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# **On the Interchange of Subdifferentiation and Conditional Expectation for Convex Functionals**

**Rockafellar, R.T. & Wets, R.J.-B.**

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AND CONDITIONAL EXPECTATION FOR CONVEX  
FUNCTIONALS

R.T. Rockafellar  
R. J-B. Wets

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS  
A-2361 Laxenburg, Austria

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## ABSTRACT

We show that the operators  $E^G$  (conditional expectation given a  $\tau$ -field  $G$ ) and  $\partial$  (subdifferentiation), when applied to a normal convex integrand  $f$ , commute if the effective domain multifunction  $\omega \rightarrow \{x \in \mathbb{R}^n \mid f(\omega, x) < +\infty\}$  is  $G$ -measurable.

ON THE INTERCHANGE OF SUBDIFFERENTIATION  
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We deal with interchange of conditional expectation and subdifferentiation in the context of stochastic convex analysis. The purpose is to give a condition that allows the commuting of these two operators when applied to convex integral functionals.

Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $\mathcal{G}$  a  $\tau$ -field contained in  $\mathcal{A}$ , and  $f$  an  $\mathcal{A}$ -normal convex integrand defined on  $\Omega \times \mathbb{R}^n$  with values in  $\mathbb{R} \cup \{\infty\}$ . The latter means that the map

$$\omega \mapsto \text{epi } f(\omega, \cdot) = \{(x, \alpha) \in \mathbb{R}^{n+1} \mid \alpha \geq f(\omega, x)\}$$

is a closed-convex-valued  $\mathcal{A}$ -measurable multifunction. See [2] and [9] for more on normal integrands and their properties. In particular recall that for any  $\mathcal{A}$ -measurable function  $x: \Omega \rightarrow \mathbb{R}^n$ , the function

$$\omega \mapsto f(\omega, x(\omega))$$

is a  $\mathcal{A}$ -measurable and the integral functional associated with  $f$  is defined by

$$I_f(x) = \int f(\omega, x(\omega)) P(d\omega) \quad .$$

To bypass some trivialities we impose the following summability conditions:

- (1) there exists a  $G$ -measurable  $x: \Omega \rightarrow \mathbb{R}^n$  such that  $I_f(x)$  is finite,
- (2) there exists  $v \in L_n^1(G) = L^1(\Omega, G, P; \mathbb{R}^n)$  such that  $I_{f^*}(v)$  is finite,

where  $f^*$  is the ( $A$ -normal) conjugate convex integrand, i.e.

$$f^*(\omega, x) = \sup_{x \in \mathbb{R}^n} [v \cdot x - f(\omega, x)] \quad .$$

Finally, we assume that  $A$  -- and hence also  $G$  -- is countably generated, and that there exists a *regular* conditional probability (given  $G$ ),  $P^G: A \times \Omega \rightarrow [0, 1]$ . Whenever we refer to the conditional expectation given  $G$ , we always mean the version obtained by integrating with respect to  $P^G$ . Consequently all conditional expectations will be regular.

In particular the conditional expectation  $E^G f$  of  $f$  is the  $G$ -normal integrand defined by

$$(E^G f)(\omega, x) = \int f(\zeta, x) P^G(d\zeta | \omega) \quad .$$

Also given  $\Gamma: \Omega \rightarrow \mathbb{R}^n$ , a closed-valued  $A$ -measurable multifunction, its conditional expectation given  $G$  is a closed-valued  $G$ -measurable multifunction obtained via a projection-type operation from a set

$$L_\Gamma^1 = \{u \in L^1(\Omega, A, P; \mathbb{R}^n) \mid u(\omega) \in \Gamma(\omega) \text{ a.s.}\} \subset L_n^1(A)$$

onto  $L_n^1(G) = L^1(\Omega, G, P; \mathbb{R}^n)$ . Valadier has shown that a regular version  $E^G \Gamma: \Omega \rightarrow \mathbb{R}^n$  is given by the expression

$$E^G \Gamma(\omega) = \text{cl}\left\{ \int u(\zeta) P^G(d\zeta | \omega) \mid u \in L_n^1(A), u(\omega) \in \Gamma(\omega) \text{ a.s.} \right\} \quad .$$

We refer to [12] and the references given therein for the properties of  $E^G f$ ; in particular to the article of Dynkin and Estigineev [3], which specifically deals with regular conditional expectations of measurable multifunctions.

