions are fulfilled:

$$g_1 \preceq g_2 \quad \text{iff} \quad r(g_1) \leq r(g_2) \quad \text{iff} \quad \sum_i P_{1i} f(x_{1i}) \leq \sum_i P_{2i} f(x_{2i}) .$$

Lottery  $g_2$  is no more risky than  $g_1$  iff the risk value of  $g_1$  is less or equal to that of  $g_2$ , which is the case iff the weighted sum of the risk values of the outcomes, i.e.,  $f(x_i)$  of  $g_1$  is not greater than that of  $g_2$ . The weights are the probabilities of occurrence, therefore, the risk function is called expected risk (function).

The axioms used to establish this result are essentially those of Fishburn (1970, Ch. 8.4.) for an expected utility representation, which are based on mixture sets. Huang's uniqueness theorem states that the expected risk function is an interval scale.

Expected risk theory is very similar to (S)EU theory; the probability component is the same, but  $u(x_i)$  is replaced by  $f(x_i)$ . Expected utility for discrete probability distributions can be written as:

$$\begin{aligned}
 EU &= \sum_{x \in X} P(x) u(x) = \sum_{x \in X} (u(x) - x) P(x) + \sum_{x \in X} xP(x) \\
 &= EV + \sum_{x \in X} (u(x) - x) P(x) \\
 &= EV + R,
 \end{aligned}$$

where R denotes the influence of a "risk" factor which relates the EV to the EU.

As was shown later by Huang (1974), EU theory is a special case of the expected risk theory. Furthermore, as one might have expected, EU theory is a special case of the Portfolio theory of Coombs (1967), where the DM prefers either minimal or maximal levels of risk, but nothing in between.

Expected utility theory is usually considered to be "the" normative theory of decision making under uncertainty. Coombs (1972, p. 2) writes in this context: "Indeed the notion of maximizing expected utility made the notion of risk superfluous. The axiomatization of utility, first by von Neumann and Morgenstern (1947), only strengthened this view". In modern and sophisticated texts on utility theory, such as Fishburn (1964, 1970), the term risk is not even mentioned in the index though the books are well indexed. Coombs concludes: "Such theories simply avoid the psychological reality" (loc. cit.). Theories of risk are designed to be alternative, and maybe psychologically more adequate, formalizations of decision making under uncertainty.

#### 4.3.4. Some Empirical Results

Coombs and Huang (1970b) tested their own polynomial risk theory. Some of the results can be summarized as follows:

- (a) Perceived risk increased with increase of a in transformation (1); 20 out of 28 Ss.
- (b) Perceived risk decreased with increase of b in transformation (2); 26 out of 28 Ss.
- (c) Parameter c acts as an "amplifier", as assumed by the distributive model.
- (d) Most Ss were reasonably consistent, there were hardly any intransitivities; the conjoint-measurement analysis strongly favored the distributive and the dual-distributive model, which could not be differentiated on the basis of the data. On psychological grounds the distributive model was favored.

A further test of the Coombs and Huang theory was undertaken by Barron (1976). The original design was expanded to test the model when odd-even effects may be operating. An odd-even effect refers to the fact that in an odd number of repetitions of a play there is no non-zero probability associated with a zero pay-off. Since people might weigh the lose-components ( $r_L$  and  $p_L$ ) more strongly than the win-component, this may, in turn, have an effect on the perceived riskiness of odd vs. even numbers of repetitions of a gamble.

The design was a complete  $3 \times 3 \times 3$  fractional design, where the factors refer to the transformation  $a(g)$ ,  $b(g)$  and  $c(g)$  of the Coombs and Huang study. The repetition factor,  $c$ , was set at 3, 4 or 5, whereas Coombs and Huang used  $c$  levels of 1 and 5.  $S_s$  task consisted in rank-ordering the 27 stimuli according to perceived risk.

Based on a conjoint measurement analysis, it was clearly evident that the distributive model was not fulfilled, since single factor independence was severely violated by most of the  $S_s$ , especially for the  $c$ -factor. One person fulfilled the requirements of the distributive model, and none any other conjoint measurement model. When all gambles with zero expected value were removed, i.e., the design was reduced to  $3 \times 2 \times 3$  factorial, five  $S_s$  fulfilled the additive model.

The results of this experiment are, of course, negative for the Coombs and Huang theory, but also other conjoint measurement models were not adequate which tells us that there was no conjoint-measurement structure which really described the data. Interestingly enough, there was a rather high similarity of the 13  $S_s$  rankings of the 27 stimuli (Kendall's  $W = .764$ ,  $\chi^2 > 258$ ,  $p \ll 0.001$ ) which indicates that risk is a construct which is used similarly by various persons (at least concerning simple gambles). The question is only how they use it.

The Pollatsek and Tversky theory could not be rejected on the basis of Huang's (1971b) data for risk derived by preferential choice, but was rejected for direct risk orderings<sup>3/</sup>. Coombs and Bowen (1971) tested VE-theory by a skewness transformation on gambles which did not affect mean and variance. The results speak against the theory. But Pollatsek and Tversky (1970) had already pointed out that skewness is a problem for their theory and that it might be necessary to restrict the range of the theory to symmetric lotteries.

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<sup>3/</sup>A direct risk ordering is an ordering of lotteries according to their perceived risk.

Huang tested the various risk theories (expected risk theory, additive risk theory of Pollatsek and Tversky, and polynomial risk theory of Coombs and Huang) in two experiments. Ss had to give a direct ordering of lotteries according to perceived risk, and to express preferential choices, from which a risk order was deduced by means of unfolding theory. Due to the complex design and data manipulation, the results are somewhat difficult to evaluate. But some points can be made:

- (1) Direct risk estimates and risk derived from preferential choices seemed to be equally valid, though they led to different risk orders in about a third of the cases.
- (2) All risk theories were more or less valid for describing the results of experiment 1, although the VE-theory is violated by the b-operation for direct risk estimates<sup>4/</sup>.
- (3) For gambles with fixed probabilities both the VE and the expected risk theory were fulfilled. These theories cannot theoretically co-exist, but they could not be separated on the basis of Huang's experimental data.

A further test of Huang's theory was undertaken by Aschenbrenner (1974, 1978). He varied also the probabilities in the lotteries he used. Of his 20 Ss, 16 were consistent enough in their risk orderings; 13 out of the 16 Ss fulfilled the conditions of the expected risk theory.

