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High Order Inverse Function Theorems

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Foreword

Many problems arising in optimization and control theory can be studied using appropriate inverse function theorems. In this paper the author proves several first and high order theorems and provides applications to questions of controllability and stability. The proofs rely on the variational principle due to Ekeland and the (constructive) Ljusternik's scheme.

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HIGH ORDER INVERSE FUNCTION THEOREMS

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Abstract   We prove several first order and high order inverse mapping theorems for maps defined on a complete metric space and provide a number of applications.

Key words Inverse function theorem, open mapping principle, set-valued map, controllability, reachable set, stability.

1   Introduction

Inverse function theorem is a natural tool to apply to many problems arising in control theory and optimization. The classical theorems, such as Ljusternik's theorem, are not always sufficient , and this because the data of the problems often happen to be "nonclassical" ones.

Such "unusual" situation does arise when one deals with

i)   A map whose domain of definition is a metric space

ii)   A map which is not single-valued

iii) A map for which the first order conditions are not sufficient to solve the problem.

This is why one has to look for different inverse function theorems adapted to new problems. During the last twenty years this task was undertaken in many papers (see for example [6], [5], [8], [9], [14], [21], [22] and bibliographies contained therein).

Let us recall first the Ljusternik theorem proved in 1934 in [23]:

Theorem 1.1 *Let $f : U \to X$ be a continuously differentiable function from a Banach space U to a Banach space X and $\bar{u} \in U$. If the derivative $f'(\bar{u})$*

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is surjective, then for all $h > 0$, $f(\overline{u}) \in \text{Int } f(B_h(\overline{u}))$ and there exists $L > 0$ such that for all $x \in X$ near $f(\overline{u})$, dist $(\overline{u}, f^{-1}(x)) \leq L \|f(\overline{u}) - x\|$.

As it was observed in [9] the assumptions of theorem imply much stronger conclusions. In fact Ljusternik's proof allows to go beyond the above result and to prove the uniform open mapping principle and regularity of the inverse map f^{-1} on a neighborhood of the point $(f(\overline{u}), \overline{u})$.

Furthermore the surjectivity assumption of the above theorem may be replaced by the assumption

$$(1) \qquad\qquad 0 \in \text{Int } \overline{f'(\overline{u})\,(B)}$$

Several extensions of Theorem 1.1 were derived in [9] via the same Ljusternik's idea. Assuming that the space U is just a complete metric space and some "covering assumptions" on f one can obtain a result similar to Theorem 1.1. However verification of covering assumptions is not always simple.

In this paper we prove a High Order Uniform Open Mapping Principle for maps defined on a complete metric space. That is, we provide a sufficient condition for the existence of $\mathcal{L} > 0$, $k \geq 1$ such that

$$(2) \qquad \forall\, u \text{ near } \overline{u} \text{ and } \forall\, h > 0, \quad f(u) + \mathcal{L}h^k B \in \text{Int } f(B_h(u))$$

To get regularity of the inverse map f^{-1} we use a high order analogue of Ljusternik's theorem proved in [19] (using again Ljusternik's scheme):

Theorem 1.2 (a general inverse function theorem). *Let G be a set-valued map from a complete metric space (U, d) to a metric space (X, d_X) having a closed graph and let $(\overline{u}, \overline{x}) \in \text{Graph } G$. Assume that for some $k > 0, \rho > 0$, $\epsilon > 0$, $0 \leq \alpha < 1$ and all $u \in B_\epsilon(\overline{u})$, $x \in G(u) \cap B_\epsilon(\overline{x})$, $h \in [0, \epsilon]$*

$$(3) \qquad\qquad \sup_{b \in B_{\rho h^k}(x)} \text{dist } (b, G(B_h(u))) \leq \alpha \rho h^k$$

Then for every $h > 0$ satisfying $h/(1 - \sqrt[k]{\alpha}) + 2\rho h^k \leq \epsilon/2$ and all $u \in B_{\epsilon/2}(\overline{u})$, $x \in G(u) \cap B_{\epsilon/2}(\overline{x})$, $y \in B_{\rho h^k}(x)$ we have

$$\text{dist } (u, G^{-1}(y)) \leq \frac{1}{1 - \sqrt[k]{\alpha}}\, h$$

In particular, for all $(u, x) \in \text{Graph } G$ near $(\overline{u}, \overline{x})$ and all y near $G(\overline{u})$

$$(4) \qquad \text{dist } (u, G^{-1}(y)) \leq \frac{1}{\sqrt[k]{\rho}(1 - \sqrt[k]{\alpha})} \sqrt[k]{d_X(x, y)}$$

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When X is a Banach space, assumption (3) can be formulated as $x + \rho h^k B \subset \overline{G(B_h(u))} + \alpha \rho h^k B$. In particular it holds true for a map satisfying the Uniform Open Mapping Principle (2) and therefore the two theorems together bring a sufficient condition for the regularity of f^{-1}. The Uniform Open Mapping Theorem is proved via the Ekeland variational principle. The curious aspect of this approach lies in the use of an apparently first order result (Ekeland's principle) to derive high order sufficient conditions.

Let us explain briefly the main ideas. When the space U is just a metric space then one can neither differentiate the function f nor speak about the continuity of the derivative. In [18], [19] we proposed to replace the derivative by the variation of the map (which can be single-valued or set-valued). The first order variation $f^{(1)}(\overline{u})$ of a single-valued map is defined in such way that for a C^1 map f between two Banach spaces the set $\overline{f'(\overline{u})B}$ is equal to it. Condition (1) together with continuity of the derivative inherit then their natural extension

$$0 \in \text{Int } \cup_{\epsilon > 0} \cap_{d(u,\overline{u}) \le \epsilon} \overline{\text{co}} \ f^{(1)}(u)$$

High order variations were introduced in [13] (see also [19], [14], [16]), where several sufficient conditions for regularity of the inverse map f^{-1} were proved. When one restricts the attention to single-valued maps only, then results of [19] can be improved and proofs can be made simpler. In this paper on one hand we prove more precise results for single-valued maps on the other we overview the applications of the inverse function theorem given in [15], [17]-[19] and provide its several new consequences.

The plan of the paper is as follows: Variations are defined in Section 2 where also several examples are given. Sections 3 and 4 are devoted to first and high order inverse function theorems for a single valued map. In Section 5 we state several theorems for set-valued maps. Their proofs can be find in [19]. Examples of applications are provided in Section 6.

2 Variations of single-valued and set-valued maps

Consider a metric space (U, d) and a Banach space X with the norm Gâteaux differentiable away from zero. For all $u \in U$, $h > 0$ let $B_h(u)$ denote the closed ball in U of center u and radius h.

We recall first the notions of Kuratowski's *limsup* and *liminf*:
Let T be a metric space and $A_r \subset X$, $r \in T$ be a family of subsets of X.

The Kuratowski *limsup* and *liminf* of A_r at r_0 are closed sets given by

$$\limsup_{r\to r_0} A_r = \{\, v \in X \ : \ \liminf_{r\to r_0} \ \text{dist}\,(v, A_r) = 0 \,\}$$
$$\liminf_{r\to r_0} A_r = \{\, v \in X \ : \ \lim_{r\to r_0} \ \text{dist}\,(v, A_r) = 0 \,\}$$

Definition 2.1 *Consider a function* $G : U \to X$ *and let* $u \in U$, $k > 0$.

i) The contingent variation of G *at* u *is the closed subset of* X *given by*

$$G^{(1)}(u) := \limsup_{h\to 0+} \frac{G(B_h(u)) - G(u)}{h}$$

ii) The k-th order variation of G *at* u *is the closed subset of* X *given by*

$$G^k(u) := \liminf_{h\to 0+, u'\to u} \frac{G(B_h(u')) - G(u')}{h^k}$$

In other words $v \in G^{(1)}(u)$ is and only if there exist sequences $h_i \to 0+$, $v_i \to v$ such that $G(u) + h_i v_i \in G(B_{h_i}(u))$. The word contingent is used because the definition reminds that of the contingent cone of Bouligand.

Similarly $v \in G^k(u)$ if and only if for all sequences $h_i \to 0+, u_i \to u$ there exists a sequence $v_i \to v$ such that $G(u_i) + h_i^k v_i \in G(B_{h_i}(u_i))$.

Clearly, $G^{(1)}(u)$ and $G^k(u)$ are closed sets starshaped at zero. When U is a Banach space and $G : U \to X$ is a Gâteaux differentiable at some $u \in U$ function, then $\overline{G'(u)(B)} \subset G^{(1)}(u)$. If moreover G is Fréchet differentiable at u then $\overline{G'(u)(B)} = G^1(u) = G^{(1)}(u)$.

The notions of variation extends to set-valued maps in a natural way. Let $G : U \to X$ be a set-valued map, that is for all $u \in U$, $G(u)$ is a (possibly empty) subset of X. The domain and the graph of G are given by

$$\text{Dom } G = \{u \in U \mid G(u) \neq \emptyset\}, \quad \text{Graph } G = \{(u,x) \mid u \in \text{Dom } G, \ x \in G(u)\}$$

Definition 2.2 *Let* $(u, x) \in \text{Graph } G$, $k > 0$.

i) The contingent variation of G *at* (u, x) *is the closed subset of* X

$$G^{(1)}(u) := \limsup_{h\to 0+} \frac{G(B_h(u)) - x}{h}$$

ii) The k-th order variation of G *at* (u, x) *is the closed subset of* X

$$G^k(u) := \liminf_{\substack{(u',x') \to_G (u,x) \\ h \to 0+}} \frac{G(B_h(u')) - x'}{h^k}$$

where $\to_G$ *denotes the convergence in* Graph G.