We consider  $I_f$  and  $I_{E^G f}$  as (integral) functionals on  $L_n^\infty(A)$  and  $L_n^\infty(G)$  respectively. The natural pairings of  $L^\infty$  with  $L^1$  and  $(L^\infty)^*$  yield for each functional two different subgradient multifunctions. We shall use  $\partial I_f$  and  $\partial I_{E^G f}$  for designating  $L^1$ -subgradients and  $\partial^* I_f$  and  $\partial^* I_{E^G f}$  for  $(L^\infty)^*$ -subgradients. Rockafellar [8, Corollary 1B] shows that when the summability conditions (1) and (2) are satisfied, one has the following representation for  $(L^\infty)^*$ -subgradients:

$$(3) \quad \partial^* I_f(x) = \{v + v_s \mid v \in \partial I_f(x), v_s \in S_n(A) \text{ with } v_s[x - x'] \geq 0 \ \forall x' \in \text{dom } I_f\}$$

where  $S_n(A)$  is the space of *singular* continuous linear functionals on  $L_n^\infty(A)$ , and

$$\text{dom } I_f = \{x \in L_n^\infty(A) \mid I_f(x) < +\infty\}$$

is the effective domain of  $I_f$ . (For the decomposition of  $(L_n^\infty)^*$  consult [2, Chapter VIII]). Furthermore the  $L^1$ -subgradient set is given by

$$(4) \quad \partial I_f(x) = \{v \in L_n^1(A) \mid v(\omega) \in \partial f(\omega, x(\omega)) \text{ a.s.}\} \quad .$$

The summability conditions (1) and (2) on  $f$  imply similar properties for  $E^G f$ , so the formulas above also apply to  $I_{E^G f}$ . Thus for  $x \in L_n^\infty(G)$  we get

$$(5) \quad \partial^* I_{E^G f}(x) = \{u + u_s \mid u \in \partial I_{E^G f}(x), u_s \in S_n(G)$$

$$\text{with } u_s[x - x'] \geq 0, \forall x' \in \text{dom } I_{E^G f}\}$$

and

$$(6) \quad \partial I_{E^G f}(x) = \{u \in L_n^1(G) \mid u(\omega) \in \partial E^G f(\omega, x(\omega)) \text{ a.s.}\} \quad .$$



We are interested in the relationship between  $\partial I_f$  and  $\partial I_{E^G f}$ . Relying on the formulas just given, Castaing and Valadier [2, Theorem VIII.37] show that if in place of the summability conditions (1) and (2), one makes the stronger assumption:

- (7) there exists  $x^0 \in L_n^\infty(G)$  at which  $I_f$  is finite and norm continuous,

then for every  $x \in L_n^\infty(G)$  one gets:

$$(8) \quad \partial I_{E^G f}(x) = E^G(\partial I_f(x)) + \text{rc}[\partial I_{E^G f}(x)] \quad ,$$

where  $\text{rc}$  denotes the recession (or asymptotic) cone [2,7]. If  $x \in \text{int dom } I_{E^G f}$ ,  $\partial I_{E^G f}(x)$  is weakly compact and then  $\text{rc}[\partial I_{E^G f}(x)] = \{0\}$ , in which case

$$(9) \quad \partial I_{E^G f}(x) = E^G \partial I_f(x) \quad .$$

This was already observed by Bismut [1, Theorem 4]. For the subspace of  $L_n^\infty$  of constant functions, Hiriart-Urruty [4] obtains a similar result for the  $\varepsilon$ -subdifferentials of convex functions.

