

A QUADRATIC APPROXIMATION METHOD BASED ON AUGMENTED  
LAGRANGIAN FUNCTIONS FOR NONCONVEX  
NONLINEAR PROGRAMMING PROBLEMS

Andrzej P. Wierzbicki

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## SUMMARY

The paper describes an algorithm for solving nonlinear programming problems of fairly general type, in particular nonconvex problems that can be convexified via augmented Lagrangian functions. The algorithm consists of two phases. In the first phase, a point in a rather large neighborhood of an optimal solution is crudely but effectively estimated by a shifted-increased penalty function algorithm. In the second phase, a saddle point of an augmented Lagrangian function and thus an optimal solution together with corresponding Lagrange multipliers are found by a rapidly convergent method of successive quadratic approximations. The switch between these two phases is adaptive. According to the existing experience in nonlinear programming algorithms, the proposed algorithm combines the best local effectiveness of a quadratic approximation method with the global robustness of a penalty method; due to the convexifying properties of augmented Lagrangian functions, the algorithm can solve nonconvex problems which satisfy second-order sufficient conditions of optimality at the solution.

For the sake of a clear presentation, a short review of some basic facts in the theory of Lagrangian functions, quadratic approximations, penalty functions and augmented Lagrangian functions is given in the first part of the paper. Then the quadratic approximations for augmented Lagrangeans are discussed in detail, in particular, in the case when these functions are not twice differentiable which corresponds to the lack of strict complementarity at the optimal solution. The double-phase algorithm is presented and commented upon. The proofs of convergence of the algorithm are given.

## 1. INTRODUCTION

In the years 1970-78, a considerable effort was made to compare various computational approaches to constrained nonlinear programming problems and to choose the most effective and reliable algorithms. It is now clear that a single algorithm of a general type cannot be effective for all cases of nonlinear programming problems. However, it is possible to look for the most reliable and effective algorithms for certain classes of problems. The following classes of nonlinear programming problems can be distinguished for this purpose :

A. Problems with a rather low number of variables and constraints (up to several hundred), with objective and constraining functions of general but smooth type, where basic difficulties are related to a strongly nonlinear and possibly nonconvex character of the functions. Requirements of a fairly high accuracy of the approximation of a solution are typical for such problems.

B. Problems with a rather high number of variables and constraints (often several thousand), but with special structural properties of the objective and constraining functions. There are several types of such problems, for example, convex problems with linear constraints; discrete-time dynamic optimization problems; decomposable nonlinear programming problems, etc. Both low and high accuracy requirements can be met in practical examples of such problems.

C. Problems with special difficulties inherent to problem formulation, for example: large problems without distinctive structure; nondifferentiable programming problems; multiobjective optimization problems; stochastic optimization problems, etc. Typically (but with notable exceptions) only low accuracy requirements can be satisfied when solving such problems.

Although basic theoretical notions and some fundamental algorithms are applicable to many of the classes and types of nonlinear programming problems mentioned above, truly effective and reliable algorithms must be chosen separately for each class or type. In this paper, only the class A shall be considered.

An extensive study of the effectiveness and reliability of many algorithms for solving problems in the class A [23] has shown that the most efficient are methods based on successive quadratic approximations to the Lagrange function and by using a subroutine of quadratic programming. Given a fairly good initial guess as to the solution and provided the problem is locally convex, quadratic approximation methods are much faster than any other methods, including various multiplier techniques, penalty and proximal point techniques of many types, gradient projection methods, etc. This practical observation has been independently confirmed by private communications from many other sources. Theoretically, the convergence of a successive quadratic approximation method to a saddle-point of the Lagrangian function can be estimated as superlinear or quadratic, depending on the appropriate assumptions [9,22]. However, the same can be proved under various assumptions for most of the other methods of nonlinear programming. It was shown in [26,27] that many penalty techniques, multiplier techniques, some gradient projection techniques, etc., are in fact special cases of a general quasi-Newton algorithm for finding a saddle-point of an augmented Lagrangian function and, as such, also possess superlinear or quadratic convergence. Therefore, the higher efficiency of a quadratic approximation method is just an empirical observation, not a theoretical result.

But a quadratic approximation method is not very robust and reliable. It may fail to find a solution if the initial guess to a complicated problem is poor, or if the Lagrangian function is locally nonconvex [23]. These disadvantages will be removed in the algorithm proposed in this paper by using an augmented Lagrangian function instead of the normal one and by adapting another fairly robust algorithm for finding a solution to an initial approximation.

Double-phase algorithms, consisting of an initial robust phase and a final fast convergent phase, have already been proposed - e.g., in [15]; but it is not quite evident what type of an algorithm should be used for the first phase. For nonconvex problems with the possibility of many local solutions, the use of a stochastic

algorithm is strongly recommended, e.g. in [4]. However, there is too little empirical data on the comparison of stochastic and deterministic algorithms. On the other hand, the results of the study [23] show that one of the most robust and fairly efficient requirements for low accuracy is a shifted-increased external penalty algorithm. Such an algorithm, originally proposed in [17] for equality constraints and further developed in [24] for inequality constraints is actually one of the first and simplest of a large family of augmented Lagrange multiplier techniques - see [2,3]. Therefore, the choice of the algorithm for the first phase is based not only on empirical results, but also on a common theoretical denominator: an augmented Lagrangian function.

## 2. PRELIMINARIES

### 2.1. Elements: normal Lagrangian function.

Consider the following problem:

$$(1) \quad \underset{x \in X_0}{\text{minimize}} \ f(x) ; \ X_0 = \{x \in \mathbb{R}^n : g(x) \leq 0 \in \mathbb{R}^m\}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  are assumed to be twice continuously differentiable but not necessarily convex. Additional constraints of the type  $x_{\min} \leq x \leq x_{\max}$  and equality constraints can also be easily incorporated into the problem (1), but are omitted here for the sake of clear presentation.

The (normal) Lagrangian function for the problem (1) is:

$$(2) \quad L(y, x) = f(x) + \langle y, g(x) \rangle = f(x) + \sum_{i \in I} y_i g_i(x)$$

where  $I = \{1, \dots, m\}$ ,  $\langle \cdot, \cdot \rangle$  denotes the scalar product and  $y \in \mathbb{R}_+^m$  is a vector of Lagrangian multipliers. The problem (1) is called normal (or regular) if a regularity condition is satisfied - for example in the form of the Slater postulate: let there exist a point  $x_1 \in \mathbb{R}^n$  such that  $g_i(x_1) < 0, i \in I$ . If the problem is normal and the functions  $f$  and  $g$  are convex, then the necessary and sufficient condition

for optimality of a solution  $\hat{x}$  is that there exists a vector of Lagrangian multipliers  $\hat{y} \in R_+^m$  such that  $L(y, x)$  has at  $(\hat{y}, \hat{x})$  its global saddle-point:

$$(3) \quad L(y, \hat{x}) \leq_{y \in R_+^m} L(\hat{y}, \hat{x}) \leq_{x \in R^n} L(\hat{y}, x)$$

Observe that the saddle-point is constrained to positive multipliers  $y \geq 0$ . The relation

$$\min_{x \in R^n} \max_{y \in R_+^m} L(y, x) = L(\hat{y}, \hat{x}) = f(\hat{x})$$

can be guaranteed for convex problems only. The difference

$$\min_{x \in R^n} \max_{y \in R_+^m} L(y, x) - \min_{y \in R_+^m} \max_{x \in R^n} L(y, x) \geq 0$$

is called the duality gap for the nonconvex problem.

Corresponding to the Slater postulate for convex problems are other regularity conditions for differentiable problems, e.g. the Fiacco-McCormick full rank postulate: let the gradient vectors  $g_{ix}(\hat{x})$  be linearly independent for all  $i$  such that  $g_i(\hat{x}) = 0$ . The necessary conditions for optimality of a solution  $\hat{x}$  in a differentiable problem are that there exists a Lagrange multiplier vector  $\hat{y}$  such that:

$$(4) \quad L_x(\hat{y}, \hat{x}) = f_x(\hat{x}) + \hat{y} g_x(\hat{x}) = f_x(\hat{x}) + \sum_{i \in I} \hat{y}_i g_{ix}(\hat{x}) = 0 \in R^n$$

and

$$(5) \quad g(\hat{x}) \leq 0 \in R^m; \langle \hat{y}, g(\hat{x}) \rangle = 0; \hat{y} \geq 0 \in R^m$$

where gradient vectors are represented as row-vectors,  $g_x(\hat{x})$  is the Jacobian matrix of  $g$  at  $\hat{x}$ . The triple condition (5) is called Kuhn-Tucker condition. This condition can also be derived for convex problems from the saddle-point condition (3). If the full rank postulate holds at  $\hat{x}$ , then the Lagrange multiplier vector  $\hat{y}$

not only exists, but is also uniquely determined by (4,5) - see, e.g., [5].

## 2.2. Second order sufficient and necessary conditions for Optimality.

The following three sets of indexes play an important role in the further analysis:

$$\begin{aligned} (6) \quad \hat{SA} = SA(y, x) &= \{i \in I : g_i(\hat{x}) = 0, \hat{y}_i > 0\} \\ \hat{WA} = WA(\hat{y}, \hat{x}) &= \{i \in I : g_i(\hat{x}) = 0, \hat{y}_i = 0\} \\ \hat{IN} = IN(\hat{y}, \hat{x}) &= \{i \in I : g_i(\hat{x}) < 0, \hat{y}_i = 0\} \end{aligned}$$

The set  $\hat{IN}$  (at the optimal point) is called the set of inactive constraints. The set  $\hat{SA}$ , with the number of elements  $\tilde{m}$ , is the set of strongly active constraints, and the set  $\hat{WA}$ , with the number of elements  $\bar{m}$ , is the set of weakly active ones. The weakly active constraints are of a peculiar type: they are active, but can be removed (for normal problems) without influencing the solution. Worse, if the weakly active constraints are perturbed, that is, changed to the form  $g_i(x) \leq \delta_i$  where  $\delta_i$  is a small number, then they become either strongly active or inactive depending on the sign of  $\delta_i$ .

The triple condition (5) is also called a complementarity relation. If  $\hat{WA} = \emptyset$ , that is there are no weakly active constraints, then it is said that strict complementarity holds.