When G is a single-valued map the point $(u,x) \in$ Graph G if and only if $G(u) = x$ and therefore in this case the variations in the sense of the first and the second definitions do coincide.

Variations of all orders can be used to prove sufficient conditions for the existence of a Hölder inverse for a single-valued and a set-valued map. They describe a local expansion of a map at a given point.

Definition 2.3 *Let $G : U \to X$ be a set-valued map and $(u,x) \in$ Graph G. The expansion of G at (u,x) is the cone spanned by all variations*

$$G^\infty(u,x) = \bigcup_{\lambda \geq 0, \, k > 0} \lambda \, G^k(u,x)$$

Let co ($\overline{co}$) denote the convex (closed convex) hull and B the closed unit ball in X. The following result was proved in [19]:

Proposition 2.4 *For every $(u,x) \in$ Graph G we have*
i) For all $0 < k \leq K$, $0 \in G^k(u,x) \subset G^K(u,x)$
ii) For all $k > 0$, $\lambda_i \geq 0$, $v_i \in G^k(u,x)$, $i = 0, ..., m$ *satisfying* $\sum_{i=0}^{m} \lambda_i = 1$

$$\sum_{i=0}^{m} \lambda_i^k \, v_i \in G^k(u,x)$$

iii) For all $k > 0$, $v \in co \, G^k(u,x)$ *there exists* $\epsilon > 0$ *such that* $\epsilon v \in G^k(u,x)$
iv) For all $k > 0$, $\bigcup_{\lambda \geq 0} \lambda co \, G^k(u,x) = \bigcup_{\lambda \geq 0} \lambda G^k(u,x)$
v) The expansion cone $G^\infty(u,x)$ *is convex and equal to*

$$G^\infty(u,x) = \bigcup_{\lambda \geq 0, \, k > 0} \lambda \, co \, G^k(u,x) = \bigcap_{n > 0} \bigcup_{\lambda \geq 0, \, k \geq n} \lambda \, co \, G^k(u,x)$$

vi) If $G^\infty(u,x) = X$ then for some $k > 0$, $\bigcup_{\lambda \geq 0} \lambda G^k(u,x) = X$.

Proposition 2.5 *Let $k > 0$. The following conditions are equivalent:*
i) $\bigcup_{\lambda \geq 0} \lambda G^k(u,x) = X$
ii) $0 \in Int \, co \, G^k(u,x)$
  Moreover for all $k > 0$, $s > 0$, $\mathbf{R}_+ G^k(u,x) \subset G^{k+s}(u,x)$ and therefore if i) or ii) holds true then $G^{k+s}(u,x) = X$.

Remark   Observe that when X is a finite dimensional space then, by the Caratheodory theorem, ii) is equivalent to

$$\exists \, v_1, ..., v_p \in G^k(u,x) \quad \text{such that} \quad 0 \in Int \, co \, \{v_1, ..., v_p\}$$

Proof Assume that i) holds true. Since $G^k(u,x)$ is starshaped at zero, for all $0 \leq \lambda \leq \mu$, $\lambda G^k(x,y) \subset \mu G^k(u,x)$. Thus $\bigcup_{\lambda \in Z} \lambda G^k(u,x) = X$. Using Baire's theorem we prove that $G^k(u,x)$ has a nonempty interior and therefore also $\mathrm{co}G^k(u,x)$ does. Assume for a moment that zero is not an interior point of $\mathrm{co}G^k(u,x)$. By the separation theorem, there exists a different from zero $p \in \left(\mathrm{co}\,G^k(u,x)\right)^+ = \left(\bigcup_{\lambda \geq 0} \lambda \mathrm{co}\,G^k(u,x)\right)^+ = \{0\}$. This proves that i) implies ii). By Proposition 2.4iv), ii) implies i). Fix $k > 0$, $s > 0$, $v \in G^k(u,x)$, $\lambda \geq 0$ and set $\bar{k} = k + s$. Let $(u_i, x_i) \to_G (u,x)$, $h_i \to 0+$, $h'_i = \sqrt[k]{\lambda}\, h_i^{\bar{k}/k}$. Then for all large i, $h'_i \leq h_i$. Let $v_i \to v$ be such that $x_i + h_i'^k v_i = x_i + h_i^{\bar{k}} \lambda v_i \in G(B_{h'_i}(u_i)) \subset G(B_{h_i}(u_i))$. This implies that $\lambda v \in G^{\bar{k}}(u,x)$ and ends the proof. $\square$

Example 1. First order variation of a set-valued map

Consider Banach spaces P, X, Y, continuously differentiable functions $g : P \times X \to Y$, $h : P \times X \to \mathbb{R}^n$ and the set-valued map $G : P \times X \to P \times Y \times \mathbb{R}^n$ defined by

$$G(p,x) = \{\,(\,p,\,g(p,x),\,h(p,x)+\rho\,) \mid \rho \in \mathbb{R}^n_+\,\}$$

Then a direct calculation yields that for all $(p,x,q) \in P \times X \times \mathbb{R}^n_+$

$$G^1(p,x,p,g(p,x),h(p,x)+q) \supset$$
$$\{(v, \tfrac{\partial g}{\partial p}(p,x)v + \tfrac{\partial g}{\partial x}(p,x)w, \tfrac{\partial h}{\partial p}(p,x)v + \tfrac{\partial h}{\partial x}(p,x)w + \rho)| \,\|(v,w)\| \leq 1, \rho \in \mathbb{R}^n_+\}$$

Example 2. Contingent variation of end points of trajectories of a control system

Let U be a topological space, X be a Banach space and $f : X \times U \to X$ be a continuous, differentiable in the first variable function.

We assume that f is locally Lipschitz in the first variable uniformly on U, i.e. for all $x \in X$ there exist $L > 0$ and $\epsilon > 0$ such that for all $u \in U, f(\cdot, u)$ is L-Lipschitz on $B_\epsilon(x)$:

$$\|f(x',u) - f(x'',u)\| \leq L \|x' - x''\|, \quad \text{for all } x',x'' \in B_\epsilon(x)$$

Fix $T > 0$ and let $\mathcal{U}$ denote the set of all (Lebesgue) measurable functions $u : [0,T] \to U$. Define a metric d on $\mathcal{U}$ by setting $d(u,v) = \mu(\{t \in [0,T] \mid u(t) \neq v(t)\})$, where μ denotes the Lebesgue measure. The space $(\mathcal{U}, d)$ is complete (see Ekeland [10]).

Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup of continuous linear operators from X to X and A be its infinitesimal generator, $x_0 \in X$. Consider the control system

(5)
$$\begin{cases} x'(t) &= Ax(t) + f(x(t),u(t)), \quad u \in \mathcal{U} \\ x(0) &= x_0 \end{cases}$$

Recall that a continuous function $x : [0,T] \to X$ is called a mild trajectory of (5) if for some $u \in \mathcal{U}$ and all $0 \leq t \leq T$

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(x(s),u(s))ds$$

We denote by x_u the trajectory (when it is defined on the whole time interval $[0,T]$ and is unique) corresponding to the control u. Define the map $G :$ $\mathcal{U} \to X$ by

$$G(u) = \{x_u(T)\}$$

Let z be a mild trajectory of (5) on $[0,T]$ and $\bar{u}$ be the corresponding control. Consider the linear control system

(6)
$$Z' = AZ + \frac{\partial f}{\partial x}(z(t),\bar{u}(t))Z$$

and let $S_{\bar{u}}(t;s)$ denote its solution operator, where $S_{\bar{u}}(s;s) = Id$, $t \geq s$. Then for all u near $\bar{u}$, G is a well defined single-valued map. Moreover for almost all $t \in [0,T]$ and for all $u \in U$

(7)
$$S_{\bar{u}}(T;t)(f(z(t),u) - f(z(t),\bar{u}(t))) \in G^{(1)}(\bar{u})$$

We refer to [11], [18] for the proof of this result.

Example 3. First order variation of end points of trajectories of a differential inclusion

Let X be a finite dimensional space, F be a set-valued map from X to X. We associate with it the differential inclusion

(8)
$$x' \in F(x)$$

An absolutely continuous function $x \in W^{1,1}(0,T)$, $T \geq 0$ (the Sobolev space) is called a trajectory of the differential inclusion (8) if for almost all $s \in [0,T]$, $x'(s) \in F(x(s))$. The set of all trajectories of (8) defined on the

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time interval $[0, T]$ and starting at ξ, $(x(0) = \xi)$ is denoted by $S_{[0,T]}(\xi)$. The reachable map of (8) from ξ is the set-valued map $R : \mathbb{R}_+ \to X$ defined by

$$R(t) = \{ x(t) \mid x \in S_{[0,t]}(\xi) \}$$

Assume that

$$\begin{cases} H_1) & \forall \, x \in X \text{ near } \xi, \ F(x) \text{ is a nonempty compact set and } 0 \in F(\xi) \\ H_2) & \exists \text{ a neighborhood } \mathcal{N} \text{ of } \xi \text{ and } L > 0 \text{ such that} \\ & \forall \, x, y \in \mathcal{N}, \ F(t, x) \subset F(t, y) + L \|x - y\| B \end{cases}$$

Hypothesis $H_2)$ means that F is Lipschitz in the Hausdorff metric on a neighborhood of ξ. Hypothesis $H_1)$ implies that $x \equiv \xi \in S_{[0,\infty)}(\xi)$.