Here we shall go one step further and provide a condition under which the  $\text{rc}$  term can be dropped from the identity (8) without requiring that  $x \in \text{int dom } I_f$ . Very simple examples show that the  $\text{rc}$  term is sometimes inescapable in (8). For instance, suppose  $G = \{\phi, \Omega\}$  (so  $E^G = E$ ) and consider  $f(\omega, \cdot) = \psi_{(-\infty, \xi(\omega)]}$ , the indicator of the unbounded interval  $(-\infty, \xi(\omega)]$ , where  $\xi$  is a random variable uniformly distributed on  $[0, 1]$ . In this case  $\psi_{(-\infty, 0]} = Ef = E^G f = I_{E^G f}$ , so that  $\partial I_{E^G f}(0) = R_+$  but  $E^G(\partial I_f(0)) = E\{0\} = \{0\}$ . Thus (8) would fail without the  $\text{rc}$  term.

*THEOREM. Suppose  $f$  is an  $A$ -normal convex integrand such that the closure of its effective domain multifunction*

$$(10) \quad \omega \mapsto D(\omega) := \text{cl dom } f(\omega, \cdot) = \text{cl } \{x \in R^n \mid f(\omega, x) < +\infty\}$$

is  $G$ -measurable. Assume that  $I_f(x) < +\infty$  for every  $x \in L_n^\infty(G)$  such that  $x(\omega) \in \text{dom } f(\omega, \cdot)$  a.s., and that there exists  $x^0 \in L_n^\infty(G)$  at which  $I_f$  is finite and norm continuous. Then for every  $x \in L_n^\infty(G)$  one has

$$(11) \quad \partial E^G f(\cdot, x(\cdot)) = E^G \partial f(\cdot, x(\cdot)) \text{ a.s. } ,$$

or in other words, the closed-valued  $G$ -measurable multifunctions

$$\omega \mapsto \partial E^G f(\omega, x(\omega))$$

and

$$\omega \mapsto E^G [\partial f(\cdot, x(\cdot))] (\omega)$$

are almost surely equal.

Proof. From (8) it follows that

$$\partial I_{E^G f}(x) \subset E^G (\partial I_f(x)) \quad .$$

In view of (6) and (4) this holds if and only if

$$\partial E^G f(\cdot, x(\cdot)) \subset E^G \partial f(\cdot, x(\cdot)) \text{ a.s. } .$$

It thus suffices to prove the reverse inclusion. Let us suppose that  $u \in \partial E^G f(\cdot, x(\cdot))$ . For every  $y \in \mathbb{R}^n$ , define

$$g(\omega, y) = f(\omega, y) - u(\omega) \cdot y \quad .$$

This is an  $A$ -normal convex integrand which inherits all the properties assumed for  $f$  in the Theorem (recall that  $u \in L_n^1(G)$ ). Moreover  $0 \in \partial E^G g(\cdot, x(\cdot))$ . We shall show that  $0 \in E^G \partial g(\cdot, x(\cdot))$ , which in turn will imply that  $u \in E^G \partial f(\cdot, x(\cdot))$  and thereby complete the proof of the Theorem.

Since almost surely  $0 \in \partial E^G g(\omega, x(\omega))$ , we know that

$$0 \in \partial I_{E^G g}(x) \subset \partial^* I_{E^G g}(x) \quad . \quad \text{Hence } x \text{ minimizes } I_{E^G g} \text{ on } L_n^\infty(G) \text{. Let}$$

inj denote the natural injection of  $L_n^\infty(G)$  into  $L_n^\infty(A)$  with

$$\omega = \text{inj} [L_n^\infty(G)] \quad .$$

Now note that  $\text{inj } \bar{x} = \bar{x}$  also minimizes  $I_{E_g}$  on  $\omega \subset L_n^\infty(A)$ , or equivalently  $I_g$  on  $\omega$ , since the two integral functionals coincide on  $\omega$  (by the definition of conditional expectation.) Thus