The conditions (4), (5) are only necessary for optimality. To become sufficient, they must be supplemented by a second-order condition. Denote by:

$$(7) \quad L_{xx}(\hat{y}, \hat{x}) = f_{xx}(\hat{x}) + \sum_{i \in I} \hat{y}_i g_{ixx}(\hat{x}) = f_{xx}(\hat{x}) + \sum_{i \in \hat{SA}} \hat{y}_i g_{ixx}(\hat{x})$$

the Hessian matrix of the Lagrangian function. Denote by

$$(8) \quad \tilde{g}_x(\hat{x}) = (g_{ix}(\hat{x}))_{i \in \hat{SA}}$$

the Jacobian matrix for strongly active constraints. Then the second-

order sufficient condition for optimality of  $\hat{x}$  is the existence of  $\hat{y}$  satisfying (4), (5), and such that:

$$(9) \quad \langle \bar{x}, L_{xx}(\hat{y}, \hat{x}) \bar{x} \rangle > 0$$

for all  $\bar{x} \neq 0$  such that

$$\tilde{g}_x(\hat{x}) \bar{x} = 0 \in \mathbb{R}^{\tilde{m}}$$

In other words, the Hessian matrix  $L_{xx}(\hat{y}, \hat{x})$  should be positive definite in the subspace  $\tilde{T} = \{\bar{x} \in \mathbb{R}^n : \tilde{g}_x(\hat{x}) \bar{x} = 0 \in \mathbb{R}^{\tilde{m}}\}$  of directions tangent to strongly active constraints. This is sufficient for the minimum of  $L(\hat{y}, \cdot)$  in the linear manifold  $\hat{x} + \tilde{T}$  and for the local optimality of  $\hat{x}$  - see, e.g. [12]. But  $L(\hat{y}, \cdot)$  need not have a minimum in directions orthogonal to  $\tilde{T}$ , that is, spanned by the vectors  $g_{ix}(\hat{x})$  for  $i \in \hat{S}A$ . On the other hand, it is possible to construct a matrix  $\tilde{g}_x^*(\hat{x}) \tilde{g}_x(\hat{x})$ , where a star denotes transposition, which is positive definite in directions orthogonal to  $\tilde{T}$  and nilpotent for  $\bar{x} \in \tilde{T}$ . Therefore, (see, e.g. [10]) there exists a number  $\delta_0 > 0$  such that, for all  $\delta \geq \delta_0$ , the sufficient condition (9) can be equivalently written in the form:

$$(10) \quad \langle \bar{x}, (L_{xx}(\hat{y}, \hat{x}) + \delta \tilde{g}_x^*(\hat{x}) \tilde{g}_x(\hat{x})) \bar{x} \rangle > 0 \text{ for all } \bar{x} \neq 0.$$

It is clear that if the positive definiteness of a matrix is sufficient for optimality, the positive semi-definiteness of this matrix should be necessary. But if there are some weakly active constraints,  $\hat{W}A \neq \emptyset$ , then the Hessian matrix  $L_{xx}(\hat{y}, \hat{x})$  need not be necessarily positive semi-definite in the entire subspace  $\tilde{T}$ ; it must be positive semi-definite only in a smaller subspace  $T = \tilde{T} \cap \bar{T}$  of directions tangent to all active constraints, where

$$\bar{T} = \{\bar{x} \in \mathbb{R}^n : \bar{g}_x(\hat{x}) \bar{x} = 0 \in \mathbb{R}^{\bar{m}}\} = \{\bar{x} \in \mathbb{R}^n : g_{ix}(\hat{x}) \bar{x} = 0 \text{ for all } i \in \hat{W}A\}$$

The second-order necessary conditions for optimality are thus



weaker than the sufficient ones and require, besides (4), (5), that

$$(11) \quad \langle \bar{x}, L_{xx}(\hat{y}, \hat{x}) \bar{x} \rangle \geq 0$$

for all  $\bar{x}$  such that

$$\tilde{g}_x(\hat{x}) \bar{x} = 0 \in R^{\tilde{m}}, \quad \bar{g}_x(\hat{x}) \bar{x} = 0 \in R^{\bar{m}}$$

where  $\bar{g}_x(\hat{x})$  is the Jacobian matrix for weakly active constraints (see, e.g., [12]). An equivalent statement to (11) is that there exists a  $\delta > 0$  such that

$$(12) \quad \langle \bar{x}, (L_{xx}(\hat{y}, \hat{x}) + \delta \tilde{g}^*(\hat{x}) \tilde{g}(\hat{x}) + \delta \bar{g}^*(\hat{x}) \bar{g}(\hat{x})) \bar{x} \rangle \geq 0$$

for all  $\bar{x} \in R^n$

The conditions (10), (12) have a straightforward interpretation in terms of the discontinuous second derivatives of an augmented Lagrangian function - see section 3.

### 2.3. Quadratic approximation methods.

Let the normal Lagrangian function be strongly convex, that is, let

$$\langle \bar{x}, L_{xx}(y, x) \bar{x} \rangle > 0 \quad \text{for all } \bar{x} \neq 0 \quad \text{and all } y \text{ and } x$$

at least in a neighbourhood of  $(\hat{y}, \hat{x})$ . Consider the second-order approximation of the Lagrangian function at a point  $(y+\bar{y}, x+\bar{x})$ :

$$\begin{aligned} (13) \quad L(y+\bar{y}, x+\bar{x}) &\cong L + L_x \bar{x} + L_y \bar{y} + \frac{1}{2} \langle \bar{x}, L_{xx} \bar{x} \rangle + \langle \bar{y}, L_{yx} \bar{x} \rangle = \\ &= f + f_x \bar{x} + \frac{1}{2} \langle \bar{x}, L_{xx} \bar{x} \rangle + \langle y+\bar{y}, g+g_x \bar{x} \rangle \end{aligned}$$

where, to simplify the notation, all values of functions  $L, f, g$  and their derivatives are supposed to be evaluated at  $(y, x)$ . Since this

is a convex function of  $\bar{x}$  and a linear one of  $\bar{y}$ , it is possible to determine a saddle-point  $(\hat{\bar{y}}, \hat{\bar{x}})$  of this function in order to approximate  $\hat{x}$  by  $x + \hat{\bar{x}}$  and  $\hat{y}$  by  $y + \hat{\bar{y}}$ . But a saddle-point is both necessary and sufficient for the optimality in an equivalent constrained optimization problem. By inspection of (13), the equivalent problem is the following quadratic programming problem:

$$(14) \underset{\bar{x} \in \bar{X}_g}{\text{minimize}} (f + f_x \bar{x} + \frac{1}{2} \langle \bar{x}, L_{xx} \bar{x} \rangle);$$

$$\bar{X}_g = \{\bar{x} \in \mathbb{R}^n : g + g_x \bar{x} \leq 0 \in \mathbb{R}^m\}$$

here  $f$  is a given constant and  $g$  a given vector in  $\mathbb{R}^m$ . The vector of Lagrangian multipliers associated with the problem is actually  $y + \bar{y}$ , not  $\bar{y}$  alone. Suppose  $f, g, L$  and their derivatives are evaluated at  $(y^k, x^k)$ , where the upper index  $k$  denotes an iteration number and suppose the problem (14) is solved to obtain  $\hat{\bar{x}}^k, \hat{\bar{y}}^k$ . Then the following iteration:

$$(15) x^{k+1} = x^k + \hat{\bar{x}}^k; y^{k+1} = y^k + \hat{\bar{y}}^k$$

converges (quadratically, if second-order derivatives  $f_{xx}, g_{ixx}$  are Lipschitz-continuous - see. e.g., [22]) to  $(\hat{y}, \hat{x})$ . Otherwise, proportionally smaller changes  $x^{k+1} = x^k + \tau^k \hat{\bar{x}}^k, y^{k+1} = y^k + \tau^k \hat{\bar{y}}^k$  with suitably chosen  $\tau^k$  provide for the convergence of the method.

The main drawback of the scheme outlined above is the necessity of programming the second-order derivatives contained in the matrix  $L_{xx}$ , although computing these derivatives and inverting the matrix  $L_{xx}$ , inherent in many quadratic programming algorithms, are lesser drawbacks when taking into account the capabilities of modern computers and comparing the necessary programming effort. Therefore, it is better to approximate the matrix  $L_{xx}$  or its inverse; usually variable metric methods are adapted for this purpose. Variable metric  $H^k \approx L_{xx}$  or  $V^k \approx L_{xx}^{-1}$  is constructed with the help of data  $\{dL_x^i, dx^i\}_{i=1}^k$  where  $dx^i = x^{i+1} - x^i$  and  $dL_x^i = L_x(y^i, x^{i+1}) - L_x(y^i, x^i)$  are finite differences of the independent variable and of the corresponding gradients, which are supposed to fulfill approximately  $dL_x^i \approx L_{xx} dx^i$ . Clearly, an approximation of  $L_{xx}$  requires at least  $n$  linearly independent steps  $dx^i$ , but variable metrics  $H^k$  or  $V^k$  are

updated iteratively after each step. Various algorithms exist for updating the approximations. Many of them require a special selection of the steps  $dx^i$  as minimizing steps in conjugate directions; many of them do not guarantee the convergence of  $H^k$  to  $L_{xx}(\hat{y}, \hat{x})$ . Since the special selection of steps  $dx^i$  cannot be guaranteed in a successive quadratic approximation method and the convergence of  $H^k$  is useful in providing a fast convergence of the method, it is better to use a variable metric scheme without these drawbacks - as, for example, a modified rank-one variable metric [11].

When using a variable metric  $H^k$  instead of  $L_{xx}$ , the approximate quadratic programming problem becomes:

$$(16) \text{ minimize } (f^k + f_{\bar{x}}^{k-} + \frac{1}{2} \langle \bar{x}, H^{k-} \bar{x} \rangle) ;$$

$$\bar{x}_g^k = \{\bar{x} \in R^n : g^k + g_x^k \bar{x} \leq 0 \in R^m\}$$

It can be proved [9,22] that the convergence of the resulting sequence  $\{y^k, x^k\}$  (with  $y^{k+1}, x^{k+1}$  defined as in (15)) to  $(\hat{y}, \hat{x})$  is linear when  $\|(H^k - L_{xx}^k) \hat{x}^k\| \leq \epsilon d^k$  with sufficiently small  $\epsilon$  and  $d^k = \|(L_x^k, \langle y^k, g^k \rangle, g_+^k)\|$ , where  $g_+^k = \max(0, g_1(x^k))$ ,  $d^k$  being a norm of the violation of the necessary conditions (4,5) at  $(y^k, x^k)$ .

Since it can also be proved that  $\|\hat{x}^k\| \leq \alpha d^k$  with some  $\alpha > 0$ , it is sufficient for the linear convergence of the method that

$\|H^k - L_{xx}^k\| < \epsilon_1$  with  $\epsilon_1 = \frac{\epsilon}{\alpha}$  being sufficiently small; but the condition

$\|(H^k - L_{xx}^k) \hat{x}^k\| \leq \epsilon d^k$  is satisfied by a wider class of variable metric approximations  $H^k$  of  $L_{xx}^k$  than only slightly stronger condition  $\|H^k - L_{xx}^k\| \leq \epsilon_1$ . The convergence of the method is superlinear when

$\|(H^k - L_{xx}^k) \bar{x}^k\| \leq \epsilon^k d^k$  with  $\epsilon^k$  converging to zero and quadratic at each  $n$ -th iteration if  $\epsilon^k$  converges to zero as fast as  $d^{k-n}$ . Observe that quadratic convergence, which would be implied if  $\epsilon^k$  would converge to zero as fast as  $d^k$ , is impossible if  $H^k$  is estimated with the help of data from earlier iterations, starting at least at  $(k-n)$ th iteration.