#### 4.4. Theories of Risk Preference

Three theories of risk preference have been proposed thus far. From them, only Coombs' theory has drawn some attention, whereas Krelle's formulations passed almost unnoticed, at least in the English-speaking world. The three theories will be described in turn.

##### 4.4.1. Coombs' Portfolio Theory

A theory of risk preference was developed by Clyde Coombs in 1967. There are some more recent formulations of it (Coombs, 1972, 1975). In this paragraph, only a rather informal description of the theory will be given.

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<sup>4/</sup> The b-operation refers to the transformation b on gamble a used by Coombs and Huang (1970b).



The central idea of the theory is easily stated: People have a single-peaked preference function over risky options with equal expected value, i.e., over lotteries. In such a situation, it is hypothesized that every person has a preferred, or "ideal" level of risk for a given level of expected value. As the risk of lotteries departs from this ideal, he likes them less. Perhaps you like some risk; too much is too frightening, but no risk is dull. Just a little excitement, that's the optimum. This notion of single-peakedness is, as behavioral scientists know, the core of Coombs' unfolding idea (1953, 1964).

More specifically, Coombs makes the following assumptions:

- A.1.: From two lotteries with equal expected value, you choose the one with the risk which you like best, i.e., which is close to your "ideal" risk level (which may be "no risk" or "maximum risk" or anything in between).
- A.2.: The risk-preference function is single-peaked: There is one and only one ideal value, from which the preference decreases on both sides<sup>5/</sup>. A third assumption need not concern us here.

If these assumptions are met, a "portfolio" structure, as Coombs calls it, can be represented in a perceived risk-expected value space as shown in Figure 2.

Coombs portfolio theory is a psychological-descriptive theory, not a normative one. As has been shown earlier, normative theories postulate that the DM wishes (or "should") to maximize his expected value (or utility), or minimize the "risk" (variance) if the expectations are equal, as assumed in Markowitz' portfolio theory (which is a theory of optimal composition of shares and has to be distinguished from Coombs' theory).

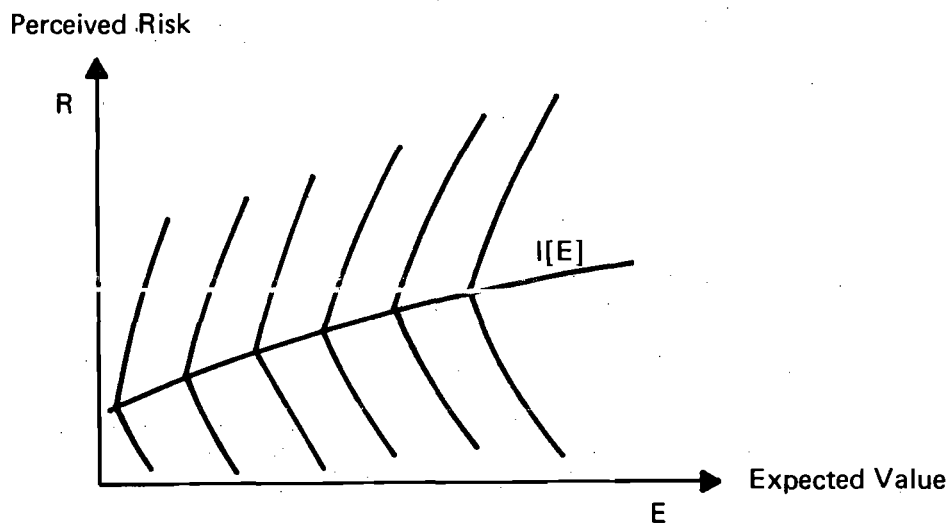
Testing the theory. As one will have noticed, "risk" is left undefined in Coombs' theory. People are supposed to have an ideal risk for any given level of expectation, but it is not spelled out what risk is. This should not irritate, since

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<sup>5/</sup> This assumption may look trivial and self-evident at a first glance. But it is not. Consider, for example, the temperature of tea as you like it best. Most people prefer hot tea to luke-warm tea and hot to steaming hot. But they prefer cold to luke-warm tea. In a recent contribution, Coombs and Avrunin, 1977a, b) study the conditions to be met for single-peakedness to arise. Especially in multi-attribute ( $n > 2$ ) situations, these conditions are very demanding.

FIGURE 2

Indifference curves and ideal crest  $I[E]$  of  
Coombs' portfolio theory



For each level  $E$  of expected value there is an "ideal" risk. The points of ideal risks are connected by the "ideal crest  $I[E]$ ". If a gamble departs from the ideal crest to the north, the increase in risk has to be compensated by an increase in expected value. If it departs from the ideal to the south, the decrease in risk has to be compensated by an increase in expected value.

there is no generally accepted definition of risk. But in order to test the theory, some "ad hoc" assumptions are necessary. To illustrate, let us consider the assumptions of the Coombs and Huang (1970a) study. The lotteries used therein had the standard format  $(r_W, \frac{1}{2}, r_L)$  with  $|r_L| = r_W - k$ , for various values of  $r_W$ ; thus, the expected value was always positive and holds constant in a set of such lotteries.

Assumptions (see also next section)

A.1. The perceived risk increases monotonically with range  $|r_W - r_L|$  and variance, e.g., the lottery  $(30, \frac{1}{2}, -10)$  is judged less risky than  $(70, \frac{1}{2}, -50)$ ;  $EV = 10$  (cents).

A.2. Repeated play of the less risky lottery is perceived to be less risky than a single play of a lottery with the same expected value and equally extreme outcomes, i.e., the same range  $|r_W - r_L|$ . Example: Playing  $(30, \frac{1}{2}, -10)$  twice should be less risky than playing  $(60, \frac{1}{2}, -20)$  once. This assumption seems to be plausible, since the extreme outcomes of the repeated gamble are also 60 and -20, but they are less likely (.25 instead of .5), with constant EV. This assumption has the additional advantage of granting the possibility of analyzing repeated games.

A.3. In certain instances, it was assumed that adding a small amount (10¢) to  $r_W$  would not increase the perceived risk, i.e.,  $(30, \frac{1}{2}, -10)$  is at least as risky as  $(40, \frac{1}{2}, -10)$ . This assumption is less elegant, but it can be replaced by others.

These assumptions seem highly plausible, especially assumptions A.1. and A.2.; therefore, it is less surprising that they are the core of the Coombs and Huang theory of risk presented earlier. It is not the place here to describe and evaluate the results of empirical studies in any detail, but the results seem to indicate that the theory may describe actual choice behavior very well, at least for simple gambles as the ones described above.