$$0 \in \partial^* (I_g + \psi_\omega)(x) \quad ,$$

where  $\psi_\omega$  is the indicator function of  $\omega$ , or equivalently:

$$0 \in \partial^* I_g(x) + \partial^* \psi_\omega(x) \quad ,$$

since  $g$  is (norm) continuous at some  $x^0 = \text{inj } x^0 \in \omega$ . By (3), this means that there exist  $v \in L_n^1(A)$ ,  $v_s \in S_n(A)$ , such that

$$(12) \quad v(\omega) \in \partial g(\omega, x(\omega)) \text{ a.s.} \quad ,$$

$$(13) \quad v_s[x - x'] \geq 0 \quad \text{for all } x' \in \text{dom } I_g \quad ,$$

and  $-(v + v_s)$  is orthogonal to  $\omega$ , i.e.

$$(14) \quad (v + v_s)[x'] = 0 \quad \text{for all } x' \in \omega \quad .$$

This last relation can also be expressed as

$$(v + v_s)[\text{inj } y] = 0 \quad \text{for all } y \in L_n^\infty(G) \quad ,$$

or still for all  $y \in L_n^\infty(G)$

$$\text{inj}^* (v + v_s)[y] = 0 \quad ,$$

where  $\text{inj}^*: (L_n^\infty(A))^* \rightarrow (L_n^\infty(G))^*$  is the adjoint of  $\text{inj}$ . Thus the continuous linear functional  $\text{inj}^* (v + v_s)$  must be identically 0 on  $L_n^\infty(G)$ , i.e. on  $L_n^\infty(G)$  one has

$$(15) \quad \text{inj}^* v_s = -\text{inj}^* v = -E^G v \quad .$$

The last equality follows from the observation that  $E^G = \text{inj}^*$  when  $\text{inj}^*$  is restricted to  $L_n^1(A)$ , cf. [2, p.265] for example.

We shall complete the proof by showing that the assumptions (12), (13) and (15) imply that

$$(16) \quad (v - E^G v)(\omega) \in \partial g(\omega, x(\omega)) \quad \text{a.s.}$$

This will certainly do, since it trivially yields the sought-for relation

$$0 = E^G(v - E^G v) \in E^G \partial g(\cdot, x(\cdot)) \quad .$$

To obtain (16), it will be sufficient to show that

$$(17) \quad E\{(-E^G v)(\omega) \cdot [x(\omega) - y(\omega)]\} \geq 0$$

for all  $y \in \text{dom } I_g \subset L_n^\infty(A)$ . To see this, recall that the relations (17) and  $v \in \partial I_g(x)$  (cf. (12)) imply that  $v - E^G v \in \partial I_g(x)$ , from which (16) follows via the representation of  $L^1$ -subgradients given by (4). In fact, because the effective domain multifunction, or more precisely its closure  $\omega \mapsto D(\omega)$ , is  $G$ -measurable, it is sufficient to show that (17) holds for every  $y \in \text{dom } I_g^{\cap \omega}$ . Suppose to the contrary that (17) holds for every  $y \in \text{dom } I_g^{\cap \omega}$  -- or equivalently because of the  $\leq$  inequality that (17) holds for every  $y \in \text{cl dom } I_g^{\cap \omega}$  -- but there exists  $\hat{y} \in L_n^1(A)$  such that  $I_g(\hat{y}) < +\infty$  and for which (17) fails, i.e. we have

$$E\{(-E^G v)(\omega) \cdot [x(\omega) - \hat{y}(\omega)]\} < 0 \quad .$$

Because  $-E^G v$  and  $x$  are  $G$ -measurable, this inequality implies that

$$(18) \quad E\{(-E^G v)(\omega) \cdot [x(\omega) - E^G \hat{y}(\omega)]\} < 0 \quad .$$