The convergence results have been obtained in [9] under the assumption of strict complementarity, by a standard reasoning based on an inverse function theorem. The convexity assumption that  $L_{xx}^k$  and  $H^k$  are positive definite was omitted in [9] by showing that the necessary condition of optimality of  $(\hat{y}, \hat{x})$  for the problem (16) have a solution close to zero if  $H^k$  is close to  $L_{xx}^k$ ,  $(y^k, x^k)$  close to  $(\hat{y}, \hat{x})$  and the second-order sufficient condition (9) is satisfied at  $(\hat{y}, \hat{x})$ . But finding the least-norm solution of the necessary conditions of optimality for the problem (16) instead of a solution to the problem itself implies some way of convexifying the problem, which was not specified in [9]. The assumption of strict complementarity was relaxed in [22] by omitting the use of an inverse function theorem; however, the convergence results in [2] are obtained under local convexity assumptions. One of the goals of this paper is to show how to obtain similar convergence results with neither strict complementarity nor convexity assumptions, by using convexification through an augmented Lagrangian function.

The convergence properties of a successive quadratic approximation method of the type (16) can be summarized as follows: the better  $H^k$  approximates  $L_{xx}$ , the faster the convergence of the method. Therefore, the initial phase algorithm must be constructed to provide not only for a starting point  $(y^0, x^0)$  sufficiently close to  $(\hat{y}, \hat{x})$  such that the successive quadratic approximation method works, but also for a starting estimate  $H^0$  of  $L_{xx}$  such that the method works fast.

#### 2.4. Shifted penalty function

If the set of admissible solutions  $X_0$  for the original problem (1) has a possibly empty interior, for example, if some equality constraints are admitted, then only external penalty functions can be used [5]. External penalty functions express additional payments for violating constraints; the objective function  $f$ , when supplemented by such penalty terms, usually has minimal points outside of the set  $X_0$ . In order to bring these points close to the set  $X_0$ , two general methods can be used. One of them is just to increase penalties via appropriate penalty coefficients. The other is to start paying penalties before the constraints are actually

violated, that is, to consider a perturbed problem:

$$(17) \quad \underset{x \in X_w}{\text{minimize}} \quad f(x) ; \quad X_{-w} = \{x \in \mathbb{R}^n : g(x) \leq -w \in \mathbb{R}^m\}$$

where  $w \geq 0 \in \mathbb{R}^m$  is a perturbation parameter, called penalty shift. The external semi-quadratic penalty function associated with the problem (17) is called shifted penalty function.

$$(18) \quad \Psi(w, \rho, x) = f(x) + \frac{1}{2}\rho \| (g(x)+w)_+ \|^2 = \\ = f(x) + \frac{1}{2}\rho \sum_{i \in I} (\max(0, g_i(x)+w_i))^2$$

where  $(g(x)+w)_+$  is the positive part of the vector  $g(x)+w$  and is composed of the elements  $(g_i(x)+w_i)_+ = \max(0, g_i(x)+w_i)$ ;  $\rho > 0$  is a penalty coefficient. A positive definite matrix  $R$  of penalty coefficients could also be used to define a penalty term  $\frac{1}{2} \langle (g(x)+w)_+, R(g(x)+w)_+ \rangle$ , but the best that can be said about the matrix  $R$  is that it should scale down constraining functions  $g_i(x)$  to a common range of values (or derivatives) - see, e.g. [12]. The question of an advantageous scaling of nonlinear programming problems is more general and has extreme practical importance but it will not be analysed in this paper. If the problem is reasonably scaled, a single penalty coefficient  $\rho$  is sufficient.

Increased penalty methods are obtained by assuming  $w=0$  and minimizing  $\Psi(0, \rho, x)$  for a sequence  $\{\rho^k\}_0^\infty$ ,  $\rho^k \rightarrow \infty$ . Rather weak assumptions suffice for convergence of such a method. If a set  $X_{\alpha p} = \{x \in \mathbb{R}^n : f(x) \leq \alpha, g(x) \leq p\}$  is bounded (hence compact for continuous  $f, g$ ) for some  $\alpha \in \mathbb{R}^1$ , e.g.  $\alpha = f(\hat{x})$ , and for some  $p \in \mathbb{R}^m$ ,  $p_i > 0$ , and if the function  $f$  is bounded from below, then the function  $\Psi(0, \rho^k, x)$  has minimal points to  $x$  in the set  $X_{\alpha p}$  for sufficiently large  $\rho^k$ ; when  $\rho^k \rightarrow \infty$ , the corresponding minimal points  $\hat{x}_\rho^k$  form a bounded sequence with accumulation points at the solutions of the original problem (1) - see [5], [12], [21]. If the solution  $\hat{x}$  of the problem (1) is unique, the sequence  $\{\hat{x}_\rho^k\}_0^\infty$  converges to this solution. If the solution  $\hat{x}$  is only locally isolated - e.g., if the second-order sufficient conditions for optimality (9) are satisfied at  $\hat{x}$  - then the increased penalty method

can be forced to converge to this solution by choosing appropriate local minima of  $\Psi(0, \rho^k, x)$ , see [5].

But the main disadvantage of the increased penalty method is that it becomes ill-conditioned as  $\rho^k$  increases. Consider the Hessian matrix for the function  $\Psi(0, \rho^k, x)$ :

$$\begin{aligned} (19) \quad \Psi_{xx}(0, \rho^k, \hat{x}^k) &= f_{xx}(\hat{x}^k) + g_{xx}(\hat{x}_\rho^k) \rho^k (g(\hat{x}_\rho^k))_+ + \rho^k \tilde{g}_x^*(\hat{x}_\rho^k) \tilde{g}_x(\hat{x}_\rho^k) = \\ &= L_{xx}(\rho^k (g(\hat{x}_\rho^k))_+, \hat{x}_\rho^k) + \rho^k \tilde{g}_x^*(\hat{x}_\rho^k) \tilde{g}_x(\hat{x}_\rho^k) \end{aligned}$$

where  $\tilde{g}_x(\hat{x}_\rho^k)$  is the Jacobian matrix composed of  $g_{ix}(\hat{x}_\rho^k)$  such that  $g_i(\hat{x}_\rho^k) > 0$  (if some  $g_i(\hat{x}_\rho^k) = 0$ , then the second derivative (19) is discontinuous at  $\hat{x}_\rho^k$ ). For sufficiently large  $\rho^k$ , the first term  $L_{xx}(\rho^k (g(\hat{x}_\rho^k))_+, \hat{x}_\rho^k)$  is close to  $L_{xx}(\hat{y}, \hat{x})$ ; but the second term  $\rho^k \tilde{g}_x^*(\hat{x}_\rho^k) \tilde{g}_x(\hat{x}_\rho^k)$  increases with  $\rho^k$  and the spectrum of  $\Psi_{xx}(0, \rho^k, \hat{x}_\rho^k)$  spreads widely, thus making the problem of minimizing  $\Psi(0, \rho^k, x)$  ill-conditioned and difficult for numerical computations - see, e.g. [12].

To overcome this difficulty, penalty shifts  $w$  can be used. Basic properties of the shifted penalty function (18) are summarized by a generalization of Everett's theorem - see, e.g. [25]; if the function (18) has an unconstrained minimum point  $\hat{x}_{\rho w}$ , then this point is actually a solution to the following perturbed constrained problem:

$$(20) \quad \underset{x \in X_p}{\text{minimize}} \quad f(x) ; \quad X_p = \{x \in \mathbb{R}^n : g(x) \leq p = (g(\hat{x}_{\rho w})_+ - w \in \mathbb{R}^m)\}$$

and determines an associated vector of Lagrangian multipliers  $\hat{y}_{\rho w}$ .

$$\begin{aligned} (21) \quad \Psi_x(w, \rho, \hat{x}_{\rho w}) &= f_{\hat{x}}(x_{\rho w}) + \rho (g(\hat{x}_{\rho w})_+ + w) + g_x(\hat{x}_{\rho w}) = 0 \Rightarrow \\ &\Rightarrow \hat{y}_{\rho w} = \rho (g(\hat{x}_{\rho w})_+ + w) + \end{aligned}$$

If a vector  $w$  such that  $p = 0$  is chosen, then

$$(22) \quad p = (g(\hat{x}_{\rho\hat{w}}) - \hat{w}) = 0 \Rightarrow \hat{x}_{\rho\hat{w}} = \hat{x}, \hat{y}_{\rho\hat{w}} = \hat{y}, \hat{w} = \frac{1}{\rho} \hat{y}.$$

Therefore, the problem of finding an adequate penalty shift  $\hat{w}$  is equivalent to the fundamental problem of finding Lagrangian multipliers  $\hat{y} = \rho\hat{w}$ . But away from the optimal solution, penalty shifts have a slightly different meaning than Lagrangian multipliers: they represent perturbations of the original problem, not the sensitivities of the original problem to perturbations. They also have slightly different properties: they are not necessarily constrained to be positive, only at the optimal point they turn out to be positive as a result of the relation (22). These different interpretations and properties make it possible to use special algorithms for finding optimal  $\hat{w}$ , see, e.g. [1],[2]. One of the oldest [17],[24] but very robust and effective methods of finding a crude approximation to  $(\hat{y}, \hat{x})$  is the following shifted-increased penalty function algorithm.

Specify  $\epsilon^0 > 0$ , the admissible violation of constraints for the first (large) iteration and  $\epsilon^{\min} \in (0; \epsilon^0)$ , the admissible violation of constraints at the end. Specify  $\gamma \in (0; 1)$ , the desired rate of convergence of violations of constraints in subsequent (large) iterations, and  $\kappa > 1$ , the rate of increase of the penalty coefficient  $\rho$  in case the desired rate of convergence is not attained. Specify  $\rho^0 > 0$ ,  $w^0 = 0$ . Set  $k = 0$  and

$$(23a) \quad \underset{x \in R^n}{\text{minimize}} \quad \psi(w^k, \rho^k, x) \text{ to obtain } x^k \in \underset{x \in R^n}{\text{Arg min}} \quad \psi(w^k, \rho^k, x),$$

$$(23b) \quad \text{compute } p^k = (g(x^k) + w^k)_+ - w^k, \quad \|p^k\|$$

$$(23c) \quad \text{if } \|p^k\| \leq \epsilon^k, \text{ set } w^{k+1} = (g(x^k) + w^k)_+, \rho^{k+1} = \rho^k, \epsilon^{k+1} = \gamma \|p^k\|,$$

$$(23d) \quad \text{if } \|p^k\| \leq \epsilon^{\min}, \text{ stop. Otherwise set } k := k+1, \text{ go to (23a).}$$

The first step (23a) calls for an iterative procedure of unconstrained minimization. Therefore, the algorithm is double-iterative

and the iterations  $k$  are called large iterations. Usually, the first step is not accurate, but this does not disturb the convergence - see, e.g. [3]. If the first step approximates a global minimum of  $\Psi(w^k, \rho^k, \cdot)$ , then the algorithm proceeds to approximate a global solution of the original problem; otherwise local minima are approximated. The second step (23b) determines the current violation of constraints. If it decreases fast enough, as in step (23c), then the penalty shift is changed (by a direct iteration solving the equation (22)), the penalty coefficient is kept constant and a better accuracy of the next iteration is required. If the violation of constraints does not decrease as required (step (23d)), then the penalty coefficient is  $\kappa$ -times increased; to keep constant the approximate value of Lagrangian multipliers, the penalty shift is  $\kappa$ -times decreased. The required accuracy is not changed, in order to attain it in the next iteration.

