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Abstract

Zero-sum games with incomplete information are formulated as linear programs in which the players' behavioral strategies appear as primal and dual variables. Known properties for these games may then be derived from duality theory.

1. Introduction

It has been known for long that any zero-sum game defined in normal form (i.e. by the payoff matrix) is equivalent to a linear program in which the variables represent the players' mixed strategies (Dantzig [1]). However, many games of interest are usually defined in extensive form (i.e. by the game tree), and then the exponential explosion of the number of pure strategies makes the normal form a pure theoretical tool inadequate for computational purposes. In the extensive form, the number of variables increases only linearly with respect to the number of information sets so that any computational procedure based on this representation is especially attractive.

The objective of this note is to show that a special class of games defined in extensive form, namely zero-sum sequential games with incomplete information (Ponssard-Zamir [2]), may indeed be directly formulated as linear programs in which the variables represent the players' behavioral strategies. Apart from its computational interest, a side product of this formulation is a new proof for the properties of these games.

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2. Recall of the Definition of the Game

The game essentially consists of four steps (a full description may be obtained in [2]).

Step 0 Chance selects a move $k \varepsilon \{1, \cdots, K\}$ according to a probability distribution $p_k^o (\forall k : p_k^o > 0; \sum_{k=1}^{K} p_k^o = 1)$.

Step 1 Player 1 selects a move $i \varepsilon \{1, \cdots, I\}$ knowing k.

Step 2 Player 2 selects a move $j \varepsilon \{1, \cdots, J\}$ knowing i but not k.

Final Step Player 2 pays an amount a_{kj}^i to Player 1. ($\forall k,i,j : a_{kj}^i$ is a real number)

3. The L-P Formulation

3.1 Definitions of the Variables

For all k and i define Player 1's behavioral strategy by

$$x_i^k = \text{Prob (move } i | \text{move } k) ,$$

and his expected security level conditional on move i by u_i. For all i and j define Player 2's behavioral strategy by

$$y_j^i = \text{Prob (move } j | \text{move } i) ,$$

and his expected security level conditional on move k by v_k.

3.2 Player 2's Problem

$$\text{Min} \sum_{k=1}^{K} v_k p_k^o$$

subject to

$$k = 1,\cdots,K;\ i = 1,\cdots,I, \qquad v_k - \sum_{j=1}^{J} a_{kj}^i y_j^i \geq 0$$

$$i = 1,\cdots,I, \qquad \sum_{j=1}^{J} y_j^i = 1$$

$$i = 1,\cdots,I;\ j = 1,\cdots,J, \qquad y_j^i \geq 0 \quad .$$

3.3 Player 1's Problem

$$\text{Max} \sum_{i=1}^{I} u_i$$

subject to

$$j = 1,\cdots,J;\ i = 1,\cdots,I, \qquad u_i - \sum_{k=1}^{K} a_{kj}^i p_k^o x_i^k \geq 0$$

$$k = 1,\cdots,K, \qquad \sum_{i=1}^{I} x_i^k = 1$$

$$k = 1,\cdots,K;\ i = 1,\cdots,I, \qquad x_i^k \geq 0 \quad .$$

3.4 A Comment on the Size of the Problem

Note that the dimensions of the matrix associated with these linear programs are $(K\times I+I) \times (K+I\times J)$ as opposed to $I^K \times J^I$ if we were to "reduce" the game to its normal form.

4. Property of the Value of the Game

Let the variables $(z_k^i)_{i = 1,\cdots,I;\ k = 1,\cdots,K}$ be defined as

$$k = 1,\cdots,K;\ i = 1,\cdots,I, \qquad z_k^i = p_k^o x_i^k \quad .$$

Then it is immediate that by this transformation, Player 1's problem is the dual of Player 2's problem (recall that for all k, $p_k^o > 0$). Hence, denoting optimal values of the variables by a bar, we obtain from the duality theory

$$\sum_{k=1}^{K} \bar{v}_k p_k^o = \sum_{i=1}^{I} \bar{u}_i \quad .$$

Thus the game has a value which is equal to the optimal values of the objective functions.

A property of this value is that it may be obtained from the concave hull of the value of an auxiliary game (see Theorem 1, page 101 in [2]). We shall now show that this property may be derived directly from our L.P. formulation.

Let us make a change of variables in Player 1's problem. Define

$$P = \{p = (p_k)_{k = 1,\cdots,K} \mid p_k \geq 0, \sum_{k=1}^{K} p_k = 1\}$$

and let the new variables

$$\lambda = (\lambda_i)_{i = 1,\cdots,I} \; ; \; (p_k^i)_{i = 1,\cdots,I,\; k = 1,\cdots,K}$$

be such that

$$i = 1,\cdots,I, \qquad \lambda_i = \sum_{k=1}^{K} p_k^o x_i^k$$

$$i = 1,\cdots,I;\; k = 1,\cdots,K, \text{ if } \lambda_i > 0,\; p_k^i = p_k^o x_i^k / \lambda_i \quad ,$$

so that $p^i = (p_k^i)_{k = 1,\cdots,K}$ is a point in P, and if $\lambda_i = 0$, let p^i be arbitrary in P.