Moreover, since  $I_g(\hat{y}) < +\infty$ , it follows that almost surely

$$\hat{y}(\omega) \in \text{dom } g(\omega, \cdot) \subset D(\omega) \quad .$$

Taking conditional expectation on both sides, we see that

$$(19) \quad (E^{\hat{G}} \hat{y})(\omega) \in E^G D(\omega) = D(\omega) \quad ,$$

because  $D$  is a closed-valued  $G$ -measurable multifunction. Naturally  $E^{\hat{G}} \hat{y} \in \mathcal{W}$ . Because  $I_g$  is by assumption finite on  $\{z \in L_n^\infty(G) \mid z(\omega) \in \text{dom } g(\omega, \cdot) \text{ a.s.}\}$ , and  $D(\omega) = \text{cl dom } g(\omega, \cdot)$ , it follows from (19) that  $E^{\hat{G}} \hat{y} \in \text{cl dom } I_g$ . Hence (17) cannot hold for every  $y \in \text{dom } I_g \cap \mathcal{W}$  since  $E^{\hat{G}} \hat{y}$  belongs to  $(\text{cl dom } I_g) \cap \mathcal{W}$  and satisfies (18).

There remains only to show that (17) holds for every  $y \in L_n^\infty(G)$  such that  $\text{inj } y = y \in \text{dom } I_g$ . But now from (13) we have that for each such  $y$

$$v_s[x-y] = v_s[\text{inj } x - \text{inj } y] \geq 0 \quad ,$$

or again equivalently: for each  $y \in \text{dom } I_g \cap L_n^\infty(G)$ ,

$$(\text{inj}^* v_s)[x-y] \geq 0 \quad .$$

But this is precisely (17), since we know from (15) that on  $L_n^\infty(G)$ ,  $\text{inj}^* v_s = -E^G v$ .  $\square$

COROLLARY. Suppose  $f$  is a  $A$ -normal convex integrand such that  $F(x) < +\infty$  whenever  $x \in \text{dom } f(\omega, \cdot)$  a.s., where

$$F(x) = E\{f(\omega, x)\} \quad .$$

Suppose moreover that there exists  $x^0 \in \mathbb{R}^n$  at which  $F$  is finite and continuous, and that the multifunction

$$\omega \mapsto D(\omega) = \text{cl dom } f(\omega, \cdot)$$

is almost surely constant. Then for all  $x \in \mathbb{R}^n$ ,

$$(20) \quad E[\partial f(\cdot, x)] = \partial F(x) \quad ,$$

where the expectation of the closed-valued measurable multi-

function  $\Gamma$  is defined by

$$E\Gamma = \text{cl}\left\{\int v(\omega)P(d\omega) \mid v \in L_n^1(A), v(\omega) \in \Gamma(\omega) \text{ a.s.}\right\}.$$

PROOF. Just apply the Theorem with  $G = \{\phi, \Omega\}$ , and identify the class of constant functions -- the  $G$ -measurable functions -- with  $R^n$ .  $\square$

This Corollary was first derived by Ioffe and Tikhomirov [5] and later generalized by Levin [6]. Note that our definition of the expectation of a closed-valued measurable multifunction is at variance with the definition now in vogue for the integral of a measurable multifunction, which does not involve the closure operation. (Otherwise the definition of the integral of a multifunction would be inconsistent with that of its conditional expectation, in particular with respect to  $G = \{\phi, \Omega\}$ , and also when  $\Gamma \rightarrow E\Gamma$  is viewed as an integral on a space of closed sets it could generate an element that it is not an element of that space.)