4.4.2. Krelle's Axiom System A: Risk-Preference Function

A further theory of risk preference is given by Krelle (1968) in his axiom system A. Its formal development is similar to standard expected utility theory, but the lotteries are constructed differently. In all other instances a lottery or gamble was defined as a pair consisting of an outcome  $x_i$  and a probability  $p_i$ ,  $\ell = (x_i, p_i)$ . Krelle defines a lottery as a pair consisting of a utility  $u_i$  and a probability  $p_i$ ,  $\ell_k = (u_i, p_i)$ . It is assumed that the  $u_i$ 's used as primitives

in the theory were derived from preference judgments in a riskless setting.

Krelle then puts forth eight axioms, two equivalence axioms, two dominance axioms, two continuity axioms and two substitution axioms. Since they are fairly standard, they will not be presented here. The critical axiom is axiom 9, which completes his axiom system A, which gives rise to a risk preference function. It reads as follows:

Axiom A9 (Independence)

Be  $L_1 = (\ell_1, \ell_2, \dots, \ell_k, \ell_{k+1}, \dots, \ell_n)$  and  
 $L_2 = (\ell_1, \ell_2, \dots, \ell_k, \bar{\ell}_{k+1}, \dots, \bar{\ell}_m)$  two lotteries  
 $\sum_i p_i = 1$ , and  $L_1 \sim L_2$ , with  $\ell_i = (u_i, p_i)$  which have the first  
 $k$  elementary gambles  $\ell$  in common. If the first  $k$  gambles are  
replaced by other gambles,  $\ell_1', \ell_2', \dots, \ell_k'$ ;  $\ell_i = (u_i', p_i')$ ,  
 $i = 1, 2, \dots, k$ , and with equal probability sum

$$\sum_{i=1}^n p_i = \sum_{i=1}^k p_i'$$

two new lotteries  $L_1' = (\ell_1', \ell_2', \dots, \ell_k', \ell_{k+1}, \dots, \ell_n)$  and  
 $L_2' = (\ell_1', \ell_2', \dots, \ell_k', \bar{\ell}_{k+1}, \dots, \bar{\ell}_m)$   
are generated and  $L_1' \sim L_2'$ .

This axiom tells that component gambles that are equal in two lotteries can be replaced by other gambles, provided that the sums of the probabilities of the components are the same (which is necessary to keep the overall probability normalized). If the lotteries were indifferent in the first place, this transformation should not alter this relation. Independence axioms are the core of every expected utility representation (Fishburn, 1970).

The development of the risk preference function now proceeds as follows. Let  $u'$  and  $u''$ ,  $u' < u''$ , be such that all utilities involved in the lotteries are in between, i.e.,  $u' \leq u \leq u''$  for all  $u$ .

Due to three axioms (Krelle's axioms A2, A7, and A8) all lotteries  $L_1, L_2, L_3, \dots$ , can be replaced by equivalent lotteries which consist of two component gambles only and are based on  $u'$  and  $u''$  as follows (solvability condition):

$$\begin{aligned} L_1 &\sim \{(u', p') ; (u'', p'')\} \\ L_2 &\sim \{(u', q') ; (u'', q'')\} \end{aligned}$$

Now, from the probability dominance axiom (A4), it follows that

$$L_1 \approx L_2 \langle \Longrightarrow \rangle p'' \geq q'' ,$$

which is immediately clear, since  $L_1$  and  $L_2$  have the same utility elements, so  $L_1$  can only be weakly preferred to  $L_2$  if the probability for the "good" outcome is at least the same.

Now, a utility index  $U$  is attached to each lottery  $L_1$ ,  $U(L_1)$ , and a componentwise evaluation of the pairs  $(u, p)$  is introduced, such that the following two conditions hold:

- i)  $L_1 \approx L_2 \langle \Longrightarrow \rangle U(L_1) \geq U(L_2)$
- ii)  $L_1 = (\ell_1, \ell_2, \dots, \ell_n) = U(L_1) = F(\ell_1) + F(\ell_2) + \dots + F(\ell_n)$ .

The function  $F$  is called evaluation function for gambles (Chancen-Bewertungsfunktion).  $F$  is now defined as follows: According to the substitution principle (A8) each individual gamble  $\ell = (u, p)$  in a lottery can be replaced by an equivalent pair  $\ell' = (u', p')$  and  $\ell'' = (u'', p'')$ . Now Krelle (1968, p. 140) defines

$$F(\ell) = p'' ,$$

without further comment. He then proceeds to show that the value of  $F(\ell) = p''$  depends only on  $\ell$  and not on the other gamble in the lottery. Then Krelle proves that proposition i and ii hold. Now the central concept is defined, the risk-preference function (Risikopräferenzfunktion).

According to axiom 2 (this says that the probabilities of identical outcomes can be added without changing the overall utility of the lottery), for  $0 \leq p + q \leq 1$ ,

$$F(u, p + q) = F(u, p) + F(u, q) ,$$

and, therefore, for each  $t \geq 0$  with  $0 \leq tp \leq 1$  we have

$$F(u, tp) = t \cdot F(u, p) .$$

Next,  $F(u, 1)$  is defined as  $\rho(u)$ ,  $F(u, 1) = \rho(u)$ .

The risk-preference function  $\rho$  has the following properties:

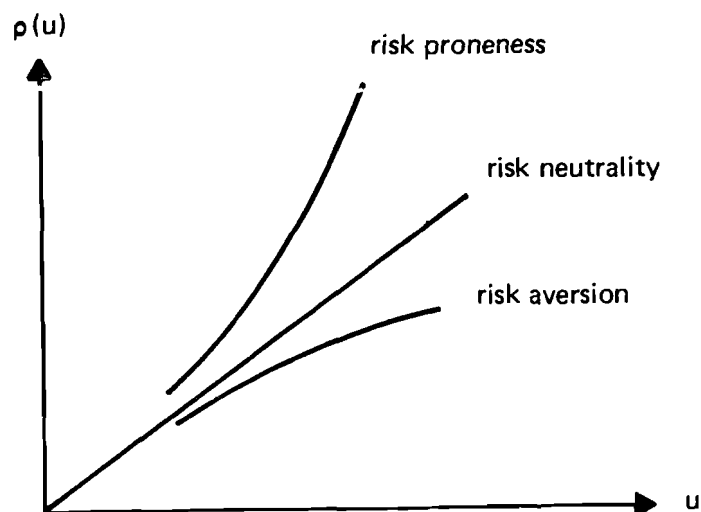
- is strictly monotonically increasing
- can assume all intermediate values
- is continuous over the entire range
- (uniqueness) With  $\rho$  being a risk-preference function,  $\rho = a + b$  with  $b > 0$  is also a risk-preference function, i.e.,  $\rho$  is an interval scale.