For all points p in P and $i = 1,\cdots,I$, let the function $w^i(p)$ be

$$w^i(p) = \min_{j = 1,\cdots,J} \sum_{k=1}^{K} a_{kj}^i p_k$$

so that

$$\lambda_i w^i(p^i) = \min_{j = 1,\cdots,J} \sum_{k=1}^{K} a_{kj}^i p_k^o x_i^k \quad .$$

Then Player 1's problem may be written as the following non-linear program: Find a convex combination $(\lambda_i)_{i = 1,\cdots,I}$ and I points $(p^i)_{i = 1,\cdots,I}$ in P such that

$$\text{Max} \quad \sum_{i=1}^{I} \lambda_i w^i(p^i)$$

subject to

$$k = 1,\cdots,K, \qquad \sum_{i=1}^{I} \lambda_i p_k^i = p_k^o \quad .$$

Let $\bar{w}(p)$ denote the concave hull of the function $w(p)$ defined as $w(p) = \max_{i = 1,\cdots,I} w^i(p)$. $w(p)$ may be interpreted as the value, as a function of p, of the game in which Player 1 moves without knowing k (see step 2 in section 3). The optimal value of this non-linear program, and thus the value of the game, may then be expressed as $\bar{w}(p^o)$.

5. Property of the Optimal Strategies

The complementary slackness conditions associated with the two linear programs give at the optimum

$$i = 1,\cdots,I, \qquad \sum_{k=1}^{K} p_k^o \bar{x}_i^k (\bar{v}_k - \sum_{j=1}^{J} a_{kj}^i \bar{y}_j^i) = 0 \quad ,$$

and

$$\sum_{j=1}^{J} \bar{y}_j^i (\bar{u}_i - \sum_{k=1}^{K} a_{kj}^i p_k^o \bar{x}_i^k) = 0 \quad ,$$

which, combined together, give

$$i = 1,\cdots,I, \quad \bar{u}_i = \sum_{k=1}^{K} \bar{v}_k p_k^o \bar{x}_i^k \quad ,$$

or in terms of the new variables defined in section 4

$$i = 1,\cdots,I, \quad \text{if } \bar{\lambda}_i > 0 \quad ,$$

$$w^i(\bar{p}^i) = \sum_{k=1}^{K} \bar{v}_k \bar{p}_k^i \quad .$$

The interpretation of this equality is as follows. If Player 2 knew Player 1's optimal strategy ($\bar{x}_i^k$), for all moves i which may occur with positive probability ($\bar{\lambda}_i > 0$), he may compute a posterior probability distribution on k ($\bar{p}^i$) and select his strategy so as to minimize Player 1's expectations conditional on move i ($w^i(\bar{p}^i)$). On the other hand, Player 2's security level associated with his optimal strategy and evaluated at $\bar{p}^i$ is $\sum_{k=1}^{K} \bar{v}_k \bar{p}_k^i$. Thus, the equality is the special formulation in the context of this game of the general minimax statement that Player 2 cannot benefit from knowing Player 1's optimal strategy.

6. Example

As an illustration, Player 2's linear program for the example presented in [2] with the specification that $p = \left(\frac{1}{3}, \frac{2}{3}\right)$ is

$$\begin{array}{llllllll}
\text{Min} & \frac{1}{3}v_1 + \frac{2}{3}v_2 & & & & & & \\
\text{s.t.} & v_1 & & -\frac{5}{2}y_1^1 + 2y_2^1 & & & & \geq 0 \\
 & & v_2 & - 1y_1^1 - 10y_2^1 & & & & \geq 0 \\
 & & & y_1^1 + y_2^1 & & & & = 1 \\
 & v_1 & & & - 12y_1^2 & & & \geq 0 \\
 & & v_2 & & & - 4y_2^2 & & \geq 0 \\
 & & & & y_1^2 + y_2^2 & & & = 1
\end{array}$$

$$(i = 1,2;\ j = 1,2;\ y_j^i \geq 0) \quad .$$

References

[1] Dantzig, G.B. "A Proof of the Equivalence of the Programming Problem and the Game Problem," in Activity Analysis of Production and Allocation, Koopmans, T.C., ed., Cowles Commission Monograph 13, Wiley, 1951.

[2] Ponssard, J.-P. and Zamir, S. "Zero Sum Sequential Games with Incomplete Information," Int. J. of Game Theory, 2, No. 2 (1973).