The derivative of F at $(\xi, 0)$ is the set-valued map $CF(\xi, 0) : X \to X$ defined by

$$\forall \, u \in X, \ CF(\xi, 0)u = \liminf_{\substack{(x,y) \to_F (\xi, 0) \\ h \to 0+}} \frac{F(x + hu) - y}{h}$$

Fix $T > 0$ and consider the single-valued map $G : W^{1,1}(0, T) \supset S_{[0,T]}(\xi) \to X$ defined by $G(x) = x(T)$. Let $K \subset coF(\xi)$ be a closed convex set having only finite number of extremal points. Then there exists $M > 0$ such that for every trajectory $w \in W^{1,1}(0, T)$ of the differential inclusion

$$(9) \qquad w' \in CF(\xi, 0)w + K, \quad w(0) = 0$$

we have $\frac{w(T)}{M} \in G^1(\xi)$. The proof follows from the results of [15].

Example 4.   High order variation of reachable map

Let F be a set-valued map satisfying all the assumptions from the Example 3. Consider again the differential inclusion (8) and the reachable map $t \to R(t)$. It was shown in [17] that for all integer $k \geq 1$

$$(10) \qquad R^k(0, \xi) = \liminf_{h \to 0+} \frac{R(h) - \xi}{h^k}$$

A very same proof implies that the above holds true for all $k > 0$.

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3 First order inverse of a single valued map

Consider a complete metric space (U, d), a Banach space X and a continuous map $G : U \to X$. We assume that the norm of X is Gâteaux differentiable away from zero. Let $\bar{u} \in U$ be a given point. We study here a sufficient condition for:

$$\forall \, h > 0, \quad G(\bar{u}) \in \text{Int} \, G(B_h(\bar{u})) \quad \text{(open mapping principle)}$$

and the regularity of the inverse map $G^{-1} : X \to U$ defined by

$$G^{-1}(x) = \{ \, u \in U \mid G(u) = x \, \}$$

on a neighborhood of $(G(\bar{u}), \bar{u})$.

Theorem 3.1 (Inverse Mapping Theorem) *If for some $\epsilon > 0$, $\rho > 0$*

$$\rho B \subset \bigcap_{d(u,\bar{u}) \leq \epsilon} \overline{co} \, G^{(1)}(u)$$

then for every $u \in B_{\frac{\epsilon}{2}}(\bar{u})$ and $h \in [0, \frac{\epsilon}{2}]$, $G(u) + h\rho \, \overset{o}{B} \subset G(B_h(u))$, (where $\overset{o}{B}$ denotes the open unit ball in X). Moreover for every $u \in B_{\frac{\epsilon}{2}}(\bar{u})$, $x \in B_{\frac{\epsilon}{4}}(G(u))$

$$\text{(11)} \qquad \text{dist} \, (u, G^{-1}(x)) \leq \frac{1}{\rho} \, \|G(u) - x\|$$

Corollary 3.2 *Assume that X is a finite dimensional space and that $0 \in \text{Int} \, \liminf_{u \to \bar{u}} \overline{co} \, G^{(1)}(u)$. Then there exist $\epsilon > 0$, $\rho > 0$ such that all conclusions of Theorem 3.1 are valid.*

Proof (of Theorem 3.1) Fix $u \in B_{\frac{\epsilon}{2}}(\bar{u})$, $0 < h \leq \frac{\epsilon}{2}$ and assume for a moment that there exists $x \in X$ satisfying

$$\text{(12)} \qquad \|x - G(u)\| < h\rho, \quad x \notin G(B_h(u))$$

Set $\Theta^2 = \|x - G(u)\| / h\rho$. Then $0 < \Theta < 1$. Applying the Ekeland variational principle [10] to the complete metric space $B_h(u)$ and the continuous function $y \to \|G(y) - x\|$ we prove the existence of $\bar{y} \in B_{\Theta h}(u)$ such that for all $y \in B_h(u)$

$$\text{(13)} \qquad \|G(\bar{y}) - x\| \leq \|G(y) - x\| + \Theta \, \rho \, d(y, \bar{y})$$

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Observe that $\overline{y} \in \text{Int } B_h(u)$ and, by (12), $x \neq G(\overline{y})$. Hence, by differentiability of the norm, there exists $p \in X^*$ of $\|p\| = 1$ such that for all $h_j \to 0+$, $v_j \to v$ we have

$$(14) \qquad \|G(\overline{y}) + h_j v_j - x\| = \|G(\overline{y}) - x\| + <p, h_j v_j> + o(h_j)$$

where $\liminf_{j \to \infty} o(h_j)/h_j = 0$. Fix $v \in G^{(1)}(\overline{y})$. Then from (13), (14) and Definition 2.1 we obtain $0 \leq <p, h_j v_j> +\Theta \rho h_j + o(h_j)$. Dividing by h_j and taking the limit yields $<p, v> \geq -\Theta \rho$. Hence

$$(15) \qquad \forall v \in \overline{co}\, G^{(1)}(\overline{y}), \quad <p, v> \geq -\Theta \rho$$

Since $d(\overline{y}, \overline{u}) \leq d(\overline{y}, u) + d(u, \overline{u}) \leq \Theta h + \frac{\epsilon}{2} < \epsilon$, by the assumption of theorem, $\rho B \subset \overline{co}\, G^{(1)}(\overline{y})$. Hence (15) yields that

$$-\rho \geq \inf_{v \in \overline{co}\, G^{(1)}(\overline{y})} <p, v> \geq -\Theta \rho$$

But $0 < \Theta < 1$ and $\rho > 0$ and we obtained a contradiction. The second statement follows from the first one and Theorem 1.2. $\square$

Remark Inequality (11) means that G is pseudo-Lipschitz at $(G(\overline{u}), \overline{u})$ with the Lipschitz constant ρ^{-1} (see Aubin [1]). $\square$

As one should expect sufficient conditions for an Open Mapping Principle to hold true are weaker than those implying the regularity of the inverse. We prove next

Theorem 3.3 (Open Mapping Principle) *Assume that there exist a compact set $Q \subset X$, $\epsilon > 0$ such that*

$$(16) \qquad \text{Int} \bigcap_{d(u,\overline{u})<\epsilon} \left(\overline{co}\, G^{(1)}(u) + Q \right) \neq \emptyset$$

If

$$(17) \qquad 0 \in \text{Int} \liminf_{u \to \overline{u}} \overline{co}\, G^{(1)}(u)$$

then for every $h > 0$, $G(\overline{u}) \in \text{Int } G(B_h(\overline{u}))$.

Proof The proof is similar to that of Theorem 3.1. We assume for a moment that for some $h > 0$, $G(\overline{u})$ is not an interior point of $G(B_h(\overline{u}))$ and we consider a sequence $\overline{x}_i \in X$, $\overline{x}_i \notin G(B_h(\overline{u}))$, $\|\overline{x}_i - G(\overline{u})\| \leq i^{-2}$, $i = 1, 2, ...$. Applying the Ekeland variational principle to the continuous function $u \to$

$\|G(u) - \bar{x}_i\|$ on the complete metric space $B_h(\bar{u})$ we prove the existence of $u_i \to \bar{u}$ such that for all $u \in B_h(\bar{u})$

$$(18) \qquad \|G(u_i) - \bar{x}_i\| \leq \|G(u) - \bar{x}_i\| + \frac{1}{i} d(u, u_i)$$

By differentiability of the norm for some $p_i \in Y^*, \|p_i\| = 1$ and for all $h_j \to 0+, v_j \to v$ we have

$$\|G(u_i) + h_j v_j - \bar{x}_i\| = \|G(u_i) - \bar{x}_i\| + <p_i, h_j v> + o_{i,v}(h_j)$$

where $\lim_{j \to \infty} o_{i,v}(h_j)/h_j = 0$. Fix $v \in G^{(1)}(u_i)$ and let $h_j \to 0+, v_j \to v$, $\bar{u}_j \in B_{h_j}(u_i)$ be such that $G(u_i) + h_j v_j = G(\bar{u}_j)$. Then replacing in (18) u by $\bar{u}_j$ we obtain

$$(19) \qquad 0 \leq <p_i, h_j v> + o_{i,v}(h_j) + \frac{h_j}{i}$$

Dividing by h_j and taking the limit when $j \to \infty$ yield

$$(20) \qquad \forall v \in G^{(1)}(u_i), \quad <p_i, v> \geq -\frac{1}{i}$$

From the Alaoglu theorem, taking a subsequence and keeping the same notations, we may assume that $\{p_i\}$ converges weakly in X^* to some p. Then (20) implies that for all $w \in \liminf_{u \to \bar{u}} \overline{co}\, G^{(1)}(u), \ <p, w> \geq 0$. This and the assumption (17) yield that $p = 0$. To get a contradiction we show next that p can not be equal to zero. Indeed let $z_i \in X, \|z_i\| \leq 1$ be such that $<p_i, z_i> \geq 1 - \frac{1}{i}$. Consider $z \in X, \rho > 0$ such that the ball $z + \rho B$ is contained in the left-hand side of (16). Let $a_i \in \overline{co}\, G^{(1)}(u_i), \ q_i \in Q$ such that for all large i, $z - \rho z_i = a_i + q_i$. Hence (20) yields $<p_i, z - \rho z_i - q_i> \geq -\frac{1}{i}$ and therefore $<p_i, z - q_i> \geq \rho - \frac{\rho}{i} - \frac{1}{i}$. Let $\{q_{i_j}\}$ be a subsequence converging to some $q \in Q$. Then, taking the limit in the last inequality we get $<p, z - q> \geq \rho > 0$. This implies that $p \neq 0$ and ends the proof. $\square$

Corollary 3.4 *Let X be a Hilbert space, H be a closed subspace of X of finite co-dimension. Assume that there exist $\rho > 0$, $z \in X$ such that*

$$\forall u \text{ near } \bar{u}, \quad z + \rho B_H \subset \overline{co}\, G^{(1)}(u)$$

where B_H denote the closed unit ball in H. If $0 \in \text{Int} \liminf_{u \to \bar{u}} \overline{co}\, G^{(1)}(u)$ then for all $h > 0$, $G(\bar{u}) \in \text{Int}\, G(B_h(\bar{u}))$.