$\rho$  was originally obtained for the interval  $u' \leq u \leq u''$ , but this range can be enlarged by a suitable transformation such that the risk-preference functions agree with each other in  $u' \leq u \leq u''$ .

A risk-preference function is defined by propositions i and ii and by two values. These supporting values  $u'$  and  $u''$  are no longer necessary now. The various risk-preference functions that can arise can be represented in a  $\rho(u) - u$  space, as shown in Figure 3.

FIGURE 3

Example of risk preference functions



A convex risk-preference function indicates risk proneness, a linear function risk neutrality, a concave function risk aversiveness. In the case of risk neutrality, the decision criterion  $\sum_i \rho_i(u_i)p_i$  is equivalent to the Bernoulli criterion of "moral expectation" (utility)  $\sum_i u_i p_i$ . This is an important fact to be kept in mind. It arises from the definition of lotteries over probabilities and utilities, instead of outcomes (say, money). In a sense,  $\rho(u)$  is a second-order utility, a utility function (called risk-preference) over certain utilities. The choice of a risk-preference function depends on the attitude toward risk of the DM, there are no prescriptions for a rational choice of a (convex, neutral, concave) risk-preference function.

Krelle (1968, p. 146f.) compares his function  $\rho$  to the traditional utility function of the "anglo-american" literature. In it, moral expectation is equivalent with expected utility,

$$\bar{U} = \sum_j u_j(x_j) \cdot p_j \quad .$$

But this does not seem to be a good guideline for decision making under uncertainty, since two persons can very well agree about the evaluation of the utility of two hypothetical events (once they have happened), but they may differ considerably concerning their willingness to take risks. One may speculate on the occurrence of the lucky, though highly unlikely event, while the other concentrates on the possibility of a catastrophe, although it is not likely to occur at all.

The evaluation of a risk, on the other hand, once one accepts axiom system A, is represented by an increasing risk-preference function  $\rho(u)$  such that the index of preference  $U$  can be given as

$$U = \sum_j \rho(u_j) \cdot p_j \quad .$$

Since the utility  $u_j$  is a function over the event to occur  $x_j$ , the above equation can be written as

$$U = \sum_j \psi(x_j) \cdot p_j \quad ,$$

where the function  $\psi$ ,  $\psi(x_j) = \rho(u_j(x_j))$ , is the Anglo-American expected utility function. It contains utility and risk-preference simultaneously, whereas in the Krelle formulation both aspects are separated which is preferable in his opinion, since both are very different phenomena.

Krelle's basic idea, separating utility from attitude towards risk, seems very reasonable. But there are some problems with his formulation. First, the measurement-theoretic status of his theory is somewhat dubious. He uses utility as a primitive--a quantity that is supposed to be measured in a fundamental way. This is done in expected utility theory. Obviously, it would not be reasonable to measure Krelle's  $u$ 's by means of expected utility theory, which leaves only one option, namely to measure it in a riskless-choice setting, e.g., by indifference curves, conjoint measurement, value differences, etc.

While in some expected utility representation probabilities enter as extraneous variables, as, for example, in the theories by Suppes (1956), Anscombe and Aumann (1963), and Pratt, Raiffa and Schlaifer (1964), in Krelle's theory probability as well as utility enter extraneously. "Extraneous" is to be understood as not measured via the axioms, but entering as a known figure from outside.

Second, besides these reservations concerning the theoretical status of axiom system A, the actual application of the system may not be very practical, since all  $u$ 's have to be measured individually in the first place.

Krelle describes a practical way to arrive at a risk-preference function as follows. Take two reasonable points A and B on the utility axis of possible outcomes (Ergebnisgrößen). Assign A the function  $p$  (see Figure 4).

Now the DM has to choose whether he prefers to get A or B with probability one half or C, which lies between A and B, for sure. It must be possible to determine the utility of C independently, so that its position on  $u$  is defined. C is raised till the DM is indifferent between C and  $(A, \frac{1}{2}, B)$ . Then C is assigned an index value of .5, which generates the point  $P_C$  on the risk-preference function.

In the next step, the DM has to choose between  $(C, \frac{1}{2}, B)$  and  $(A, \frac{1}{2}, D)$ , with  $D > B$ . D is raised until the DM is indifferent, and D is assigned the value 1.5, defining  $P_D$  on  $p$ ; etc.

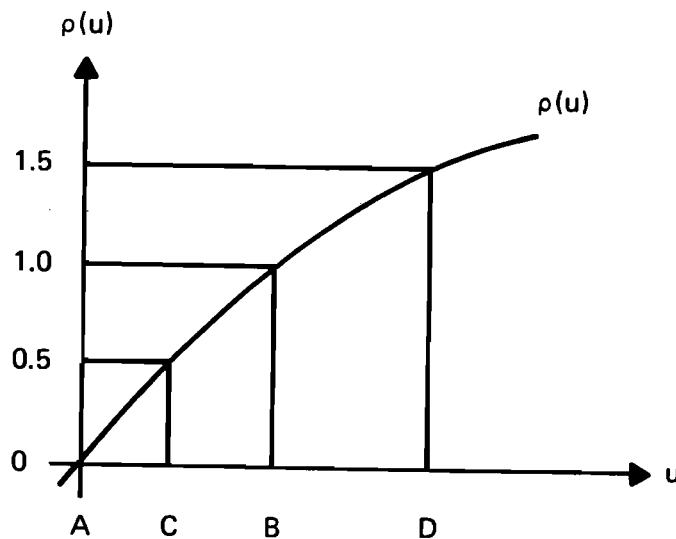
If the abscissa is defined as outcomes  $x_i$  instead of in terms of utilities  $u$ , and the ordinate is defined as  $u(x)$  instead of  $p(u)$ , the procedure is a standard one to generate a utility function. Since it is now made clear where the  $u$ 's come from, one might be suspicious whether Krelle's axiom system A is really something different from an expected utility theory. But this is an open question as yet.



If one takes Krelle's theory for granted, one is faced with a dilemma: Either one has expected utility, in which utility proper and risk preference are confounded, or one has the risk-preference function, which assumes that utility has been measured independently already. A somewhat similar procedure was proposed recently by v. Winterfeldt (1978), the author first constructs a "difference value judgment model", which is then in a second step transformed into an expected utility model.