It is easily seen that the algorithm (23) converges for the wide class of problems, for which the classical increased penalty method works. For a smaller class of problems (such that the Lagrangian multipliers  $\hat{y}$  depend Lipschitz-continuously on the perturbation parameters  $p$ ) the algorithm (23) converges without increasing the penalty coefficient  $\rho$ , if it was large enough at the beginning - see, e.g. [24], [25].

In practical applications, the algorithm (23) is very robust, it is rather difficult to find practical examples of problems for which this algorithm does not work, as long as the required accuracy is not too high. Usually, a few (two to five) large iterations provide for a reasonable estimate of the optimal  $\hat{x}, \hat{y} = \rho \hat{w}$ . Moreover, if a variable metric method is used for the unconstrained minimization, an estimation of the Hessian matrix  $L_{xx}(\hat{y}, \hat{x})$  can be obtained. In fact, if there are no weakly active constraints at the optimal solution, then

$$(23) \quad \Psi_{xx}(\hat{w}, \rho, \hat{x}) = L_{xx}(\hat{y}, \hat{x}) + \rho \tilde{g}_x^*(\hat{x}) \tilde{g}_x(\hat{x})$$

where  $\tilde{g}_x(\hat{x})$  is composed of gradients  $g_{ix}(\hat{x})$  of strongly active constraints,  $i \in SA(\hat{y}, \hat{x})$ . If there are some weakly active constraints



then  $\Psi_{xx}(\hat{w}, \rho, \cdot)$  is discontinuous at  $\hat{x}$ . Hence, when using a variable matrix method, it is better to estimate  $L_{xx}(y, x)$  separately and augment it by terms  $\rho g_{ix}^*(x) g_{ix}(x)$  for (all) currently active constraints. The only difficulty in such an approach is that  $L_{xx}(\hat{y}, \hat{x})$  might not be positive definite for nonconvex problems, even though the second-order sufficient condition for optimality (10) implies that  $\Psi_{xx}(w, \rho, x)$  is positive definite for all  $(w, x)$  close to  $(\hat{w}, \hat{x})$  - also if it is discontinuous, since due to weakly active constraints only nonnegative terms can be added to  $\Psi_{xx}$ , compare (12). Therefore, special variable metric approximations must be used for estimating  $L_{xx}(y, x)$  in the nonconvex case.

## 2.5. Augmented Lagrangian functions

The similarity of the shifted penalty function to the Lagrangian function suggests the questions : is  $\Psi(w, \rho, x)$  a kind of Lagrangian function or not? In fact, it has only to be slightly modified to obtain the following augmented Lagrangian function:

$$\begin{aligned} (25) \quad \Lambda(y, \rho, x) &= \Psi\left(\frac{y}{\rho}, \rho, x\right) - \rho \left\| \frac{y}{\rho} \right\|^2 = f(x) + \frac{1}{2} g \|(\rho g(x) + y)_+\|^2 - \frac{1}{2} \|y\|^2 = \\ &= f(x) + \frac{1}{2} g \sum_{i \in I} ((\max(0, \rho g_i(x) + y_i))^2 - (y_i)^2) \end{aligned}$$

where the variable  $y$  can be interpreted as well as a vector of Lagrange multipliers as, when subdivided by  $\rho$ , a penalty shift  $w = \frac{y}{\rho}$ .

If only equality constraints were considered,  $g(x) = 0$ , then the operations  $(\cdot)_+$  and  $\max(0, \cdot)$  would have to be omitted in the definition (25) and the augmented Lagrangian function would be more easily interpreted as the normal Lagrangian function plus a quadratic penalty term :

$$\begin{aligned} (26) \quad \Lambda(y, \rho, x) &= f(x) + \sum_{i \in I} y_i g_i(x) + \frac{1}{2} \rho \sum_{i \in I} (g_i(x))^2 = \\ &= L(y, x) + \frac{1}{2} \rho \|g(x)\|^2, \text{ for constraints of the type} \\ &\quad g(x) = 0 \text{ only.} \end{aligned}$$

In this form the augmented Lagrangian function was introduced originally by Hestenes [10]. However, in the more general case of inequality constraints  $\Lambda(y, \rho, x) \neq L(y, x) + \frac{1}{2}\rho \| (g(x))_+ \|^2$  and the more complicated expression (25) is needed; this has been introduced by many authors - see, e.g., [7], [13], [19], [25]. The reason for the more complicated definition (25) is that only in this form does the Lagrangian function possess all strong properties of the normal Lagrangian function - and a few more. First, it is easily proved [19] that for  $\rho > 0$ :

$$(27) \quad \min_{x \in R^n} \max_{y \in R^m} \Lambda(y, \rho, x) = \min_{x \in X_0} f(x)$$

and the original problem is expressed as the primal problem for the augmented Lagrangian function; observe that the vector  $y$  of Lagrangian multipliers in (27) is not constrained to be positive. Secondly, the saddle-point relation:

$$(28) \quad \min_{x \in R^n} \max_{y \in R^m} \Lambda(y, \rho, x) = f(\hat{x}) = \Lambda(\hat{y}, \rho, \hat{x}) = \max_{y \in R^m} \min_{x \in R^n} \Lambda(y, \rho, x)$$

is not only a sufficient condition of optimality of  $(\hat{y}, \hat{x})$  for arbitrary problems of the type (1) and a necessary condition of optimality for normal convex problems, but also, as proved by Rockafellar [19], a necessary condition of optimality for a large class of nonconvex problems (characterized by the possibility of supporting the primal parametric function  $\hat{f}(p) = \inf_{x \in X_p} f(x)$  by a quadratic function)

which include also all nonconvex problems with solutions satisfying the second-order sufficient conditions of optimality (10). Hence the duality gap is closed for a large class of nonconvex problems when using the augmented Lagrangian function; moreover, the duality relation (28) imposes no sign constraints on the dual variables  $y$ . The positive sign of these variables results from the first-order necessary conditions for the saddle-point:

$$(29) \quad \Lambda_x(\hat{y}, \rho, \hat{x}) = f_x(\hat{x}) + (\rho g(\hat{x}) + \hat{y})_+ g_x(\hat{x}) = 0 \in \mathbb{R}^n$$

$$(30) \quad \Lambda_y(\hat{y}, \rho, \hat{x}) = \frac{1}{\rho} ((\rho g(\hat{x}) + \hat{y})_+ - y) = 0 \in \mathbb{R}^m$$

which are fully equivalent to the first-order necessary conditions of optimality (4), (5). The fact that the complementarity condition (5) of Kuhn-Tucker type is actually equivalent to equation (30), although rather elementary, was not perceived in the theory of non-linear programming [14], [20], [24] for a long time.

If the functions  $f, g$ , are twice differentiable, then the augmented Lagrange function is twice differentiable in  $x$  and  $y$  if, and only if, there is no component  $(\rho g_i(x) + y_i)$  equal to zero. If the relation (30) is satisfied, then this is equivalent to the requirement that there are no weakly active constraints at  $(\hat{y}, \hat{x})$ , since then either  $g_i(\hat{x}) < 0$  or  $\hat{y}_i > 0$ . More generally, it is convenient to define currently strongly active, weakly active and inactive constraints by:

$$(31) \quad SA(y, x) = \{i \in I : \rho g_i(x) + y_i > 0\}$$

$$WA(y, x) = \{i \in I : \rho g_i(x) + y_i = 0\}$$

$$IN(y, x) = \{i \in I : \rho g_i(x) + y_i < 0\}$$

This definition is consistent with (6) although it implies that a constraint might be made currently inactive by assuming a sufficiently negative value of  $y_i$ , even if  $g_i(x) > 0$ . But the positiveness of  $y_i$  is guaranteed in most computational algorithms and this definition does not induce any difficulties.

If  $WA(y, x) = \emptyset$ , then the augmented Lagrange function (25) is twice differentiable and its second-order derivatives have the form:

$$\begin{aligned}
 (32a) \quad \Lambda_{xx}(y, \rho, x) &= f_{xx}(x) + (\rho g(x) + y)_+ g_{xx}(x) + \rho \tilde{g}_x^*(x) \tilde{g}_x(x) = \\
 &= L_{xx}((\rho g(x) + y)_+, x) + \rho \tilde{g}_x^*(x) \tilde{g}_x(x)
 \end{aligned}$$

where  $\tilde{g}_x(x)$  is composed of  $g_{ix}(x)$  for  $i \in SA(y, x)$ ,

$$(32b) \quad \Lambda_{xy}(y, \rho, x) = (\tilde{g}_x^*(x), 0)$$

where the 0 matrix is related to inactive constraints,  $i \in IN(y, x)$ , and

$$(32c) \quad \Lambda_{yy}(y, \rho, x) = \begin{bmatrix} 0 & 0 \\ 0 & \text{diag}(-\frac{1}{\rho}) \end{bmatrix}$$

where 0 is related to active constraints,  $i \in SA(y, x)$ , and to all off-diagonal elements.

Observe that the augmented Lagrangian function  $\Lambda(y, \rho, x)$  is always concave in  $y$  - linear for active constraints and quadratic with a negative definite Hessian matrix for inactive constraints. This observation can be proved more generally [19], [20] and does not depend on the assumption  $WA(y, x) = \phi$ . Therefore, for the existence of a saddle-point at a pair  $(\hat{y}, \hat{x})$  satisfying (29), (30), it is sufficient that  $\Lambda(\hat{y}, \rho, \cdot)$  has a minimum at  $\hat{x}$ ; and if  $(\hat{y}, \hat{x})$  is a saddle-point, then  $\hat{x}$  is an optimal solution of the original problem (1).