#### APPLICATION

Consider the *stochastic optimization problem*:

$$(21) \text{ find } \inf E[f(\omega, x_1(\omega), x_2(\omega))] \text{ over all } x_1 \in L_{n_1}^\infty(G), x_2 \in L_{n_2}^\infty(A),$$

where  $A$  and  $G$  are as before, and  $f$  is an  $A$ -normal convex integrand which satisfies the norm-continuity condition:

$$(22) \quad \text{there exists } (x_1^0, x_2^0) \in L_{n_1}^\infty(G) \times L_{n_2}^\infty(A) \\ \text{at which } I_f \text{ is finite and norm continuous.}$$

Suppose also that the effective domain multifunction

$$\omega \rightarrow \text{dom } f(\omega, \cdot, \cdot) = \{(x_1, x_2) \in R^{n_1} \times R^{n_2} \mid f(\omega, x_1, x_2) < +\infty\}$$

is uniformly bounded and that there exists a summable function  $h \in L^1(A)$  such that  $(x_1, x_2) \in \text{dom } f(\omega, \cdot, \cdot)$  implies that

$|f(\omega, x_1, x_2)| \leq h(\omega)$ . Finally suppose that the multifunction

$$\omega \mapsto D_1(\omega) = \text{cl} \{x_1 \in \mathbb{R}^{n_1} \mid \exists x_2 \in \mathbb{R}^{n_2} \text{ such that } f(\omega, x_1, x_2) < +\infty\}$$

is  $G$ -measurable. For a justification and discussion of these assumptions cf. [11, Section 2]. From Theorem 1 of [11], it follows that the problem

$$(23) \quad \text{find } \inf E[g(\omega, x_1(\omega))] \text{ over all } x_1 \in L_{n_1}^\infty(G),$$

where

$$g(\omega, x_1) = E^G \left\{ \inf_{x_2 \in \mathbb{R}^{n_2}} f(\cdot, x_1, x_2) \right\}(\omega),$$

is equivalent to (21) in the sense that if  $(\bar{x}_1, \bar{x}_2)$  solves (21), then  $\bar{x}_1$  solves (23), and similarly any solution  $x_1$  of (23) can be "extended" to a solution  $(x_1, x_2)$  of (21). Both problems also have the same optimal value.

The hypotheses imply that

$$(\omega, x_1) \mapsto \inf_{x_2} f(\omega, x_1, x_2)$$

is an  $A$ -normal convex integrand, since the multifunction  $\omega \mapsto \text{epi}(\inf_{x_2} f(\omega, x_1, x_2))$  is closed-convex-valued and  $A$ -measurable. Its effective domain multifunction, or more precisely

$$\omega \mapsto D_1(\omega) := \text{cl dom } g(\omega, \cdot),$$

is  $G$ -measurable. Combining (11) with the representation for the subgradients of infimal functions [13, VIII.4], we have that for every  $x_1 \in L_{n_1}^\infty(G)$

$$\begin{aligned} \partial g(\cdot, x_1(\cdot)) &= E^G \{v(\omega) \mid (v(\omega), 0) \in_{\text{a.s.}} \partial f(\omega, x_1(\omega), x_2) \\ &\quad \text{for some } x_2 \in \mathbb{R}^{n_2}\}(\cdot), \end{aligned}$$

from which Theorem 2, the main result of [11], follows directly.

REMARK. If the underlying probability measure  $P$  has finite support, then  $(L_n^\infty)^* = L_n^1$ , and (11) and (20) are satisfied without any other restriction.

On the other hand, if  $P$  is nonatomic, and the effective domain multifunction (or its closure) is not  $G$ -measurable, then the identities (11) and (20) do not apply. More precisely, suppose that there exists a subset  $C$  of  $R^n$  such that the  $A$ -measurable set

$$\{\omega \mid \text{dom } f(\omega, \cdot) \cap C \neq \emptyset\}$$

has (strictly) positive mass and is not  $G$ -measurable. Then the term  $\text{rc}[\partial I_{E^{Gf}}(x)]$  can never be dropped from the representation of  $\partial I_{E^{Gf}}$  given by (8), as can be seen from an adaptation of the arguments in Section 4 of [10]. In those cases the inclusion  $E^G \partial f \subset \partial E^G f$  will be strict for at least some  $x \in L_n^\infty(G)$ .



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