Proof Observe that $\rho B \subset \rho B_H + \rho B_{H\perp}$. Thus for all u near $\bar{u}$, $z + \rho B \subset z + \rho B_H + \rho B_{H\perp} \subset \overline{co}\, G^{(1)}(u) + \rho B_{H\perp}$. Since $B_{H\perp}$ is a compact set, Theorem 3.3 ends the proof. $\square$

4 High order inverse of a single valued map

Let U be a complete metric space, X be a uniformly smooth Banach space (see [7]) and $G : U \to X$ be a continuous function. In this section we prove higher order sufficient conditions for openness of G and regularity of the inverse map G^{-1}.

Theorem 4.1 (Inverse Mapping Theorem) *Let $\overline{u} \in U$ and assume that for some $k \geq 1$, $M > 0$, $\rho > 0$ and for all $u \in U$ near $\overline{u}$ and all small $h > 0$*

$$(21) \qquad \rho B \subset \overline{co} \left(\frac{G(B_h(u)) - G(u)}{h^k} \cap MB + G^{(1)}(u) \right)$$

Then there exists $\mathcal{L} > 0$ such that for all $u \in U$ near $\overline{u}$, for all $x \in X$ near $G(\overline{u})$ and all $h > 0$ small enough

$$G(u) + \mathcal{L}h^k B \subset G(B_h(u)); \quad \text{dist}\,(u, G^{-1}(x)) \leq \frac{1}{\sqrt[k]{\mathcal{L}}} \|G(u) - x\|^{\frac{1}{k}}$$

Corollary 4.2 *Assume that X is a finite dimensional space and for some $k \geq 1$, the convex cone spanned by $G^k(\overline{u}, \overline{x})$ is equal to X. Then the conclusions of Theorem 4.1 hold true.*

Proof From Proposition 2.5 and the Remark following it we deduce that for some $v_i \in G^k(\overline{u}, \overline{x})$, $i = 1, ..., p$ we have $0 \in \text{Int}\, co\{v_1, ..., v_p\}$. Then for some $\epsilon > 0$ and for all $v_i' \in B_\epsilon(v_i)$ we have $0 \in \text{Int}\, co\{v_1', ..., v_p'\}$. On the other hand, by the definition of variation for every $1 \leq i \leq p$ there exists $\delta_i > 0$ such that

$$d(\overline{u}, u) + h \leq \delta_i \implies \text{dist}\left(v_i, \frac{G(B_h(u)) - G(u)}{h^k} \right) \leq \epsilon$$

Hence the assumption of Theorem 4.1 is satisfied. $\square$

Proof(of Theorem 4.1) Assume for a moment that there exist $\overline{u}_i \to \overline{u}$, $\overline{h}_i \to 0+$, $x_i \in G(\overline{u}_i) + 2^{-k}i^{-k}\overline{h}_i^k B$ satisfying

$$(22) \qquad x_i \notin G(B_{\overline{h}_i}(\overline{u}_i))$$

Applying the Ekeland variational principle to the complete metric space $B_{\overline{h}_i}(\overline{u}_i)$ and the continuous function $u \to \sqrt[k]{\|G(u) - x_i\|}$ we prove the existence of $u_i \in B_{\frac{\overline{h}_i}{2}}(\overline{u}_i)$ such that for all $u \in B_{\overline{h}_i}(\overline{u}_i)$

$$(23) \qquad \sqrt[k]{\|G(u_i) - x_i\|} \leq \sqrt[k]{\|G(u) - x_i\|} + \frac{1}{i}\, d(u, u_i)$$

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By (22) and the smooth differentiability of the norm, there exist $p_i \in X^*$ of $\|p_i\| = 1$ and a function $o : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{h \to 0+} o(h)/h = 0$ such that for all $u \in B_{\bar{h}_i}(\bar{u}_i)$ we have

$$\|G(u) - x_i\| \leq \|G(u_i) - x_i\| + <p_i, \ G(u) - G(u_i)> + o(\|G(u) - G(u_i)\|)$$

Hence

$$
\left\{
\begin{array}{l}
\sqrt[k]{\|G(u) - x_i\|} \leq \\[2mm]
\sqrt[k]{\|G(u_i) - x_i\| + <p_i, \ G(u) - G(u_i)> + o(\|G(u) - G(u_i)\|)} = \\[2mm]
\sqrt[k]{\|G(u_i) - x_i\|} \left(1 + <p_i, \ \frac{G(u) - G(u_i)}{\|G(u_i) - x_i\|}> + \frac{o(\|G(u) - G(u_i)\|)}{\|G(u_i) - x_i\|}\right)^{1/k} \leq \\[2mm]
\sqrt[k]{\|G(u_i) - x_i\|} \left(1 + \frac{1}{k} <p_i, \ \frac{G(u) - G(u_i)}{\|G(u_i) - x_i\|}> + \bar{o}\left(\frac{\|G(u) - G(u_i)\|}{\|G(u_i) - x_i\|}\right)\right)
\end{array}
\right.
$$

where $\bar{o} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\lim_{h \to 0+} \bar{o}(h)/h = 0$. This and (23) yield that for all $u \in B_{\bar{h}_i}(\bar{u}_i)$

$$
\begin{array}{ll}
0 \leq & \frac{1}{k} \|G(u_i) - x_i\|^{\frac{1}{k} - 1} <p_i, G(u) - G(u_i)> + \\[2mm]
& \|G(u_i) - x_i\|^{\frac{1}{k}} \ \bar{o}\left(\frac{\|G(u) - G(u_i)\|}{\|G(u_i) - x_i\|}\right) + \frac{1}{i} d(u, u_i)
\end{array}
$$

Hence for all $u \in B_{\bar{h}_i}(\bar{u}_i)$

$$
(24) \quad
\left\{
\begin{array}{l}
0 \leq \ <p_i, \ G(u) - G(u_i)> + \|G(u_i) - x_i\| \ \bar{o}\left(\frac{\|G(u) - G(u_i)\|}{\|G(u_i) - x_i\|}\right) \\[2mm]
\quad + \frac{k}{i} \|G(u_i) - x_i\|^{\frac{k-1}{k}} d(u, u_i)
\end{array}
\right.
$$

Fix $v \in G^{(1)}(u_i)$ and let $h_j \to 0+, v_j \to v$ be such that $G(u_i) + h_j v_j \in G(B_{h_j}(u_i))$. Then from (24) we obtain

$$0 \leq \ <p_i, \ h_j v_j> + o(h_j) + \frac{k}{i} \|G(u_i) - x_i\|^{\frac{k-1}{k}} h_j$$

Dividing by h_j and taking the limit yields

$$(25) \qquad \forall \, v \in G^{(1)}(u_i), \quad <p_i, \ v> \geq -\frac{k}{i} \|G(u_i) - x_i\|^{\frac{k-1}{k}}$$

On the other hand, by (24), for $h_i = \sqrt[k]{\|G(u_i) - x_i\|}/\sqrt[2k]{i}$ and for all $v \in (G(B_{h_i}(u_i)) - G(u_i))/h_i^k$ we have

$$(26) \qquad 0 \leq \ <p_i, \ v> + \sqrt{i} \ \bar{o}\left(\frac{\|v\|}{\sqrt{i}}\right) + k \, i^{-\frac{k+1}{2k}}$$

Adding (25) and (26) yields

$$
(27) \quad \begin{cases} \forall \, v \in \dfrac{G(B_h(u_i)) - G(u_i)}{h_i^k} \cap MB + G^{(1)}(u_i) \\[2mm] <p_i, v> \; \geq \; -\frac{k}{i} \, \|G(u_i) - x_i\|^{\frac{k-1}{k}} - \sqrt{io} \left(\frac{1}{\sqrt{i}}\right) - ki^{-\frac{k+1}{2k}} \end{cases}
$$

Observe that for all large i, $d(u_i, \bar{u}) \leq d(u_i, \bar{u}_i) + d(\bar{u}_i, \bar{u}) \leq \bar{h}_i/2 + d(\bar{u}_i, \bar{u}) < \epsilon$. Hence assumption (21) of theorem contradicts (27). This proves that (22) can not hold. The second statement of theorem follows from the first one and Theorem 1.2. $\square$

Theorem 4.3 (An Open Mapping Principle) *Assume that there exist $\rho > 0$, $M > 0$ $k \geq 1$ and a compact subset $Q \subset X$ such that for all $u \in U$ near $\bar{u}$ and all small $h > 0$*

$$
(28) \quad \rho B \; \subset \; \overline{co} \left(\frac{G(B_h(u)) - G(u)}{h^k} \cap MB + G^{(1)}(u) \right) + Q
$$

If $G^\infty(\bar{u}) = X$, then for all $h > 0$, $G(\bar{u}) \in \mathrm{Int}\, G(B_h(\bar{u}))$.