This provides for a rather straightforward interpretation of the second-order sufficient (10) and necessary (12) conditions of optimality in terms of the augmented Lagrange function. Observe that these conditions are actually related to various approximations of the Hessian matrix  $\Lambda_{xx}(y, \rho, x)$  at  $(\hat{y}, \hat{x})$ . Even if there are some weakly active constraints,  $\hat{WA} \neq \phi$ , and  $\Lambda_{xx}$  cannot be defined at  $(\hat{y}, \hat{x})$ , there are points  $x$  arbitrarily close to  $\hat{x}$  such that  $WA(\hat{y}, x) = \phi$  and  $\Lambda_{xx}$  can be defined at  $(\hat{y}, x)$ . If (10) is satisfied, then  $\Lambda_{xx}$  is positive definite at  $(\hat{y}, x)$  such that all weakly active constraints become inactive. If a weakly active

constraint becomes strongly active for some  $x$  close to  $\hat{x}$ , then  $\Lambda_{xx}$  is only increased by a positive semidefinite matrix  $\rho g_{ix}^*(x)g_{ix}(x)$ . Hence, (10) implies that all approximations of  $\Lambda_{xx}$  close to  $(\hat{y}, \hat{x})$  are positive definite which is sufficient for a minimum of  $\Lambda(\hat{y}, \rho, \cdot)$  at  $\hat{x}$  and thus for the optimality of  $\hat{x}$ , if (29), (30) are satisfied. Conversely, the necessary conditions (12) are also necessary for a saddle-point of the augmented Lagrange function: for, if (12) were violated, then a second-order approximation of  $\Lambda(\hat{y}, \rho, x) - \Lambda(\hat{y}, \rho, \hat{x})$  would be negative at a point  $(\hat{y}, x)$  such that all weakly active constraints become strongly active and  $\Lambda(\hat{y}, \rho, \cdot)$  could not have a minimum at  $\hat{x}$ ; see [19].

If  $\hat{W}\hat{A} = \phi$  and (10) is satisfied, then the matrix

$$(33) \quad M = \begin{bmatrix} \Lambda_{xx} & \Lambda_{xy} \\ \Lambda_{yx} & \Lambda_{yy} \end{bmatrix} \quad (y, \rho, x)$$

is invertible for  $(y, x)$  in some neighborhood of  $(\hat{y}, \hat{x})$ . Hence it is possible to solve the necessary conditions (29), (30) by a Newton-like method or even by a quasi-Newton method, that is, with second-order derivatives only approximated, not computed. It was shown in [26], [27] that a sufficiently general quasi-Newton method for solving (29), (30) includes all fundamental classes of constrained nonlinear programming algorithms, such as gradient projection methods, multiplier methods, penalty methods and also quadratic approximation methods.

### 3. QUADRATIC APPROXIMATIONS OF AUGMENTED LAGRANGIAN FUNCTIONS

If it were known a priori which constraints are active at the optimal solution, then the nonlinear programming problem (1) would be fairly easy - since it would be equivalent to a problem with a smaller number of equality constraints. However, the lack of such a priori knowledge constitutes one of the main difficulties of the problem (1). Therefore, each practical algorithm for solving (1) has first to include a procedure for determining probably active constraints and then to account for activity changes. Moreover,

some practical algorithms are constructed under the assumption that there are no weakly active constraints at  $(\hat{y}, \hat{x})$  and thus no activity changes occur in a neighborhood of  $(\hat{y}, \hat{x})$ . Such an assumption is not entirely unjustified, since problems with weakly active constraints are not very probable in practice. Still, it is better to have an algorithm which works without this assumption; and, by using quadratic approximations to the augmented Lagrangean function, such an algorithm can be constructed. But for the sake of a clear presentation, it is better first to investigate the implications of the assumption  $WA(\hat{y}, \hat{x}) = \phi$ , then to relax this assumption and only then to discuss a method for determining probably active constraints.

### 3.1. Quadratic approximation in the smooth case

Suppose there are no weakly active constraints at the optimal solution  $(\hat{y}, \hat{x})$  of the problem (1),  $\hat{WA} = \phi$ . Then there exists a neighborhood  $U(\hat{y}, \hat{x})$  such that the augmented Lagrangean function (25) can be written for all  $(y, x) \in U(\hat{y}, \hat{x})$  as:

$$(34) \quad \Lambda(y, \rho, x) = f(x) + \sum_{i \in \hat{SA}} (y_i g_i(x) + \rho g_i^2(x))$$

since  $y_i$  can be assumed identically equal zero for  $i \in \hat{IN}$ . Suppose  $(y+\bar{y}, x+\bar{x}) \in U(\hat{y}, \hat{x})$  and consider the following approximation:

$$(35) \quad \Lambda(y+\bar{y}, \rho, x+\bar{x}) \cong \Lambda + \Lambda_x \bar{x} + \Lambda_y \bar{y} + \frac{1}{2} \langle \bar{x}, \Lambda_{xx} \bar{x} \rangle + \langle \bar{y}, \Lambda_{yx} \bar{x} \rangle$$

where  $\bar{y}_i$  are also assumed to be zero for  $i \in \hat{IN}$ ; therefore, it is possible to consider only the active parts  $\tilde{g}$  and  $\tilde{y}$  of  $g$  and  $y$ . Since  $\Lambda = f + \langle \tilde{y}, \tilde{g} \rangle + \frac{1}{2} \rho \|\tilde{g}\|^2$ ,  $\Lambda_x \bar{x} = f_x \bar{x} + \langle \tilde{y}, \tilde{g}_x \bar{x} \rangle + \langle \rho \tilde{g}, \tilde{g}_x \bar{x} \rangle$ ,  $\Lambda_y \bar{y} = \langle \tilde{g}, \bar{y} \rangle$  and  $\langle \bar{y}, \Lambda_{yx} \bar{x} \rangle = \langle \bar{y}, \tilde{g}_x \bar{x} \rangle$  where all functions and derivatives are evaluated at  $(y, x)$ , one has:

$$(36) \quad \Lambda(y+\bar{y}, \rho, x+\bar{x}) \cong f + \frac{1}{2} \rho \|\tilde{g}\|^2 + f_x \bar{x} + \langle \rho \tilde{g}, \tilde{g}_x \bar{x} \rangle + \frac{1}{2} \langle \bar{x}, \Lambda_{xx} \bar{x} \rangle \\ + \langle \tilde{y} + \bar{y}, \tilde{g} + \tilde{g}_x \bar{x} \rangle$$

But this, with the exception of a constant term, is the normal Lagrangian function for the problem:

$$(37) \quad \underset{\bar{x} \in \bar{X}_g}{\text{minimize}} \quad (f_x \bar{x} + \langle \rho \tilde{g}, \tilde{g}_x \bar{x} \rangle + \frac{1}{2} \langle \bar{x}, \Lambda_{xx} \bar{x} \rangle) ;$$

$$\bar{X}_{\tilde{g}} = \{ \bar{x} \in \mathbb{R}^n : \tilde{g} + \tilde{g}_x \bar{x} \leq 0 \in \mathbb{R}^m \}$$

If the second-order sufficient condition for optimality (10) is satisfied, then  $\Lambda_{xx}$  is positive definite and the saddle-point of (36) is equivalent to the optimal solution of (37).

The necessary (and sufficient, since  $\Lambda_{xx}$  is positive definite) conditions for  $\hat{\bar{x}}, \hat{\bar{y}} = \tilde{y} + \hat{\bar{y}}$  being the optimal solution of (37) are:

$$(38a) \quad f_x + (\rho \tilde{g} + \hat{\bar{y}}) g_x + \hat{\bar{x}} \Lambda_{xx} = 0$$

$$(38b) \quad \tilde{g} + \tilde{g}_x \hat{\bar{x}} \leq 0 ; \quad \langle \hat{\bar{y}}, \tilde{g} + \tilde{g}_x \hat{\bar{x}} \rangle = 0 ; \quad \hat{\bar{y}} \geq 0$$

Suppose now that  $(\hat{\bar{y}}, x + \hat{\bar{x}}) \in U(\hat{y}, \hat{x})$  where the activity of constraints does not change (this assumption will be relaxed below, hence there is no need here to examine the precise conditions under which this is valid). Since then  $\hat{\bar{y}}_i > 0$  for all  $i \in SA(\hat{y}, \hat{x})$ , one has  $\tilde{g} + \tilde{g}_x \hat{\bar{x}} = 0$  and (38a,b) (recall that gradients  $\Lambda_x, \Lambda_y$  are represented as row vectors and thus  $\Lambda_x^*, \Lambda_y^*$  are column vectors) can be reformulated to:

$$(39a) \quad \Lambda_x^* + \Lambda_{xx} \hat{\bar{x}} + \Lambda_{yx} \hat{\bar{y}} = 0$$

$$(39b) \quad \Lambda_y^* + \Lambda_{yx} \hat{\bar{x}} = 0$$

Now, (39a,b) is a Newton-like approximation for solving the necessary conditions of optimality (29), (30). If  $\Lambda_{xx}$  is positive definite and  $\Lambda_{yx}$  is of full rank, then the equations (39a,b) have a unique solution:

$$(40a) \quad \hat{\bar{y}} = (\Lambda_{yx} \Lambda_{xx}^{-1} \Lambda_{xy})^{-1} (\Lambda_y^* - \Lambda_{yx} \Lambda_{xx}^{-1} \Lambda_x^*)$$

$$(40b) \quad \hat{\bar{x}} = \Lambda_{xx}^{-1} ((\Lambda_{xy} (\Lambda_{yx} \Lambda_{xx}^{-1} \Lambda_{xy})^{-1} \Lambda_{yx} \Lambda_{xx}^{-1} - I) \Lambda_x^* - \Lambda_{xy} (\Lambda_{yx} \Lambda_{xx}^{-1} \Lambda_{xy})^{-1} \Lambda_y^*)$$

Since the resulting  $\hat{\bar{y}}, \hat{\bar{x}}$  are linear in  $\Lambda_y, \Lambda_x$ , they are clearly bounded by the norm of  $\Lambda_y, \Lambda_x$ : there exists a constant  $\alpha_1 > 0$  such that  $\|(\hat{\bar{y}}, \hat{\bar{x}})\| \leq \alpha_1 \|(\Lambda_y, \Lambda_x)\|$ , where any norm in  $R^{\bar{m}+n}$  can be used. The constant  $\alpha_1$  corresponds to the norm of the matrix  $\tilde{M}^{-1}$ , where

$$\tilde{M} = \begin{bmatrix} \Lambda_{xx} & \Lambda_{x\bar{y}} \\ \Lambda_{\bar{y}x} & 0 \end{bmatrix}.$$

Similar estimation can be obtained for the distance of a pair  $(y, x)$  from the optimal pair  $(\hat{y}, \hat{x})$  for the original problem. In fact, the following elementary lemma holds.