FIGURE 4

Method for assessing  $\rho(u)$



### Generalization 1 of axiom system A

If persons exhibit probability preferences, axiom system A can be transferred to account for them. The procedure (Krelle's axiom system C) is as follows:

Call  $p^*$  attractiveness of a probability  $p$ ,

$$p^* = \psi(p) ,$$

and  $p^*$  is a monotonic and strictly increasing function. Since  $\sum_i p_i^*$  is not necessarily equal to one, as required by axiom system A, the following condition must be introduced to account for that. Let

$$\begin{aligned} L_1 &= \{(u_i, p_i^*)\} \quad \text{and} \\ L_2 &= \{(u_i, q_i^*)\} \quad , \quad i = 1, 2, \dots, n \quad , \end{aligned}$$

be two lotteries with the same utilities and modified probabilities  $p_i^*$  and  $q_i^*$  such that

$$(i) \quad \sum_{i=1}^n p_i^* = \sum_{i=1}^n q_i^* .$$

The original axiom A4 has to be modified as follows

A4\*: If

$$\begin{aligned} L_1 &= \{(u_1, p_1^*), (u_2, p_2^*), \dots, (u_n, p_n^*)\} \quad \text{and} \\ L_2 &= \{(u_1, q_1^*), (u_2, q_2^*), \dots, (u_n, q_n^*)\} \end{aligned}$$

are two lotteries and condition (i) is fulfilled, and the utilities are ordered such that  $u_1 \leq u_2 \leq \dots \leq u_n$ , and if the probability  $p_i^*$  are shifted towards the more favorable  $u$ 's, i.e., if

$$\sum_{i=1}^k p_i^* \leq \sum_{i=1}^k q_i^* \quad \text{for all } k, \quad k = 1, 2, \dots, n \quad ,$$

then  $L_1 \succsim L_2$  .

If there is at least one  $k$ ,  $k = 1, 2, \dots, n-1$ , such that

$$\sum_{i=1}^k p_i^* < \sum_{i=1}^k q_i^* \quad \text{and } u_k < u_{k+1}$$

then  $L_1 \succ L_2$ . The other axioms can remain unaltered, with  $p$  replaced by  $p^*$ .

The result is the generalized risk-preference function:

$$\begin{aligned} u^*(L) &= \sum_{i=1}^n \rho(u_i) \cdot p_i^* \\ &= \sum_{i=1}^n \rho(u_i) \cdot \psi(p_i) \end{aligned}$$

This generalization was first proposed by Georges Bernard (1964, 1965).

#### Generalization 2 of axiom system A

Bernard proposes to define  $\rho$  and  $\psi$  further as follows:

$$u_B^*(L) = \sum_{i=1}^n u_i^a \cdot p_i^c,$$

where

- a is the elasticity of the overall utility of a lottery with respect to utilities ( $a > 0$ ;  $u > 0$ )<sup>6/</sup>,
- c is the elasticity of the overall utility of a lottery with respect to probabilities.

According to Krelle, the basic idea behind this formulation is to take increasing or decreasing marginal utility (or attractiveness) of the probabilities into account.

If very small probabilities are (almost) neglected, as it seems to be the case in taking insurances, then  $c > 1$ . If the reverse is true, if very small probabilities are taken very seriously, then  $0 < c < 1$ . The latter could refer to the debate of "risk" in the atomic issue. There, failure probabilities are generally assumed to be very small, but may be taken to be subjectively larger as they numerically are. The same line of reasoning may hold true for utilities.

Bernard's formulation contradicts axiom system A unless  $x$  is a linear function of  $p$ , which can be seen from A2. Is this axiom system acceptable for a "rational" DM? The axioms might seem reasonable as long as the attractiveness of probabilities  $p_i^*$  refers to singular events, or almost singular events. As soon as there are many replications, one should assume that

$$c \rightarrow 1 \quad \text{for} \quad z \rightarrow \infty,$$

where  $z$  denotes the number of replications, a reasonable assumption for a Bayesian.

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<sup>6/</sup>This reminds somewhat of Stevens (1959) proposal to measure utility as a power function of money,  $u = x^a$ , with  $a$  typically being in the neighborhood of .5.

For any practical application the problem arises how to measure the  $p_i^*$ . This question is not discussed by Krelle. It must be noted that the  $p_i^*$  (as the  $p_i$  in axiom system A) as well as the  $u_i$  enter into the lotteries as extraneous variables, they have to be measured in the first place. Therefore, always a two-step procedure has to be performed:

(a) Axiom system A: risk-preference function

1. Measure the individual utilities  $u_i$ , in a relevant range in a riskless setting, provide  $p_i$ 's (assumed to be known);
2. deduce risk-preference function from preference judgments between lotteries  $L = \{(u_i, p_i)\}$ , if conditions are fulfilled, i.e., if the DM "accepts" the axioms.

(b) Axiom system A: modification 1

1. Measure individual utilities (as above); measure the attractiveness of probabilities  $p_i^*$  by some adequate procedure;
2. deduce generalized risk-preference function from preference between lotteries  $L = \{(u_i, p_i^*)\}$ .

(c) Axiom system A: modification 2 (Bernard's rule)

1. Measure the  $u_i$ 's and  $p_i$ 's ;
2. determine the coefficients  $a$  and  $c$  by some adequate procedure.

As will be seen from the description in the procedures, some steps are not operationally defined, indicated by phrases such as "by some adequate procedure". There are, to be sure, some procedures which may lead to the desired results. But it must be stressed again that Krelle's "axiomatizations" are not axiom systems of fundamental measurement. Despite this fact, they might be applicable in some situations, an answer to this (implicit) question has to be found empirically. Krelle might be right when he states that Bernard's criterion may be a strong simplification, but that it might not be possible to measure risk-preferences to a finer degree anyway.

4.4.3. Krelle's Axiom System B: Dispersion-Preference Formulation

Axioms A1 to A8 hold also for axiom system B, but axiom 9A is replaced by

Axiom 9B (dispersion-preference):

The utility index  $U$  of a lottery  $L = \{(u_1, p_1) \dots, (u_n, p_n)\}$ ,  $U(L)$ , depends only on the expected utility  $\bar{u}$  (the utilities are again assumed to be given)

$$i) \quad \bar{u} = \sum_{i=1}^n u_i p_i$$

and a measure of dispersion

$$ii) \quad u^* = \sum_{i=1}^n f(|u_i - \bar{u}|) \cdot p_i ,$$

$f$  being a non-negative, strictly monotonous increasing function with  $f(0) = 0$  and  $x > 0 \Rightarrow f(x) > 0$ .