When the space X is finite dimensional then the assumption (28) is always satisfied with Q equal to the unit ball.

Proof By Propositions 2.4vi) and Proposition 2.5 there exists $k' > 0$ such that $0 \in \mathrm{Int}\, coG^{k'}(\bar{u}, \bar{x})$. Set $\bar{k} = \max(k, k')$. Then, by Proposition 2.4i), $0 \in \mathrm{Int}\, co\, G^{\bar{k}}(\bar{u}, \bar{x})$. Moreover for all $u \in U$ near $\bar{u}$ and all small $h > 0$ and $h' = h^{\bar{k}/k}$ we have

$$
\begin{cases} \rho B & \subset \; \overline{co}\left(\frac{G(B_{h'}(u)) - G(u)}{h'^k} \cap MB + G^{(1)}(u) \right) + Q \\[3mm] & \subset \; \overline{co}\left(\frac{G(B_h(u)) - G(u)}{h^{\bar{k}}} \cap MB + G^{(1)}(u) \right) + Q \end{cases}
$$

Hence we may assume that k in the assumption (28) is so that

$$
(29) \quad\quad\quad\quad 0 \; \in \; \mathrm{Int}\, co\, G^k(\bar{u}, \bar{x})
$$

Assume for a moment that there exist $\bar{h}_i \to 0+$, $x_i \in G(\bar{u}) + 2^{-k} i^{-k} \bar{h}_i^k B$ satisfying (22). Setting $\bar{u}_i = \bar{u}$ in the proof of Theorem 4.1 from (27) we get

$$
(30) \quad \begin{cases} \forall \, v \in \overline{co} \left(\frac{G(B_{h_i}(u_i)) - G(u_i)}{h_i^k} \cap MB + G^{(1)}(u_i) \right) \\[2mm] <p_i, v> \; \geq \; -\frac{k}{i} \, \|G(u_i) - x_i\|^{\frac{k-1}{k}} - \sqrt{io} \left(\frac{1}{\sqrt{i}}\right) - ki^{-\frac{k+1}{2k}} \end{cases}
$$

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Let $z_i \in B$ be such that $< p_i, z_i > \geq 1 - \frac{1}{i}$ and

$$q_i \in Q, \quad v_i \in \overline{co} \left(\frac{G(B_{h_i}(u_i)) - G(u_i)}{h_i^k} \cap MB + G^{(1)}(u_i) \right)$$

be such that $-\rho z_i = v_i + q_i$ (they do exist because of (28)). Then

$$\rho(1 - \tfrac{1}{i}) + < p_i, q_i > \quad \leq \quad < p_i, \, \rho z_i + q_i >$$
$$\leq \quad \tfrac{k}{i} \|G(u_i) - x_i\|^{\frac{k-1}{k}} + \sqrt{io} \left(\tfrac{1}{\sqrt{i}}\right) + k i^{-\frac{k+1}{2k}}$$

Taking subsequences and keeping the same notations we may assume that $q_i \to q$ and p_i converges weakly in X^* to some p. Thus $\rho + < p, q > \leq 0$ and consequently $p \neq 0$. On the other hand (30) implies that for all $v \in G^k(\overline{u}, \overline{x})$ of $\|v\| \leq \frac{M}{2}$ we have $< p, v > \geq 0$. This and (29) yield $p = 0$. The obtained contradiction ends the proof. □

5   Inverse of a set-valued map

Consider again a complete metric space (U, d) and a Banach space X with the norm Gâteaux differentiable away from zero. Let G be a set-valued map from U to X, whose graph is closed in $U \times X$. Consider a point $(\overline{u}, \overline{x}) \in$ Graph G. We proved in [19] a sufficient condition for

$$(31) \qquad \qquad \forall \, h > 0, \quad \overline{x} \in \text{Int} \, G(B_h(\overline{u}))$$

and the regularity of the inverse map

$$G^{-1}(x) \;=\; \{ \, u \in U : x \in G(u) \, \}$$

on a neighborhood of $(\overline{x}, \overline{u})$. In this case the results are more restrictive, we only state them (the corresponding proofs can be found in [19]).

Theorem 5.1 (Open Mapping Principle) *Assume that for some* $\epsilon >$ 0, $M > 0$ *and a compact set* $Q \subset X$

$$(32) \quad \text{Int} \bigcap_{\substack{(u,x) \,\in\, \text{Graph } G \\ d(u,\overline{u}) \leq \epsilon, \|x - \overline{x}\| \leq \epsilon}} \left(\overline{co} \, (G^{(1)}(u,x) \cap MB) + Q \right) \;\neq\; \emptyset$$

If

$$0 \in \text{Int} \liminf_{(u,x) \to o(\overline{u},\overline{x})} \overline{co} \, \left(G^{(1)}(u,x) \cap MB \right)$$

then for all $h > 0$, $\overline{x} \in \text{Int} \, G(B_h(\overline{u}))$.

Observe that when X is a finite dimensional space then the condition (32) is always satisfied with Q equal to the unit ball and $M = 1$. Hence

Corollary 5.2 *Assume that X is a finite dimensional space and for some $M > 0$*

$$0 \in \text{Int} \liminf_{(u,x) \to_G (\bar{u}, \bar{x})} \overline{\text{co}} \left(G^{(1)}(u,x) \cap MB \right)$$

Then for all $h > 0$, $\bar{x} \in \text{Int} \, G(B_h(\bar{u}))$.

Theorem 5.3 (Inverse Function Theorem) *If for some $\epsilon > 0$, $M > 0$*

$$0 \in \text{Int} \bigcap_{\substack{(u,x) \in \text{Graph } G \\ d(u,\bar{u}) \leq \epsilon, \, \|x - \bar{x}\| \leq \epsilon}} \overline{\text{co}} \left(G^{(1)}(u,x) \cap MB \right)$$

then there exists $L > 0$ such that for all $(u,x) \in \text{Graph } G$ near $(\bar{u}, \bar{x})$ and all $y \in X$ near $\bar{x}$

$$\text{dist} \left(u, \, G^{-1}(y) \right) \leq L \, \|x - y\|$$

The high order results also have their analogs in the set-valued case.

Theorem 5.4 (High Order Open Mapping Principle) *Assume that X is a uniformly smooth Banach space and that there exist $p > 0, k > 0, M > 0$ and a compact subset $Q \subset Y$ such that for all $(u,x) \in \text{Graph } G$ near $(\bar{u}, \bar{x})$ and all small $h > 0$*

$$pB \subset \overline{\text{co}} \left(\frac{G(B_h(u)) - x}{h^k} \cap MB + G^{(1)}(u) \cap M \, B \right) + Q$$

If $G^\infty(\bar{u}, \bar{x}) = X$, then for all $h > 0$, $\bar{x} \in \text{Int} \, G(B_h(\bar{u}))$.

Theorem 5.5 (High Order Inverse Function Theorem) *Assume that all the assumptions of Theorem 5.4 are satisfied with $k \geq 1$ and $Q = 0$. Then there exists $L > 0$ such that for all $(u,x) \in \text{Graph } G$ near $(\bar{u}, \bar{x})$ and for all $y \in X$ near $\bar{x}$*

$$\text{dist} \left(u, \, G^{-1}(y) \right) \leq L \sqrt[k]{\|x - y\|}$$

Corollary 5.6 *Assume that X is a finite dimensional space and that for some $k \geq 1$ the convex cone spanned by $G^k(\bar{u}, \bar{x})$ is equal to X. Then the conclusions of Theorem 5.5 are valid.*

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6 Applications

6.1 Taylor coefficients and inverse of a vector-valued function

Consider a function f from a Banach space X to a Hilbert space Y and a point $\bar{x} \in X$. We assume that $f \in C^k$ at $\bar{x}$ for some $k \geq 1$. Then for a neighborhood $\mathcal{N}$ of $\bar{x}$ and for al all $x \in \mathcal{N}$ there exist i-linear forms $A_i(x)$, $i = 1, ..., k$ such that $A_i(\cdot)$ is continuous at $\bar{x}$ and

$$\lim_{h \to 0+} \sup_{\|w\| \leq 1} \left(f(x + hw) - f(x) - \sum_{i=1}^{k} \frac{h^i}{i!} A_i(x)w...w \right) / h^k = 0$$

uniformly in $x \in \mathcal{N}$.