*Lemma 1. Suppose  $\hat{x}$  is an optimal solution of the problem (1) and  $\hat{y}$  is a corresponding vector of Lagrangian multipliers satisfying (4), (5). Suppose the full rank postulate holds at  $\hat{x}$ ,  $g_{ix}(\hat{x})$  be linearly independent for all  $i$  such that  $g_i(\hat{x}) = 0$  and there are no weakly active constraints,  $y_i > 0$  for these  $i$  (in different notation,  $\hat{W}\hat{A} = \emptyset$ ). Suppose the second-order sufficient conditions for optimality (9) are satisfied at  $(\hat{y}, \hat{x})$  and for a sufficiently large  $\rho > 0$ , an augmented Lagrangian function  $\Lambda(y, \rho, x)$  (25) is formulated for the problem. Then there exists a neighborhood  $U(\hat{y}, \hat{x})$  and a constant  $\delta > 0$  such that*

$$(41) \quad \|(y - \hat{y}, x - \hat{x})\| \leq \delta \|(\Lambda_y(y, \rho, x), \Lambda_x(y, \rho, x))\| \text{ for all } (y, x) \in U(\hat{y}, \hat{x}).$$

*Proof:* Consider  $\Lambda_x, \Lambda_y$  to be given vectors. Then the equations:

$$(42) \quad \Lambda_y = \Lambda_y(y, \rho, x) \quad : \quad \Lambda_x = \Lambda_x(y, \rho, x)$$

define  $(y, x)$  implicitly as a function of  $\Lambda_y, \Lambda_x$ . In fact, these equations have the solution  $(\hat{y}, \hat{x})$  at  $(\Lambda_y, \Lambda_x) = (0, 0)$  since (4), (5) are equivalent to (29), (30). Moreover, the right-hand sides have jointly an invertible operator of Frechet derivatives; the



inversion of this operator corresponds to the solution of (39a,b) with the sign  $\Lambda_y, \Lambda_x$  changed, where  $\Lambda_{xx}$  is positive definite due to (9), (10) and  $\Lambda_{yx}$  has full rank due to the full-rank postulate. Hence, the implicit function theorem can be applied: the pair  $(y, x)$  is a Frechet-differentiable function of  $(\Lambda_y, \Lambda_x)$ , hence also locally Lipschitz-continuous, which proves (41).

### 3.2. Quadratic approximations in non-smooth case

If there are some weakly active constraints at the optimal solution,  $\hat{W}A \neq \emptyset$ , then the second-order derivatives of the augmented Lagrangian function are discontinuous at  $(\hat{y}, \hat{x})$  - in any neighborhood of  $(\hat{y}, \hat{x})$  at those points  $(y, x)$  which satisfy  $\rho g_i(x) + y_i = 0$  for  $i \in \hat{W}A$ . But, in the neighborhood of  $(\hat{y}, \hat{x})$ , there are disjoint open sets in which the second-order derivatives  $\Lambda_{xx}(y, \rho, x)$  and  $\Lambda_{yx}(y, \rho, x)$ , not counting trivial  $\Lambda_{yy}(y, \rho, x)$ , are defined and continuous. In fact, let  $\Omega$  be any (possibly empty) subset of  $\hat{W}A$  and define:

$$(42a) \quad \Gamma_{\Omega} = \{(y, x) \in R^{m+n} : \rho g_i(x) + y_i > 0 \text{ for all } i \in \Omega, \\$$

$$\rho g_i(x) + y_i < 0 \text{ for all } i \in \hat{W}A \setminus \Omega\}$$

$$(42b) \quad \Gamma_0 = \{(y, x) \in R^{m+n} : \rho g_i(x) + y_i < 0 \text{ for all } i \in \hat{W}A\}$$

$$(42c) \quad \Gamma_M = \{(y, x) \in R^{m+n} : \rho g_i(x) + y_i > 0 \text{ for all } i \in \hat{W}A\}$$

Thus,  $\Gamma_0$  corresponds to  $\Omega = \emptyset$  and  $\Gamma_M$  to  $\Omega = \hat{W}A$ . If the full rank postulate is fulfilled and  $g_{ix}(x)$  are linearly independent for  $i \in \hat{W}A$ , then it is easy to show that each of the sets  $\Gamma_0, \Gamma_M, \Gamma_{\Omega}$ , for all  $\Omega$ , is nonempty and contains points arbitrarily close to  $(\hat{y}, \hat{x})$ . In each of these sets,  $\Lambda(y, \rho, x)$  is twice differentiable, provided no other constraints change their activity. Therefore, define a neighborhood  $U(\hat{y}, \hat{x})$  such that  $IN(y, x) = \hat{IN}$  and  $SA(y, x) = \hat{SA}$  for all  $(y, x) \in U(\hat{y}, \hat{x})$  and observe that:

$$(43) \quad \Lambda(y, \rho, x) = f(x) + \frac{1}{2\rho} \sum_{i \in \hat{S}\hat{A}} (\rho g_i(x) + y_i)^2 + \\ + \frac{1}{2\rho} \sum_{i \in \hat{W}\hat{A}} (\max(0, \rho g_i(x) + y_i))^2 - \frac{1}{2\rho} \sum_{i \in I} (y_i)^2$$

for all  $(y, x) \in U(\hat{y}, \hat{x})$ , and

$$(44a) \quad \Lambda(y, \rho, x) = \Lambda^\Omega(y, \rho, x) = f(x) + \frac{1}{2\rho} \sum_{i \in \hat{S}\hat{A} \cup \Omega} (\rho g_i(x) + y_i)^2 - \\ - \frac{1}{2\rho} \sum_{i \in I} (y_i)^2$$

for all  $(y, x) \in U(\hat{y}, \hat{x}) \cup \Gamma_\Omega$ ,

$$(44b) \quad \Lambda(y, \rho, x) = \Lambda^0(y, \rho, x) = f(x) + \frac{1}{2\rho} \sum_{i \in \hat{S}\hat{A}} (\rho g_i(x) + y_i)^2 - \\ - \frac{1}{2\rho} \sum_{i \in I} (y_i)^2$$

for all  $(y, x) \in U(\hat{y}, \hat{x}) \cup \Gamma_0$ ,

$$(44c) \quad \Lambda(y, \rho, x) = \Lambda^M(y, \rho, x) = f(x) + \frac{1}{2\rho} \sum_{i \in \hat{S}\hat{A} \cup \hat{W}\hat{A}} (\rho g_i(x) + y_i)^2 - \\ - \frac{1}{2} \sum_{i \in I} (y_i)^2$$

for all  $(y, x) \in U(\hat{y}, \hat{x}) \cup \Gamma_M$ .

Each of the functions  $\Lambda^\Omega, \Lambda^0, \Lambda^M$  is twice differentiable. Moreover, the following inequalities hold for  $(y, x) \in U(\hat{y}, \hat{x})$  :

$$(45a) \quad \Lambda^0(y, \rho, x) \leq \Lambda(y, \rho, x) \leq \Lambda^M(y, \rho, x)$$

$$(45b) \quad \Lambda^0(y, \rho, x) \leq \Lambda^\Omega(y, \rho, x) \leq \Lambda^M(y, \rho, x)$$

$$(45c) \quad \Lambda^0(\hat{y}, \rho, \hat{x}) = \Lambda(\hat{y}, \rho, \hat{x}) = \Lambda^\Omega(\hat{y}, \rho, \hat{x}) = \Lambda^M(\hat{y}, \rho, \hat{x})$$

If the second-order sufficient condition for optimality (10) is satisfied, then the function  $\Lambda^0$  is locally strictly convex in  $x$  at  $(\hat{y}, \hat{x})$ , has a minimum in  $x$  at  $\hat{x}$  for  $y=\hat{y}$ , and a saddle-point at  $(\hat{y}, \hat{x})$ . Because of the relations (45a,b,c), the functions  $\Lambda, \Lambda^\Omega, \Lambda^M$  must also have a minimum in  $x$  at  $\hat{x}$  for  $y=\hat{y}$  and thus a saddle-point at  $(\hat{y}, \hat{x})$ . This way, a family of differentiable approximations to  $\Lambda(y, \rho, x)$  has been constructed, with the lower approximation  $\Lambda^0(y, \rho, x)$  and the upper approximation  $\Lambda^M(y, \rho, x)$ . The properties of this family imply the following lemma:

*Lemma 2.* The assumption that  $\hat{W}\hat{A} = \emptyset$  can be omitted in Lemma 1, and the conclusion of Lemma 1 still holds.

*Proof.* For all  $\Omega$  ( $O$  and  $M$  are included as special cases of  $\Omega$  here) repeat the proof of Lemma 1 to obtain

$$\|(y-\hat{y}, x-\hat{x})\| \leq \delta_\Omega \|(\Lambda_Y^\Omega(y, \rho, x), \Lambda_X^\Omega(y, \rho, x))\|$$

for all  $(y, x) \in U^\Omega(\hat{y}, \hat{x})$ . Take  $U(\hat{y}, \hat{x}) = \bigcap_\Omega U^\Omega(\hat{y}, \hat{x})$ ; it is a nonempty neighborhood, since there is a finite number of sets  $\Omega$ . Take  $\delta = \max_\Omega \delta_\Omega$ . Now suppose  $(y, x) \in \Gamma_\Omega \cap U(\hat{y}, \hat{x})$ . Then

$$\begin{aligned} \|(y-\hat{y}, x-\hat{x})\| &\leq \delta_\Omega \|(\Lambda_Y^\Omega(y, \rho, x), \Lambda_X^\Omega(y, \rho, x))\| \leq \\ &\leq \delta \|(\Lambda_Y(y, \rho, x), \Lambda_X(y, \rho, x))\| \end{aligned}$$

since  $\Lambda_Y(y, \rho, x) = \Lambda_Y^\Omega(y, \rho, x)$  and  $\Lambda_X(y, \rho, x) = \Lambda_X^\Omega(y, \rho, x)$  in this case. But there are points  $(y, x)$  in  $U(\hat{y}, \hat{x})$  which do not belong to any of the sets  $\Gamma_\Omega$ ; in such a case, however, these points must belong to an intersection of the closures  $\bar{\Gamma}_\Omega$ , say  $(y, x) \in \bar{\Gamma}_{\Omega_1} \cap \bar{\Gamma}_{\Omega_2}$ . At such a point,  $\Lambda_Y(y, \rho, x) = \Lambda_Y^{\Omega_1}(y, \rho, x) = \Lambda_Y^{\Omega_2}(y, \rho, x)$  and  $\Lambda_X(y, \rho, x) = \Lambda_X^{\Omega_1}(y, \rho, x) = \Lambda_X^{\Omega_2}(y, \rho, x)$  since the first derivatives of  $\Lambda$  are continuous and the conclusion (41) of Lemma 1 holds. It can easily be checked that, since the gradients  $g_{ix}(\hat{x})$  are linearly independent for  $i \in \hat{W}\hat{A}$ , the sets  $\Gamma_\Omega$  and the intersections of their closures cover all neighborhood  $U(\hat{y}, \hat{x})$ .

Observe that the inequalities (45a,b,c) imply an easy estimate of  $\|x-\hat{x}\|$ , and that the above lemma gives even more - an estimate of  $\|(y-\hat{y}, x-\hat{x})\|$ .