In other words,

$$U(L) = \varnothing(\bar{u}, u^*) .$$

$\varnothing$  is called the dispersion-preference function of a person. Of course,  $f$  and  $\varnothing$  must be defined such that axioms A1 A8 still hold true.

All technicalities are omitted here, instead a specific function will be given fulfilling the assumptions. If one defines  $u^*$  as the mean absolute deviation,

$$u^* = \sum_{j=1}^n |u_j - \bar{u}| \cdot p_j ,$$

one has an example of a permitted dispersion index. Interestingly enough, the variance is not a permitted measure of dispersion in this theory (Krelle, 1968, p. 152). Now the function  $\varnothing$  has to be made explicit. Krelle shows that the simplest function  $\varnothing$ , together with  $u^*$  as given above, which fulfills axiom system B, is the following:

$$\varnothing(u, u^*) = \bar{u} + a \cdot u^* ,$$

with the restriction  $|a| < 1/2$ . If the DM is risk averse,  $a$  must be negative, i.e., the expected utility  $\bar{u}$  is reduced by  $a$  times the mean absolute deviation, which functions as a measure of "risk" in this case.

In this instance, the lottery with smaller expected utility can be preferred. If  $a = 0$ , the decision criterion is simply the expected utility, which is equivalent to risk neutrality in axiom system A. Finally, if  $a$  is greater than 0, the DM is risk prone, but the definition is somewhat different

from that of system A (see Krelle, 1968, pp. 159-163, for some more detail).

It is interesting to compare Krelle's dispersion-preference function  $\rho$  with Coombs' portfolio theory of risk preference. Coombs theory states that a person has a risk preference for each expected value level. Risk is left undefined, but some assumptions are necessary to empirically test the theory. Krelle's system B is more general in that it incorporates a direct expected utility-uncertainty trade-off. Uncertainty was given an operational definition in terms of mean absolute deviation. Psychologically, this assumes that expected utility and "risk" are compensatory, to some extent at least.

What is the relation between axiom systems A and B? B is simpler and may be intuitively quite appealing. A restriction is introduced by using a single index for characterizing the dispersion, thus abstracting from the whole distributional form of the lottery. According to Krelle, this may be acceptable in the case of symmetric distributions, but a person is not necessarily assumed to accept the B axioms if the options are as in the following example (Krelle, 1968, p. 159):

$$\begin{aligned} L_1 &= \{(1, .2), (2, .3), (3, .3), (4, .2)\} \\ L_2 &= \{(-2, .1), (2.5, .4), (3, .3), (4, .2)\} \end{aligned}$$

Both lotteries have an expected utility  $\bar{u}$  value of 2.5 and a mean absolute deviation of .9. If axiom system B is accepted and  $u^*$  is defined as mean absolute deviation, then  $L_1$  should be equivalent to  $L_2$ , i.e.,  $L_1 \sim L_2$ . Krelle states that many risk-averse persons will prefer  $L_2$  to  $L_1$ , but one could not call them irrational. Krelle's system B has striking similarities with the risk theory of Pollatsek and Tversky (1970), presented earlier.

## 5. DISCUSSION OF THE RISK DEFINITIONS AND RISK THEORIES

Different aspects of the various risk definitions and theories arose in the foregoing sections. The distinction between risk and risk preference is the most fundamental one. While risk refers to the perception of the riskiness of an option, risk preference refers to the DM's preference along the risk dimension. Furthermore, it is evident that risk is a property of decision making under uncertainty in which options (or alternatives) are characterized by probability distributions over outcomes, which were called lotteries or gambles. Probability distributions, in turn, can be characterized either by their components or by their moments. Therefore, it is almost trivial to state that "risk" must be a function of the components or of the moments. But it is as yet unclear (1) whether to base the aggregation on the risk

components or on the moments, and (2) what the functional form is, or, to put it differently, which aggregation rule delivers an acceptable definition of risk, which may then lead to an acceptable theory of risk in a next step. As to the status of a theory, one has to distinguish between axiomatic and "other" theories.

There are definitions of risk which rely on one component only, mostly on  $p_L$ , the probability (or probabilities) to "lose". Instances of such definitions are "pure risk", or the definitions of Norusis (1973) and of Starr, et al. (1976, p. 640). This use is much in line with an everyday use of the term. Since it is a definition (at best), it is obvious that it is not axiomatically founded, it is normative (at least for the author who uses it) and no aggregation rule is involved, since only one component is considered.

In the "low probability-high loss" formulation of the problem which is often used to characterize a typical "risk" situation both kinds of components are used. Since there are two aspects involved, losses and their respective probabilities, there must be a composition rule to combine the two. This is accomplished by expectancy theories, above all mathematical statistics in which risk is defined as expected loss. The aggregation rule is, of course, the expectation operator. Mathematical statistics is clearly a normative theory and, depending on its formulation, it has or has not a measurement-theoretic background.

It has to be re-emphasized that "loss" has no specific psychological meaning in mathematical decision theory and statistics. It is completely equivalent to an expected value theory (or expected utility, or subjective expected utility).

Finally, a last set of definitions which are only in terms of the left side of the distribution is the semi-variance  $V_b$  or  $V_o$  as defined earlier. Here, the variability over losses is taken as a characteristic of "risk". In Markowitz' portfolio theory, the semi-variance can be used instead of the more frequently used variance.

Now we come to the situation in which losses as well as gains are considered. An obvious thing to do is to consider the win and lose components and to try to find an aggregation (and weighting) rule for them. This was first done by Slovic (1967), who proposed an additive rule and estimated the weights by least squares. This approach could be criticized since the aggregation rule is not derived, but simply assumed by the researcher. Within the functional measurement theory approach of Norman Anderson, the functional form can be found out more directly, although this kind of measurement is not really axiomatic measurement (but comes somewhat close to it; see Wallsten, 1976, for a critical discussion). This approach was

taken by Anderson and Shanteau (1970) who found the additive model to be less satisfactory, compared with the multiplying model. Further research along these lines, or adapting the stricter version of conjoint measurement, is clearly needed.

The next class of definitions and theories is based on representation of lotteries in terms of moments. The most named moment in relation to risk is clearly the variance, although it was obvious from experiments and theoretical considerations that the first and third moment have some impact, too. Again, the problem of the aggregation rule comes up. Markowitz, in his portfolio theory avoided that issue by the following decision rule:

- a. from two lotteries with the same expected value, take the one with the smaller variance;
- b. from two lotteries with the same variance, take the one with higher expected value.