Theorem 6.1 (Second order invertibility condition) *Assume that $f \in C^2$ at $\bar{x}$, that $\text{Im} f'(\bar{x})$ is a closed subspace of Y and for some $\alpha > 0$, $\epsilon > 0$ and all $x \in B_\epsilon(\bar{x})$ the following holds true*

$$(33) \quad \begin{cases} \forall y \in \text{Im } f'(\bar{x})^\perp \text{ of } \|y\| = 1 \ \exists \ w \in X \text{ of } \|w\| \leq 1 \text{ such that} \\ < y, f''(\bar{x})ww > \ \geq \ \alpha, \quad < y, f'(x)w > \ \geq \ \alpha \|f'(x)w\| \end{cases}$$

Then there exists $L > 0$ such that for all x near $\bar{x}$ and for all y near $f(\bar{x})$

$$\text{dist}\,(x,\ f^{-1}(y)) \ \leq \ L\sqrt{\|f(x) - y\|}$$

Proof It is not restrictive to assume that $\alpha \leq 1$. Since $f \in C^2$ at $\bar{x}$, there exists a function $o : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lim_{h \to 0+} o(h^2)/h^2 = 0$ and for all x near $\bar{x}$ and all $w \in X$ of $\|w\| \leq 1$ we have

$$\frac{f'(x)w}{h} + \frac{1}{2} f''(\bar{x})ww \subset \frac{f(B_h(x)) - f(x)}{h^2} + \left(\frac{o(h^2)}{h^2} + \|f''(x) - f''(\bar{x})\| \right) B$$

Hence, by the separation theorem and Theorem 4.1 it is enough to show that for some $M > 0$, $\rho > 0$, $\bar{\epsilon} > 0$ and all $y \in Y$ of $\|y\| = 1$, $x \in B_{\bar{\epsilon}}(\bar{x})$, $h > 0$

$$\sup \left\{ < y, e > \ | \ e \in \left\{ \frac{f'(x)w}{h} + \frac{1}{2} f''(\bar{x})ww \ | \ \|w\| \leq 1 \right\} \cap MB + f'(x)B \right\} \geq \rho$$

Set $H = \text{Im} f'(\bar{x})$, $M = 1 + \|f''(\bar{x})\|$ and let $\gamma > 0$ be such that $\gamma B_H \subset f'(\bar{x})B$. Then for some $0 < \bar{\epsilon} < \epsilon$ and all $x \in B_{\bar{\epsilon}}(\bar{x})$

$$(34) \quad \begin{cases} \forall \ y \in H \quad \text{of} \quad \|y\| = 1, \quad \sup_{\|w\| \leq 1} < y, f'(x)w > \ \geq \frac{\gamma}{2} \\ \forall \ y \in H^\perp \quad \text{of} \quad \|y\| = 1, \quad \sup_{\|w\| \leq 1} < y, f'(x)w > \ \leq \frac{\alpha\gamma}{16M} \end{cases}$$

Set $\rho = \min\{\frac{\alpha\gamma}{16M}, \frac{\alpha}{8}\}$. Fix $x \in B_{\bar{r}}(\bar{x})$, $h > 0$, $y \in Y$ of $\|y\| = 1$ and let $y_1 \in H, y_2 \in H^{\perp}$ be such that $y = y_1 + y_2$. If $\|y_1\| \geq \frac{\alpha}{4M}$ then from (34) we obtain

$$\sup_{\|w\|\leq 1} < y, f'(x)w > \; \geq \sup_{\|w\|\leq 1} < y_1, f'(x)w > - \sup_{\|w\|\leq 1} < y_2, f'(x)w >$$
$$\geq \tfrac{7}{2} \|y_1\| - \tfrac{\alpha\gamma}{16M} \geq \rho$$

If $\|y_1\| \leq \frac{\alpha}{4M}$ then $\|y_2\| \geq 1 - \frac{\alpha}{4M}$ and, by (33), for some $w \in X$ of $\|w\| \leq 1$

$$< y_2, f'(x)w > \; \geq \; \alpha \|y_2\| \, \|f'(x)w\|, \quad < y_2, f''(\bar{x})ww > \; \geq \; \alpha \|y_2\|$$

If $\left\| f'(x)\frac{w}{h} \right\| \leq 1$ then $\left\| f'(x)\frac{w}{h} + \tfrac{1}{2}f''(\bar{x})ww \right\| \leq M$ and

$$< y, f'(x)\tfrac{w}{h} + \tfrac{1}{2}f''(\bar{x})ww > \; \geq \; < y_2, \tfrac{1}{2}f''(\bar{x})ww > - \|y_1\| M$$
$$\geq \tfrac{\alpha}{2} \|y_2\| - \tfrac{\alpha}{4} \geq \tfrac{\alpha}{4} - \tfrac{\alpha^2}{8M} \geq \rho$$

If $\left\| f'(x)\frac{w}{h} \right\| \geq 1$ then setting

$$\bar{w} \; = \; \frac{hw}{\|f'(x)w\|}$$

we obtain $\|\bar{w}\| \leq 1$, $\left\| f'(x)\frac{\bar{w}}{h} + \tfrac{1}{2}f''(\bar{x})\bar{w}\,\bar{w} \right\| \leq M$ and

$$< y, f'(x)\tfrac{\bar{w}}{h} + \tfrac{1}{2}f''(\bar{x})\bar{w}\,\bar{w} > \; \geq \; < y_2, f'(x)\tfrac{\bar{w}}{h} > - \|y_1\| M \geq$$
$$\alpha \|y_2\| - \tfrac{\alpha}{4} \geq \alpha\left(1 - \tfrac{\alpha}{4M}\right) - \tfrac{\alpha}{4} \geq \rho$$

The proof is complete. $\square$

Theorem 6.2 (A high order condition) *Assume that for some $\lambda > 0$ and for all x near $\bar{x}$ and $y \in Y$ of $\|y\| \leq 1$ there exists $\lambda' \geq \lambda$ such that*

$$\lambda' y \in \overline{\text{co}} \left\{ A_i(x)w...w \mid \|w\| \leq 1, \; A_j(x)w...w = 0 \text{ for } 1 \leq j < i \leq k \right\}$$

Then f^{-1} is pseudoholderian on a neighborhood of $(f(\bar{x}),\bar{x})$ with the Holder exponent $\frac{1}{k}$ in the sense of Theorem 4.1.

Proof   Observe that for all $x \in \mathcal{N}$, $w \in X$ of $\|w\| \leq 1$

$$\frac{\sum_{i=1}^{k} h^i A_i(x)w...w}{i!\, h^k} \; \subset \; \frac{f(B_h(x)) - f(x)}{h^k} \; + \; \frac{o(h^k)}{h^k}B$$

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where $\lim_{h \to 0+} \frac{o(h^k)}{h^k} = 0$. Hence, by Theorem 4.1, it is enough to show that for some $M > 0$, $\rho > 0$, $\epsilon > 0$ and all $x \in B_\epsilon(\bar{x})$, all small $h > 0$ and every $y \in Y$ of $\|y\| = 1$

$$\sup \left\{ <y, e> \ \Big| \ e \in \left\{ \sum_{i=1}^{k} \frac{h^i A_i(x)w...w}{i!\, h^k} \ \Big| \ \|w\| \le 1 \right\} \cap MB \right\} \ \ge \ \rho$$

Let $\epsilon > 0$ be such that $\max_{1 \le i \le k} \sup_{x \in B_\epsilon(\bar{x})} \|A_i(x)w...w\| < \infty$ and such that the assumption of theorem is satisfied on $B_\epsilon(\bar{x})$. Fix $y \in Y$ of $\|y\| = 1$ and $x \in B_\epsilon(\bar{x})$, $0 < h \le 1$ and let $w_x \in B$, $1 \le s \le k$ be such that $< y, A_s(x)w_x...w_x > \ge \lambda$ and for all $1 \le j < s$, $A_j(x)w_x...w_x = 0$. Setting $w = h^{\frac{k}{s}-1} w_x$ we obtain $\|w_x\| \le 1$ and

$$\sum_{i=1}^{k} \frac{h^i A_i(x)w_x...w_x}{i!\, h^k} = \frac{1}{s!} A_s(x)w_x...w_x + \sum_{i=s+1}^{k} \frac{h^{\frac{i-s}{s}k}}{i!} A_i(x)w_x...w_x$$

Then

$$< y, \sum_{i=1}^{k} \frac{h^i A_i(x)w_x...w_x}{i!\, h^k} > \ \ge \ \frac{\lambda}{k!} - \left\| \sum_{i=s+1}^{k} \frac{h^{\frac{i-s}{s}k}}{i!} A_i(x)w_x...w_x \right\|$$

Since the right-hand side of the above inequality converges to $\lambda/k!$ when $h \to 0+$ uniformly in $x \in B_\epsilon(\bar{x})$ we end the proof. $\square$

6.2 Stability

Consider a Banach space X, finite dimensional spaces P, Y and continuously differentiable functions $g : P \times X \to Y$, $h : P \times X \to \mathbb{R}^n$. For all $p \in P$ define the set

$$(35) \qquad D_p \ = \ \{ \ x \in X \ | \ g(p,x) = 0, \ h(p,x) \le 0 \ \}$$

Let $(\bar{p}, \bar{x})$ be such that

$$g(\bar{p}, \bar{x}) = 0, \ h(\bar{p}, \bar{x}) \ \le \ 0$$

We study here the map $p \to D_p$ on a neighborhood of $(\bar{p}, \bar{x})$.

Theorem 6.3 (first order condition) *Assume that for some $\bar{w} \in X$*

$$\frac{\partial g}{\partial x}(\bar{p}, \bar{x}) \, \bar{w} \ = \ 0, \qquad \frac{\partial h}{\partial x}(\bar{p}, \bar{x}) \, \bar{w} \ < \ 0$$

Then there exist $\epsilon > 0$, $L > 0$ such that for all $p, p' \in B_\epsilon(\bar{p})$, $x \in D_p \cap B_\epsilon(\bar{x})$

$$\text{dist}\,(x, D_{p'}) \ \le \ L \, \|p - p'\|$$

Remark   The above result is the well known Mangasarian and Fromowitz condition for stability (see [24]). It was also proved in [25] via an inverse mapping theorem involving the inverse of a closed convex process. The proof given below uses the variational inverse function theorem (Corollary 4.2).