### 3.3. Estimation of active and strongly active constraints

When solving a nonlinear programming problem - particularly if quadratic approximations described in previous paragraphs are applied - it is useful not only to know which constraints are active at a given point  $(y,x)$  but also to predict which constraints will be strongly and weakly active at the solution  $(\hat{y},\hat{x})$ .

Suppose a sequence  $\{y^k, x^k\}$  is converging to  $(\hat{y},\hat{x})$  and define neighborhoods  $U_{\epsilon^k}$  of  $(\hat{y},\hat{x})$  by  $U_{\epsilon^k} = \{(y,x) \in \mathbb{R}^{m+n} : \|(\Lambda_y(y,\rho,x), \Lambda_x(y,\rho,x))\| < \epsilon^k\}$ . According to lemmas 1,2,  $\|(y^k-\hat{y}, x^k-\hat{x})\| \leq \delta \epsilon^k$  for  $(y^k, x^k) \in U_{\epsilon^k}$ .

Define the index sets  $A^k$  of constraints that are probably active at  $(\hat{y},\hat{x})$  as seen from  $(y^k, x^k)$  by:

$$(46a) \quad A^k = \{i \in I : g_i(x^k) + \frac{y_i^k}{\rho} > -\eta_g^k\}$$

and the index sets  $S^k$  of constraints that are probably strongly active at  $(\hat{y},\hat{x})$  as seen from  $(y^k, x^k)$  by :

$$(46b) \quad S^k = \{i \in I : \frac{y_i^k}{\rho} > +\eta_y^k\}$$

where  $\{\eta_g^k\}_0^\infty$ ,  $\{\eta_y^k\}_0^\infty$  are chosen sequences,  $\eta_g^k > 0$ ,  $\eta_g^k \rightarrow 0$ ,  $\eta_y^k > 0$ ,  $\eta_y^k \rightarrow 0$ .

Now observe that  $g_i(x^k) + \frac{y_i^k}{\rho} = g_i(\hat{x})(x^k-\hat{x}) + o(x^k-\hat{x}) + \frac{1}{\rho}(\hat{y}_i + y_i^k - \hat{y}_i)$  where  $o(\cdot)$  is a function such that

$$\lim_{\|x^k-\hat{x}\| \rightarrow 0} \frac{o(x^k-\hat{x})}{\|x^k-\hat{x}\|} = 0.$$

Hence there exists a constant  $\xi$  such that, for sufficiently small

$\epsilon^k$  and  $U_{\epsilon^k}$  :

$$(47a) \quad g_i(x^k) + \frac{y_i^k}{\rho} \leq g_i(\hat{x}) + \xi \|y^k - \hat{y}, x^k - \hat{x}\| < -\omega + \xi \delta \epsilon^k$$

for all  $i \in \hat{IN} \quad (y^k, x^k) \in U_{\epsilon^k}$ ,

where

$$-\omega = \max_{i \in \hat{IN}} (g_i(\hat{x})).$$

If, for example, the maximum norm is used in (47a), then  $\xi > \max(\frac{1}{\rho}, \xi_i)$  where  $\xi_i = \|g_{ix}(\hat{x})\|$ ; to estimate  $\xi$  more precisely for a given  $\epsilon^k$ , the norms  $\|g_{ixx}(\hat{x})\|$  would be needed. But it is more practical to assume that  $\xi$  cannot be known precisely a priori. Neither can  $\omega$ ; however, for practical purposes, it is possible to assume an arbitrary bound,  $\bar{\omega}$  and to count all constraints with  $g_i(\hat{x}) > -\bar{\omega}$  as not distinguishable from active constraints.

Similarly to (47a) it can be obtained that:

$$(47b) \quad g_i(x^k) + \frac{y_i^k}{\rho} \geq -\xi \delta \epsilon^k \text{ for all } i \in \hat{SA} \cup \hat{WA} \text{ and } (y^k, x^k) \in U_{\epsilon^k},$$

since  $g_i(\hat{x}) = 0$  for all  $i \in \hat{SA} \cup \hat{WA}$ . Moreover

$$(48a) \quad \frac{y_i^k}{\rho} \leq \frac{\delta \epsilon^k}{\rho} \text{ for all } i \in \hat{WA} \cup \hat{IN} \text{ and } (y^k, x^k) \in U_{\epsilon^k}$$

since  $\hat{y}_i = 0$  for all  $i \in \hat{WA} \cup \hat{IN}$  and

$$(48b) \quad \frac{y_i^k}{\rho} \geq \frac{\hat{y}_i}{\rho} - \frac{\|(y^k - \hat{y}, x^k - \hat{x})\|}{\rho} \geq \frac{\tau}{\rho} \frac{\delta \epsilon^k}{\rho}$$

for all  $i \in \hat{SA}, (y^k, x^k) \in U_{\epsilon^k}$

where  $\tau = \min_{i \in \hat{SA}} \hat{y}_i$ ; again,  $\tau$  is not known a priori, but some  $\bar{\tau}$  can be assumed as a practical bound for counting a constraint to be strongly active.

An exact estimation of the activity of constraints at  $(\hat{y}, \hat{x})$ , that is,  $A^k = \hat{SA} \cup \hat{WA}$  and  $S^k = \hat{SA}$  could be obtained if  $\epsilon^k, \eta_g^k, \eta_y^k$

would satisfy the following inequalities:

$$(49a) \quad \xi \delta \epsilon^k \leq \eta_g^k \leq \omega - \xi \delta \epsilon^k$$

$$(49b) \quad \frac{\delta}{\rho} \epsilon^k \leq \eta_y^k \leq \tau \frac{\delta}{\rho} \epsilon^k$$

For arbitrary positive  $\omega, \tau, \xi, \rho, \delta$ , these inequalities are satisfied for sufficiently large  $k$ , if the sequences  $\{\eta_g^k\}$   $\{\eta_y^k\}$  converge to zero more slowly than the sequence  $\{\epsilon^k\}$ . For example, if  $\eta_g^k = \xi_g (\epsilon^k)^{\frac{1}{2}}$  and  $\eta_y^k = \xi_y (\epsilon^k)^{\frac{1}{2}}$  with some positive  $\xi_g, \xi_y$  are chosen, then the inequalities (49a,b) are satisfied if:

$$(50a) \quad \epsilon^k \leq \min \left( \frac{\xi_g^2}{\xi^2 \delta^2}, \left( \frac{(\xi_g^2 + 4 \xi \delta \omega)^{\frac{1}{2}} - \zeta_g}{\xi \delta} \right)^2 \right) \leq \frac{\omega^2}{\zeta_g^2} \text{ for small } \omega,$$

$$(50b) \quad \epsilon^k \leq \min \left( \frac{\zeta_y^2 \rho^2}{\rho}, \left( \frac{(\zeta_y^2 \rho^2 - 4 \delta \tau)^{\frac{1}{2}} - \zeta_y \rho}{2 \delta} \right)^2 \right) \leq \frac{\tau}{\delta^2 \zeta_y^2} \text{ for small } \tau.$$

If some small values  $\bar{\omega}$  and  $\bar{\tau}$  are arbitrarily specified, and

$$\zeta_g \geq (\xi \delta \bar{\omega})^{\frac{1}{2}}, \zeta_y \geq \frac{\tau \delta}{\rho^2}^{\frac{1}{2}}$$

are assumed, then for  $\epsilon^k \leq \frac{\omega^2}{\zeta_g^2}$  and  $\epsilon^k \leq \frac{\bar{\tau}^2}{\rho^2 \zeta_y^2}$ , the sets of strongly

active and active constraints at the solution are estimated at a point  $(y^k, x^k) \in U$ , up to the accuracy  $\bar{\omega}$  in constraining function values and the accuracy  $\bar{\tau}$  in multiplier values. Since the true values of  $\omega$  and  $\tau$  are finite,  $\epsilon^k \rightarrow 0$  always results eventually in  $A^k = \hat{S}A \cup \hat{W}A$  and  $S^k = \hat{S}A$ .

### 3.4. Properties of approximative quadratic programming problems for augmented Lagrangians.

Consider the following augmented approximate quadratic programming problem:

$$(51) \text{ minimize } (f_x^k \bar{x} + \frac{1}{2} \langle \bar{x}, H^k \bar{x} \rangle + \rho \sum_{i \in S^k} (g_i^k g_{ix}^k \bar{x} + \frac{1}{2} (g_{ix}^k \bar{x})^2))$$

$$\bar{x}^k = \{\bar{x} \in R^n : g_i^k + g_{ix}^k \bar{x} \leq 0, i \in A^k\}$$

Here the sets  $S^k$  and  $A^k$  are not necessarily defined by (46a,b) although such a choice is actually advantageous. The set  $A^k$  is a set of (indexes of) possibly active constraints and the set  $S^k$  is a set of constraints used for augmenting the Lagrangian function and convexifying the quadratic programming problem (51). The values of functions and gradients  $g_i^k, g_{ix}^k, f_x^k$  are evaluated at  $(y^k, x^k)$  and  $H^k$  is an approximation to the Hessian matrix

$$L_{xx}^k = L_{xx}((\rho g^k + y^k)_+, x^k) .$$

Denote the sets of currently strongly and weakly active constraints at  $(y^k, x^k)$  defined by (31), by  $SA^k = SA(y^k, x^k)$ ,  $WA^k = WA(y^k, x^k)$  and assume that  $S^k \subset SA^k$ ,  $(WA^k \cup SA^k) \subset A^k$ ; this assumption is satisfied if  $S^k$  and  $A^k$  are defined by (46a,b). Let  $A^k$  and  $S^k$  contain  $m^k$  and  $\tilde{m}^k$  elements respectively and denote by  $g^k$  the  $m^k$ -vector composed of  $g_i^k$  and by  $g_x^k$  the  $(m^k \times n)$ -matrix composed of row vectors  $g_{ix}^k$  for  $i \in A^k$ , by  $\tilde{g}^k$  the  $\tilde{m}^k$ -vector of  $g_i^k$  and by  $G^k$  the  $(\tilde{m}^k \times n)$ -matrix of  $g_{ix}^k$  for  $i \in S^k$ . Then the problem (51) can be equivalently written as :

$$(52) \text{ minimize } (f_x^k \bar{x} + \langle \rho \tilde{g}^k, G^k \bar{x} \rangle + \frac{1}{2} \langle \bar{x}, (H^k + \rho G^{k*} G^k) \bar{x} \rangle ;$$

$$\bar{x} \in \bar{X}^k$$

$$\bar{X}^k = \{\bar{x} \in R^n : g^k + g_x^k \bar{x} \leq 0 \in R^{m^k}\}$$

The set  $\bar{X}^k$  is usually nonempty but unbounded and the existence of a solution  $\hat{\bar{x}}$  to this problem can be guaranteed if the matrix  $H^k + \rho G^{k*} G^k$  is positive definite; this is the main reason for augmenting the approximate quadratic programming problem.