But with this formulation, the value of "risk" is not determined and must be added to the model externally.

Two further classes are evident from the literature. First, a linear combination of moments, expected value and variance as is most obvious from the regular risk system of Pollatsek and Tversky (1970), but also inherent in the Coombs and Huang theory. The second principle used is again the expectation rule, which underlies EV, EU, SEV, and SEU theory. These theories have been critisized, especially by Coombs, since they do not deal with risk in any psychological transparent and relevant fashion.

Tversky (1975, IV, p. 5) says: "Expected utility theory deals with the problem of risk through the slope of the utility function for the respective attribute, e.g., money. In this respect, utility theory does not permit attitudes toward risk per se, only attitudes toward money". In his contribution, Tversky shows that people consistently violate expected utility theory, since they over-evaluate options which do not have any uncertainty associated with them.

A generalization of SEU theory has been proposed by Huang in her expected risk theory. In it, expectation is not taken over utility values, but risk values. That is, instead of a utility function that assigns a utility to each outcome, there is a risk function that assigns a risk value to each outcome. The status of this theory is still a rather open question, theoretically and empirically. The notion of risk values of outcomes may not be a very appealing concept, and expectation theories have been critisized on theoretical grounds, e.g., Allais (1953) and Krantz and Tversky (1965).

My impression is that SEU theory is inadequate psychologically, and most likely even normatively, and even Huang's theory, though elegant, is too restrictive to cover anything



beyond symmetric gambles with fixed probabilities.

A last word on risk preference. Coombs' theory is a descriptive theory, and it has a measurement-theoretic foundation. It is a generalization of SEU theory, too. It seems to be very reasonable as long as single-peakedness is given. But, as has been shown by Coombs and Avrunin (1977a, b), single-peakedness is not a trivial concept and is not likely to arise in multi-attribute situations, which restricts the range of the theory. This remains an empirical question, too. In contrast to this completely descriptive theory, a "good" risk theory should have a normative touch so that it can be used for decision making under "risk". The two axiomatizations of Krelle have a stronger normative flavor--once one is willing to accept the axioms. The measurement-theoretic status of these theories is not quite clear, as had been pointed out. No experimental results are known which tested Krelle's system.

#### 6. TECHNOLOGICAL AND SOCIETAL RISK: A NEW DIMENSION?

Does "risk" in a technological and societal context gain a new dimension? To be really enabled to answer this question, we have to know what we mean by "risk" in a more elementary context, say, in laboratory experiments on lotteries. Since we really do not know that too well, we cannot answer the second question. But this statement is too trivial a way out. We know something in the first case, and we can speculate on the second. Let's do that.

The risk issue arises primarily with hazards, natural and man-made. I define hazards as potential loss (which is to be differentiated from expected loss). Natural hazards are possible damages done by the elements, like floods, and earthquakes. Man-made and technological hazards are due to this (mis-) functioning of technologies, such as energy plants, chemical plants, planes and cars, etc. Cigarette smoking, alcohol and drugs pose a hazard, too. They are man-made, but not technological. They could be called cultural hazards. Dangerous initiation rites would be another example, perhaps not too far from the first. Hazards due to wars fall into the same category. Let us concentrate on technological hazards.

If we think again in terms of risk components, i.e., probabilities for win and lose, and amounts to win and lose, some very obvious relationships emerge: The more hazardous (the results of a technology are), the less likely. Car accidents are quite likely, followed by plane accidents, and then major accidents of a nuclear power plant, which are generally considered to be among the least likely events. Now to the hazards. Let us assume that the worst thing that can happen is that you lose your life. This is common to all events. If one computes the expected loss, say, in terms of number of deaths per year, the figures differ markedly.

The more likely events have very high figures, the low probability events have very low figures<sup>7/</sup>. Or, in other words, deaths due to car accidents occur every day, but there has not been any really large nuclear accident. This reasoning can be translated into a simple model with some probability theory flavor. The probability of occurrence of a noxious event might be an exponential function of the expected loss. Taking logarithms,

$$\log p = k - cm \quad ,$$

with

p probability of occurrence of event,  
m magnitude of expected loss,  
k, c constants to be determined empirically .

Now one can apply extreme value distributions to that. Mostly, the asymptotic distribution of maximal values in a sample is supposed to follow an exponential distribution function of the form

$$F(x) = \exp -e^{-k(x-c)} \quad ,$$

with density

$$f(x) = k \exp -k(x-c) -e^{-k(x-c)} \quad .$$

See, e.g., Apostolakis (1974).

Varying along the dimension of expected loss, as we just saw, is the dimension of severity. An accident that is likely relative to others, such as a car accident, will affect only very few people, whereas a major power plant accident may result in a mass catastrophe. This might make a difference psychologically--the aspect of 'catastrophic' might be more salient than expected loss. Due to nature  $p_L$  and  $r_L$  seem to be correlated, so one cannot determine what is the more influential in actual risk assessments which are done intuitively. With lotteries, one could, in principle, vary  $p_L$  and  $r_L$  independently of each other, although the resulting lotteries may not look very reasonable, if too extremely deviating from lotteries modeled after 'nature'.

As you will have noticed, nothing has been said about  $p_W$  and  $r_W$  thus far. It will have to be analyzed in detail which role the 'positive' components play. Following earlier

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<sup>7/</sup>There is quite an extended literature on such models, see, e.g., Farmer (1967) or Slesin and Ferreira (1976).

observations by Slovic one might conclude that the negative aspects are more important than the positive ones. If this holds true, the variance or the mean absolute deviation would not be adequate measures, since they are symmetrical, weighting positives and negatives equally. Something between the variance and the semi-variance  $V_0$  (which takes only the negatives into account) may be appropriate.

From a behavioral decision theory point of view, at least two questions must be raised: (1) How can components be estimated, and (2) How are the estimated components aggregated into an overall judgment? Ad 1: People may not be very good in estimating extreme probabilities and ultimate losses. As it is known from the experimental literature, probabilities of extreme events are very badly assessed. Even if those probabilities are assessed by experts, one could have doubts as to their accuracy as well. For example, when assessing failure rates of a system, certain failure possibilities may be overlooked, or failures may be correlated (common mode failures), resulting in too low estimates if not taken into account. Some peculiarities of man's reasoning in the situation of uncertainty and "risk" are discussed in Slovic, Fischhoff and Lichtenstein (1976). One might further speculate that it is very difficult to assess the impact of a really serious loss. Ad 2: Some aggregating rules have been discussed in Ch. 5. There is no single rule which is really satisfactory, to my judgment.