Proof   Define the set-valued map $G : P \times X \to P \times Y \times \mathbb{R}^n$ by

$$G(p,x) \;=\; \{\,(\,p,\, g(p,x),\, h(p,x)+\rho\,) \mid \rho \in \mathbb{R}^n_+ \,\}$$

and set $\xi = (\overline{p},\overline{x},\overline{p},g(\overline{p},\overline{x}),h(\overline{p},\overline{x}))$. Then, by Example 1,

$$G^1(\xi) \;\supset$$
$$\{(v, \tfrac{\partial g}{\partial p}(\overline{p},\overline{x})v + \tfrac{\partial g}{\partial x}(\overline{p},\overline{x})w,\; \tfrac{\partial h}{\partial p}(\overline{p},\overline{x})v + \tfrac{\partial h}{\partial x}(\overline{p},\overline{x})w + \rho)\mid \|(v,w)\| \le 1, \rho \in \mathbb{R}^n_+\}$$

Let $\overline{w}$ be as in the assumptions of theorem. Without any loss of generality we may assume that $\|\overline{w}\| \le 1$. Then $(0,0,\tfrac{\partial h}{\partial x}(\overline{p},\overline{x})\overline{w}) \in G^1(\xi)$. Hence for all $\mu \ge 0$ and all $(v,w) \in P \times X$

$$\{(v, \tfrac{\partial g}{\partial p}(\overline{p},\overline{x})v + \tfrac{\partial g}{\partial x}(\overline{p},\overline{x})w,\; \tfrac{\partial h}{\partial p}(\overline{p},\overline{x})v + \tfrac{\partial h}{\partial x}(\overline{p},\overline{x})(w + \mu\overline{w}) + \rho)\mid \rho \in R^n_+\}$$

$$\subset \;\bigcup_{\lambda \ge 0} \lambda co\, G^1(\xi)$$

Fix $(v,y,z) \in P \times Y \times \mathbb{R}^n$. Since $\tfrac{\partial g}{\partial x}(\overline{p},\overline{x})$ is surjective, there exists $w \in X$ such that

$$y - \frac{\partial g}{\partial p}(\overline{p},\overline{x})v \;=\; \frac{\partial g}{\partial x}(\overline{p},\overline{x})w$$

On the other hand, since $\tfrac{\partial h}{\partial x}(\overline{p},\overline{x})\overline{w} < 0$ there exists $\mu > 0$ such that

$$z - \frac{\partial h}{\partial p}(\overline{p},\overline{x})v - \frac{\partial h}{\partial x}(\overline{p},\overline{x})w \;\in\; \mu\frac{\partial h}{\partial x}(\overline{p},\overline{x})\overline{w} \;+\; \mathbb{R}^n_+$$

Therefore $\bigcup_{\lambda \ge 0} \lambda co G^1(\xi) = P \times Y \times \mathbb{R}^n$ and we may apply Corollary 4.2. Thus there exists $L > 0$ such that for all (p,x) near $(\overline{p},\overline{x})$ and all z near $(p,g(p,x),h(p,x))$

$$\text{dist}\,((p,x),G^{-1}(z)) \le L\,\|(p,g(p,x),h(p,x)) - z\|$$

Fix (p,x) sufficiently close to $(\overline{p},\overline{x})$ with $x \in D_p$, p' sufficiently close to $\overline{p}$ and $z = (p',0,h(p,x))$ and let $(p'',x') \in G^{-1}(z)$ be such that $\|(p,x) - (p'',x')\| \le L\|p - p'\|$ Then $z \in G(p'',x') = (p'',g(p'',x'),h(p'',x')) + \mathbb{R}^n_+$. This yields that $p'' = p'$, $g(p',x') = 0$ and $h(p,x) \in h(p',x') + \mathbb{R}^n_+$. Therefore $h(p',x') \le h(p,x) \le 0$. Consequently $x' \in D_{p'}$ and

$$\|x - x'\| \le \|(p,x) - (p',x')\| \le L\|p - p'\|$$

Second order sufficient conditions require a more fine analysis. Theorem 6.1 can be applied to the case when equality constraints only are present. We state next a result for inequality constraints.

Theorem 6.4 (Second order condition) *Assume that $g \equiv 0$, $h \in C^2$ at $(\overline{p}, \overline{x})$ and let H be the largest subspace contained in $\mathrm{Im}\, \frac{\partial h}{\partial x}(\overline{p}, \overline{x}) + \mathbb{R}^n_+$. If there exists $\alpha > 0$ such that for all (p, x) near $(\overline{p}, \overline{x})$*

$$\begin{cases} \forall\, y \in H^\perp \text{ of } \|y\| = 1, \ \exists\, w \in X, \ \|w\| \le 1, \ \rho_i \in \mathbb{R}^n_+, \ i = 1, 2, \ \|\rho_i\| \le 1, \\ < y, \frac{\partial^2 h}{\partial x^2}(\overline{p}, \overline{x})ww + \rho_1 > \ \ge \alpha, \quad < y, \frac{\partial h}{\partial x}(p, x)w + \rho_2 > \ \ge \alpha \left\| \frac{\partial h}{\partial x}(p, x)\overline{w} \right\| \end{cases}$$

then there exist $\epsilon > 0$, $L > 0$ such that for all p, $p' \in B_\epsilon(\overline{p})$, $x \in D_p \cap B_\epsilon(\overline{x})$

$$\mathrm{dist}\,(x, D_{p'}) \ \le \ L\sqrt{\|p - p'\|}$$

Proof   Consider the set-valued map $G : P \times X \to P \times \mathbb{R}^n$ defined by

$$G(p, x) \ = \ \{\, (p, h(p, x) + \rho) \mid \rho \in \mathbb{R}^n_+ \,\}$$

Then for all $e \ge h(p, x)$ and for all (p, x) near $(\overline{p}, \overline{x})$, $t > 0$

$$\left\{ (v, \tfrac{\partial h}{\partial p}(p, x)v + \tfrac{\partial h}{\partial x}(p, x)\tfrac{w}{t} + \tfrac{1}{2}\tfrac{\partial^2 h}{\partial x^2}(\overline{p}, \overline{x})ww + \rho) \mid \|(v, w)\| \le 1, \rho \in \mathbb{R}^n_+ \right\}$$
$$\subset \ \tfrac{G((p,x)+tB)-(p,e)}{t^2} + \left(\tfrac{o(t^2)}{t^2} + \left\| \tfrac{\partial^2 h}{\partial x^2}(\overline{p}, \overline{x}) - \tfrac{\partial^2 h}{\partial x^2}(p, x) \right\| \right) B$$

where $o(t^2)/t^2 \to 0$ when $t \to 0+$ and

$$\left(0, \frac{\partial h}{\partial x}(p, x)B + \mathbb{R}^n_+ \right) \ \subset \ G^{(1)}(p, x, p, e)$$

Let $\delta > 0$ be such that

$$M_1 := \sup_{(p,x) \in B_\delta(\overline{p}, \overline{x})} \left\| \frac{\partial h}{\partial p}(p, x) \right\| \ < \ \infty$$

Fix $(a, q) \in P \times Y$ of $\|(a, q)\| = 1$. If $\|q\| \le \frac{1}{2(M_1+1)}$ then $\|a\| \ge 1 - \frac{1}{2(M_1+1)}$ and

$$\sup_{\|v\| \le 1} < (a, q), (v, \frac{\partial h}{\partial p}(p, x)v) > \ \ge \ \|a\| - \|q\| M_1 \ \ge \ 1 - \frac{1 + M_1}{2(M_1 + 1)} \ = \ \frac{1}{2}$$

By Theorem 5.5 it remains to show the existence of $\gamma > 0$, $M > 0$ such that for all $y \in Y$ of $\|y\| = 1$ and all (p, x) near $(\overline{p}, \overline{x})$ and all small $t > 0$

$$\sup \left\{ < y, e > \mid e \in \left\{ \tfrac{\partial h}{\partial x}(p, x)\tfrac{w}{t} + \tfrac{1}{2}\tfrac{\partial^2 h}{\partial x^2}(\overline{p}, \overline{x})ww + \rho \mid \|w\| \le 1, \rho \in \mathbb{R}^n_+ \right\} \cap MB \right.$$
$$\left. + \left(\tfrac{\partial h}{\partial x}(p, x)B + \mathbb{R}^n_+ \right) \cap MB \right\} \ \ge \ \gamma$$

But this follows from the assumptions by the arguments similar to the proof of Theorem 6.1. $\square$

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6.3 Interior points of reachable sets of a control system

We consider the control system described in Example 2 and we impose the same assumptions on f, X, S. For all $T > 0$ denote by $R(T)$ the reachable set of (5) at time T, i.e.,

$$R(T) = \{x(T) \mid x \text{ is a mild trajectory of (5)}\}$$

Let z be a mild trajectory of (5) on $[0, T]$ and $\bar{u}$ be the corresponding control. We provide here a sufficient condition for $z(T) \in \text{Int } R(T)$ and study how much we have to change controls in order to get in neighboring points of $z(T)$.

Consider the linear control system

$$(36) \qquad \begin{cases} w'(t) = Aw(t) + \frac{\partial f}{\partial x}(z(t), \bar{u}(t))w(t) + v(t) \\ w(0) = 0; \quad v(t) \in \overline{co}\, f(z(t), U) - f(z(t), \bar{u}(t)) \end{cases}$$

and let $R_{\bar{u}}^L(T)$ denote its reachable set by the mild trajectories at time T.

Theorem 6.5 *Under the above assumptions assume that for all $x \in X$ the set $f(x, U)$ is bounded and for all $u \in U$, $t \in [0, T]$, $\frac{\partial f}{\partial x}(\cdot, u)$ is continuous at $z(t)$. If $0 \in \text{Int } \overline{R_{\bar{u}}^L(T)}$ then $z(T) \in \text{Int } R(T)$ and there exist $\epsilon > 0$, $L > 0$ such that for every control $u \in \mathcal{U}$ satisfying $d(u, \bar{u}) \leq \epsilon$ and all $b \in B_\epsilon(z(T))$ there exists a trajectory-control pair (x_v, v) which verifies*

$$x_v(T) = b; \quad \mu(\{t \in [0, T] \mid u(t) \neq v(t)\}) \leq L \, \|b - x_u(T)\|$$

In particular for every $b \in B_\epsilon(z(T))$ there exists a control $u \in \mathcal{U}$ such that

$$x_u(T) = b; \quad \mu(\{t \in [0, T] \mid u(t) \neq \bar{u}(t)\}) \leq L \, \|b - z(T)\|$$

The above result was proved in [18] therefore we only sketch the idea of the proof. From (7) we deduce that for almost all $t \in [0, T]$

$$S_{\bar{u}}(T; t)\, (\overline{co}\, f(z(t), U) - f(z(t), \bar{u}(t))) \subset \overline{co}\, G^{(1)}(\bar{u})$$

and for the same reasons for all u near $\bar{u}$ and for almost all $t \in [0, T]$

$$(37) \qquad S_u(T; t)\, (\overline{co} f(x_u(t), U) - f(x_u(t), u(t))) \subset \overline{co}\, G^{(1)}(u)$$

On the other hand

$$R_u^L(T) = \{ \int_0^T S_u(T; t)v(t)dt : v(t) \in \overline{co} f(x_u(t), U) - f(x_u(t), u(t)) \text{ is measurable}\}$$

and therefore integrating (37) we obtain $R_u^L(T) \subset T\overline{co}G^{(1)}(u)$. Hence

$$\frac{1}{T}\overline{R_u^L(T)} \subset \overline{co}\, G^{(1)}(u)$$

Since $0 \in \mathrm{Int}\overline{R_u^L(T)}$ and $\overline{R_u^L(T)}$ is a closed convex set, the separation theorem and regularity of the data imply that for some $\delta > 0$

$$0 \in \mathrm{Int} \bigcap_{d_T(u,\overline{u})\leq\delta} \overline{R_u^L(T)}$$

This allow to apply Theorem 3.1 (see [18] for details of the proof). $\square$

6.4   Local controllability of a differential inclusion

We consider the dynamical system described by a differential inclusion from the Example 3 and we assume H_1) and H_2). Let $T > 0$ be a given time. We study here sufficient conditions for $\xi \in \mathrm{Int}\, R(T)$ and the regularity of the "inverse". Consider the following linearized inclusion

(38) $\qquad w' \in CF(\xi,0)w + co\, F(\xi); \quad w(0) = 0$

and let $R^L(T)$ denote its reachable set at time T.

Theorem 6.6 *If $0 \in \mathrm{Int}\, R^L(T)$ then $\xi \in \mathrm{Int}\, R(T)$ and there exists a constant $L > 0$ such that for every $x \in S_{[0,T]}(\xi)$ sufficiently close to the constant trajectory ξ (in $W^{1,1}(0,T)$) and for all b near ξ there exists $y \in S_{[0,T]}(\xi)$, satisfying*

$$y(T) = b; \quad \|y - x\|_{W^{1,1}(0,T)} \leq L\|b - x(T)\|$$

In particular this implies that for all b near ξ there exists $x \in S_{[0,T]}(\xi)$ with

$$x(T) = b; \quad \|x - \xi\|_{W^{1,1}(0,T)} \leq L\|b - \xi\|$$

Proof   The map G defined in Example 3 is continuous on its domain of definition. It was proved in [15] that if $0 \in \mathrm{Int}\, R^L(T)$ then there exists a compact convex set $K \subset coF(\xi)$ having only finite number of extremal points such that the reachable set $R_K^L(T)$ at time T of the differential inclusion (9) satisfies $0 \in \mathrm{Int}\, R_K^L(T)$. From the Example 3 we also know that for some $M > 0$

$$\frac{1}{M}\, R_K^L(T) \subset G^1(\xi)$$

Applying Corollary 4.2 we end the proof.

6.5 Small time local controllability of differential inclusions

Consider again dynamical system described in Example 3 and satisfying H_1), H_2). We study here sufficient conditions for

$$\forall T > 0, \quad \xi \in \text{Int } R(T)$$

Set

$$V = \{ v \in X \mid \exists k > 0 \text{ such that } \forall t \geq 0, \xi + t^k v \in R(t) + o(t^k) \}$$

Theorem 6.7 *Under the above assumptions assume that for every* $x \in X$, $F(x)$ *is a convex set. If the convex cone spanned by* V *is equal to* X, *then for all* $T > 0$, $\xi \in \text{Int } R(T)$.

The above result was proved in [17]. It follows from Theorem 5.4, (10) and from the existence of $\epsilon > 0$ such that $\text{Graph} R \cap B_\epsilon(0, \xi)$ is a closed set.

References

[1] AUBIN J.-P. (1982) *Comportement lipschitzien des solutions de problèmes de minimisation convexes*. CRAS, PARIS, 295, 235-238

[2] AUBIN J.-P. (1984) *Lipschitz behavior of solutions to convex minimization problems*. Math. Operations Research, 8, 87-111

[3] AUBIN J.-P. & CELLINA A. (1984) DIFFERENTIAL INCLUSIONS. Springer Verlag

[4] AUBIN J.-P. & EKELAND I. (1984) APPLIED NONLINEAR ANALYSIS. Wiley-Interscience

[5] AUBIN J.-P. & FRANKOWSKA H. (1987) *On the inverse function theorem*. J. Math. Pures & Appliquées, 66, 71-89

[6] AMBROSETTI A. & PRODI G. (1973) *On the inversion of some differential mappings with singularities between Banach spaces*. Ann. Math. Pure Appl., 93, 231-247

[7] BEAUZAMY P. (1985) INTRODUCTION TO BANACH SPACES AND THEIR GEOMETRY. North Holland, Math. Study, 68

[8] CLARKE F. H. (1983) OPTIMIZATION AND NONSMOOTH ANALYSIS. Wiley-Interscience

[9] DMITRUK A.V., MILYUTIN A.A. & OSMOLOVSKII N.P. (1980) *Ljusternik's theorem and the theory of extrema*. Uspekhi Mat. Nauk, 35:6, 11-46 / Russian Math Surveys, 35:6, 11-51

[10] EKELAND I. (1979) *Nonconvex minimization problems*. Bull. Am. Math. Soc., 1, 443-474

[11] FATTORINI H. (1987) *A unified theory of necessary conditions for nonlinear nonconvex systems*. Appl. Math. & Opt., 2, 141-184

[12] FATTORINI H. & FRANKOWSKA H. (1987) *Necessary conditions for infinite dimensional control problems*. (to appear)

[13] FRANKOWSKA H. (1986) *Théorème d'application ouverte pour des correspondances*. CRAS, PARIS 302, 559-562

[14] FRANKOWSKA H. (1987) *An open mapping principle for set-valued maps*. J. Math. Analysis & Appl., 127, 172-180

[15] FRANKOWSKA H. (1987) *Local controllability and infinitesimal generators of semigroups of set-valued maps*. SIAM J. Control & Optimization, 25, 412-432

[16] FRANKOWSKA H. (1987) *Théorèmes d'application ouverte et de fonction inverse*. CRAS, Paris, 305, 773-776

[17] FRANKOWSKA H. (1986) *Local controllability of control systems with feedbacks*. J. Opt. Th. & Appl., (to appear)

[18] FRANKOWSKA H. (1987) *On the linearization of nonlinear control systems and exact reachability*. Proceedings of IFIP Conference on Optimal Control of Systems Governed by Partial Differential Equations, Santiago de Compostela, Spain, July 6-9, 1987, Springer Verlag (to appear)

[19] FRANKOWSKA H. *Some Inverse Mapping and Open Mapping Theorems*. (to appear)

[20] GRAVES L.M. (1950) *Some mapping theorem*. Duke Math. J., 17, 111-114

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[21] IOFFE A. D. (1987) *On the local surjection property.* J. Nonlinear Analysis, 11, 565-592

[22] IOFFE A.D. & TIKHOMIROV V.M. (1974) THEORY OF EXTREMAL PROBLEMS. Nauka, Moskow

[23] LJUSTERNIK L.A. (1934) *Conditional extrema of functionals.* Mat. Sb., 41, 390-401

[24] MANGASARIAN O.L., & FROMOVITZ S. (1967) *The Fritz John necessary optimality conditions in the presence of equality and inequality constraints.* J. Math. Analysis & Appl., 17, 37-47

[25] ROBINSON S. (1976) *Stability theory for systems of inequalities, part II: differentiable nonlinear systems.* SIAM J. Numerical Analysis, 13, 497-513

[26] ROCKAFELLAR R.T. (1986) *Lipschitzian stability in optimization: the role of nonsmooth analysis.* IIASA WP-8646

[27] ROCKAFELLAR R.T. (1987) *Second-order optimality conditions in nonlinear programming obtained by way of pseudoderivatives.* (preprint)