But there may be other factors besides these risk components which influence the perceived riskiness of an option. Various such factors have been proposed and to some extent empirically established, like voluntary vs. involuntary (Starr, 1969), immediacy of effect, personal and scientific knowledge about hazards and their probability of occurrence, control (car vs. plane), active-passive, newness, chronic-catastrophic (already discussed above), common-dread, etc. See Slovic et al. (1977), or Otway and Fishbein (1977). Furthermore, some people are afraid that there are some new dimensions (of risk) associated with some of the modern technologies. Extreme safety measures considered necessary by the public, might imply extreme control<sup>8/</sup>. Extreme control, however, may be beyond the possibilities of a democratic government, so that a more authoritarian kind of government might turn out to be necessary.

Can all these aspects be taken into account to formulate a single theory of risk? Which are the mutual interconnections? First of all, it may be safe to assume that the probability

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<sup>8/</sup>The important question what the safety requirements of the public are, is left out here. See, "How Safe is Safe Enough", by Fischhoff et al. (1977) and Rowe (1977). It is evident that the issue is closely related to risk and risk preference.

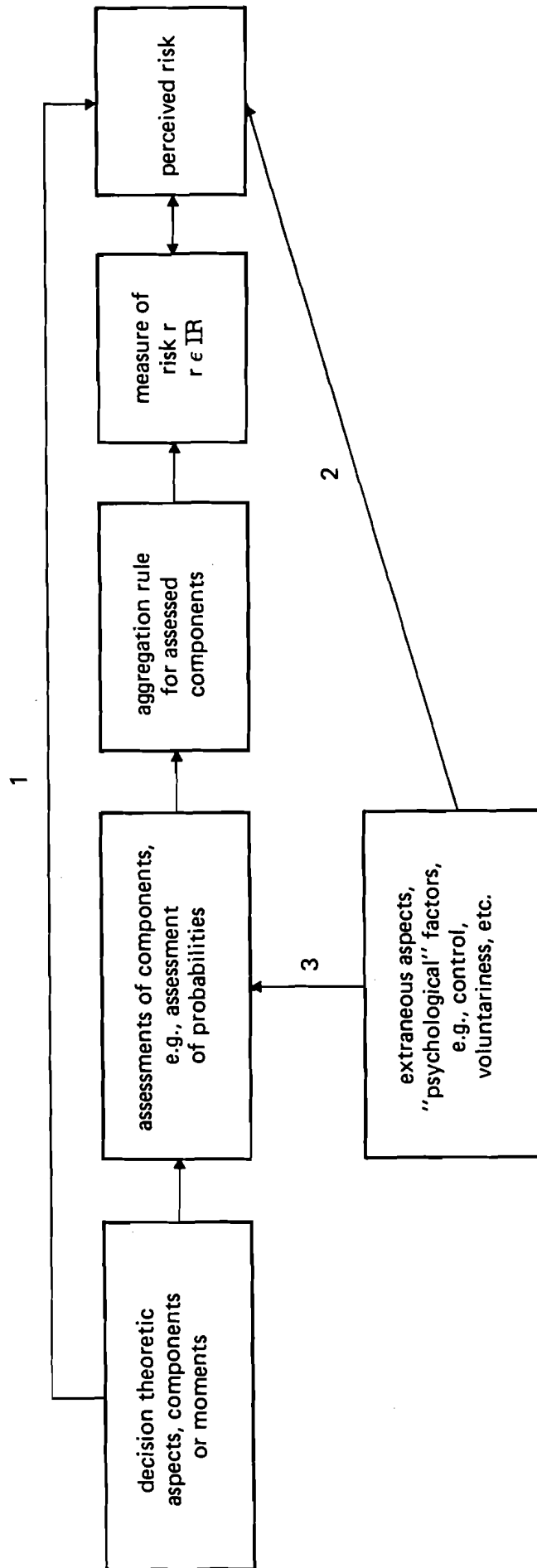
distribution over outcomes (may it be defined in terms of components or moments) has an influence on perceived risk. This is indicated by arrows in Figure 4 below. But the "psychological" aspects, i.e., other than those modeled by a probability distribution over outcomes, may influence perceived risk as well. If the influence is direct, arrow two is appropriate. Then, this influence must be modelled externally and additionally. But it could also be the case that these psychological factors influence the perception and therefore the assessments of the components, as indicated by arrow three. If this is the route of influence, a decision-theoretic model, if adequately formulated, is sufficient. It is further evident that a correspondence must exist between a measure of risk and perceived risk. This correspondence, however, need not be a one-to-one correspondence, since a "good" risk theory must also be a normatively appealing and normatively valid theory, and perceptions will probably not follow prescriptions completely. On the other hand, there must be a considerable correspondence, since otherwise no one would accept the risk measure (and the theory it is derived from) as a prescriptive or "normative" theory. Perhaps some kind of path-analytic modeling, which is suggested by the diagram, might help to clarify which way the psychological factors affect risk perception, if they do.

For completeness sake, let's go the remaining path through our little model. After the aspects of the situations have been formulated--again component vs. moment--the necessary assessments (estimates) have to be made. Next, the assessed components must be aggregated to yield a measure of risk. The aggregation is not independent of the modeling of the situation, of course.

How to proceed further? My personal feeling would be to concentrate, in a first step, on aspects of lotteries, to improve assessments of the components, i.e., estimates for small probabilities and for large losses, and to improve our understanding of how they combine into an overall measure of risk.

But let's come back to what we have to do. We have to decide. How do we want to decide? The goal is, of course, to make "good" decisions, to settle for the best option. Decision theory, based on expectation theory, is a normative approach for decision making under uncertainty. Is it also appropriate for decision making under "risk", i.e., a situation in which negative consequences are really negative, although not likely to occur at all? My reservations center around the fact that individual decision making under uncertainty is an expectancy theory, "risk" is integrated out. EU or SEU theory may (or may not) be perfect for repeatable events such as investments or insurances, but may be very inappropriate under the conditions described above. So what we really have to do in the long run is to develop a better normative theory of decision making under "risk", which must be different from the existing theory of decision making under uncertainty.

FIGURE 5  
Model of Relationship Between Decision Theoretic Aspects, Psychological Aspects,  
and Perceived Risk



Explanations:

A one-headed arrow " → " indicates influence,

a two-sided arrow " ↔ " correspondence.

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