There is still another useful equivalent formulation of the problem (51) in relation to the augmented Lagrangian (25) and its quadratic approximations. Suppose that the  $m^k$ -vector of Lagrangian multipliers for the problem (51) has the form  $y = y^k + \bar{y}$ ; since  $(WA^k \cup SA^k) \subset A^k$  was assumed, hence  $y_i^k = 0$ ,  $g_i^k < 0$  for  $i \in I \setminus A^k$ , and

these constraints can be disregarded. Similarly, assume that  $y_i^k = 0$  for  $i \in A^k \setminus SA^k$ ; even if originally there were  $y_i^k > 0$  for a constraint that is not currently strongly active, then it is possible to change  $y_i^k$  to  $y_i^k = 0$  without influencing other constraints nor the problem (51), where only the interpretation of  $\bar{y}_i$  is changed. Under these assumptions:

$$(53a) \quad \Lambda_x(y^k, \rho, x^k) = \Lambda_x^k = f_x^k + \sum_{i \in SA^k} (y_i^k + \rho g_i^k) g_{ix}^k$$

$$(53b) \quad \Lambda_{yi}(y^k, \rho, x^k) = \Lambda_{yi}^k = \begin{cases} g_i^k & , \quad i \in SA^k \\ 0 & , \quad i \in I \setminus SA^k \end{cases}$$

Define also

$$(53c) \quad \Lambda_{xx}^k = L_{xx}^k + \rho \sum_{i \in SA^k} g_{ix}^{k*} g_{ix}^k ; \quad \Lambda_{yix}^k = \begin{cases} g_{ix}^k & , \quad i \in SA^k \\ 0 & , \quad i \in I \setminus SA^k \end{cases}$$

If  $WA^k \neq \emptyset$ , the  $\Lambda_{xx}^k$  is not the second derivative  $\Lambda_{xx}(y^k, \rho, x^k)$ , but only one of its "one-sided" approximations; similarly  $\Lambda_{yx}^k$ .

The problem (51) or (52) can be equivalently rewritten as:

$$(54) \quad \underset{x \in \bar{X}^k}{\text{minimize}} \quad ((\Lambda_x^k + \Delta_x^k) \bar{x} - \Lambda_{xy}^k y^k + \frac{1}{2} \langle \bar{x}, (\Lambda_{xx}^k + \Delta_{xx}^k) \bar{x} \rangle)$$

$$\bar{X}^k = \{ \bar{x} \in \mathbb{R}^n : \Lambda_y^{k*} + \Delta_y^{k*} + (\Lambda_{yx}^k + \Delta_{yx}^k) \bar{x} \leq 0 \in \mathbb{R}^{m^k} \}$$

where  $\Delta_x^k, \Delta_{xx}^k, \Delta_y^k, \Delta_{yx}^k$  express various differences between the problem (51) and a quadratic approximation to the augmented Lagrangian  $\Lambda(y, \rho, x)$

$$(55a) \quad \Delta_x^k = - \rho \sum_{i \in SA^k \setminus S^k} g_i^k g_{ix}^k$$

is due to a possible difference of  $SA^k$  and  $S^k$ ,



$$(55b) \Delta_{xx}^k = (H^k - L_{xx}^k - \rho \sum_{i \in SA^k \setminus S^k} g_{ix}^{k*} g_{ix}^k)$$

expresses, beside this difference, the errors of approximation of  $L_{xx}^k$  by  $H^k$ ,

$$(55c) \Delta_{yi}^k = \begin{cases} 0, & i \in SA^k \\ g_i^k, & i \in A^k \setminus SA^k \end{cases} \quad \Delta_{yix}^k = \begin{cases} 0, & i \in SA^k \\ g_{ix}^k, & i \in A^k \setminus SA^k \end{cases}$$

are related to the fact that more constraints are taken into account than actually strongly active at  $(y^k, x^k)$ .

Even when accounting for these differences, the following lemma can be proved:

*Lemma 3. Suppose, as in Lemma 2, that  $\hat{x}$  is an optimal solution and  $\hat{y}$  a corresponding vector of Lagrange multipliers for the problem (1), satisfying the necessary conditions (4,5). Let the full-rank postulate hold at  $\hat{x}$ ,  $g_{ix}(\hat{x})$  be linearly independent for all  $i$  such that  $g_i(\hat{x}) = 0$ , let the second-order sufficient conditions (9) be satisfied at  $(\hat{y}, \hat{x})$  and let  $\rho > 0$  be such that the conditions (9) have the form (10). Suppose  $S^k, A^k$  are such that  $S^k = \hat{S}A$  and  $A^k = \hat{S}A \cup \hat{W}A$  for  $(y^k, x^k)$  in a neighborhood of  $(\hat{y}, \hat{x})$ . Then there exist a neighborhood  $U(\hat{y}, \hat{x})$  and a number  $\alpha > 0$  such that for all  $(y^k, x^k) \in U(\hat{y}, \hat{x})$  with*

$$H^k + \rho \sum_{i \in S^k} g_{ix}^{k*} g_{ix}^k$$

*positive definite and bounded, the problem (51)  $\Leftrightarrow$  (52)  $\Leftrightarrow$  (54) has a solution  $\bar{x}^k$  with the corresponding Lagrange multiplier vector  $y^k, \hat{y}^k$  satisfying the following inequality:*

$$(56) \|(\hat{y}^k, \hat{x}^k)\| \leq \alpha \|(\Lambda_y^k, \Lambda_x^k)\|$$

*Proof.* It is sufficient to investigate neighborhoods of  $(\hat{y}, \hat{x})$  such that  $S^k = \hat{S}A$ ,  $A^k = \hat{S}A \cup \hat{W}A$ , although the conclusions of the lemma can be clearly extended to large neighborhoods without conceptual

difficulties. Since  $g_{ix}(\hat{x})$  are linearly independent for  $i \in \hat{S} \cup \hat{W}$  and  $g_{ix}(\cdot)$  are continuous, hence  $g_{ix}^k$  stay linearly independent for  $i \in A^k$  and  $(y^k, x^k)$  in a neighborhood of  $(\hat{y}, \hat{x})$ . In this case, the set  $\bar{X}^k$  is nonempty. Since

$$H^k + \rho \sum_{i \in S^k} g_{ix}^{k*} g_{ix}^k$$

is assumed to be positive definite, the problem (51)  $\Leftrightarrow$  (52)  $\Leftrightarrow$  (54) has a unique solution  $\hat{x}^k$  with multipliers  $y^k + \hat{y}^k$ , even if the set  $\bar{X}^k$  is unbounded.

The necessary and sufficient conditions of optimality of  $(\hat{y}^k, \hat{x}^k)$  are:

$$(57a) \quad f_x^k + (H^k + \sum_{i \in S^k} g_{ix}^{k*} g_{ix}^k) \hat{x}^k + \sum_{i \in A^k} g_{ix}^{k*} (y_i^k + \hat{y}_i^k) = 0$$

$$(57b) \quad g_i^k + g_{ix}^k \hat{x}^k \leq 0, \quad i \in A^k; \quad \sum_{i \in A^k} (g_i^k + g_{ix}^k \hat{x}^k) (y_i^k + \hat{y}_i^k) = 0;$$

$$; \quad y_i^k + \hat{y}_i^k \geq 0, \quad i \in A^k$$

The solution of this system of equations and inequalities depends Lipschitz continuously on  $f_x^k, g_i^k, g_{ix}^k$  as long as  $g_{ix}^k$  are linearly independent and

$$H^k + \rho \sum_{i \in S^k} g_{ix}^{k*} g_{ix}^k$$

is positive definite and bounded - see, e.g. [8], [18]. If  $f_x^k = f_x(\hat{x})$ ,  $g_i^k = g_i(\hat{x})$ ,  $g_{ix}^k = g_{ix}(\hat{x})$  and  $y_i^k = \hat{y}_i$ , then  $\hat{y}^k = 0$ ,  $\hat{x}^k = 0$  satisfies (57a,b). Since  $f_x^k = f_x(x^k)$ ,  $g_i^k = g_i(x^k)$ ,  $g_{ix}^k = g_{ix}(x^k)$  and these functions are differentiable, hence there is a neighborhood of  $(\hat{y}, \hat{x})$  and a constant  $\alpha_1 > 0$  such that

$$(58) \quad \|(\hat{y}^k, \hat{x}^k)\| \leq \alpha_1 \| (y^k - \hat{y}, x^k - \hat{x}) \|$$

Since the conclusions of Lemma 2 hold under the assumptions of Lemma 3, (58) can be combined with (45) to yield (56) with  $\alpha = \alpha_1 \delta$ .

Observe that no assumptions were made in the lemma and proof above about the sets of active constraints for the problem (51)  $\Leftrightarrow$  (52)  $\Leftrightarrow$  (54). Denote these sets by:

$$(59a) \quad K^k = \{i \in A^k : g_i^k + g_{ix}^k \hat{x}^k = 0, y_i^k + \hat{y}_i^k > 0\}$$

and

$$(59b) \quad \bar{K}^k = \{i \in A^k : g_i^k + g_{ix}^k \hat{x}^k = 0\}$$

It can be concluded from the proof of Lemma 3 that  $S^k \subset K^k \subset \bar{K}^k \subset A^k$  for  $(y^k, x^k)$  sufficiently close to  $(\hat{y}, \hat{x})$ . Clearly,  $K^k \subset \bar{K}^k \subset A^k$ ; to show that  $S^k \subset K^k$  assume  $S^k = \hat{S}A$  and suppose that  $\hat{S}A \not\subset K^k$ , that is, there is an  $i$  with  $\hat{y}_i > 0$  such that  $y_i^k + \hat{y}_i^k = 0$ . Hence,  $|\hat{y}_i^k| = y_i^k$ ; if any  $y_i^k$  with  $|y_i^k - \hat{y}_i| < \epsilon$  is chosen, then  $|\hat{y}_i^k| > (1-\epsilon)\hat{y}_i$ , which contradicts (58) for sufficiently small  $\epsilon$ .

The assumption that

$$H^k + \rho \sum_{i \in S^k} g_{ix}^{k*} g_{ix}^k$$

is positive definite is actually used twice in the proof of Lemma 3. If this assumption is not satisfied, then, first, it may happen that a solution to the problem (51) does not exist; the conditions (57a,b) are only necessary in such a case and a point  $(\hat{y}^k, \hat{x}^k)$  satisfying these conditions might not correspond to a solution of (51). Secondly, it may happen that there are many points  $(\hat{y}^k, \hat{x}^k)$  satisfying (57a,b); the one that is closest to  $(0,0)$  among them might satisfy (58), but a solution of (51) might not satisfy (58) and (56). The convexifying term

$$\rho \sum_{i \in S^k} g_{ix}^{k*} g_{ix}^k$$

is used for these two reasons, since it provides only for the sufficiency of the necessary conditions (57a,b). Observe that the

necessary condition (57a) can be equivalently written as:

$$(60) \quad f_x^{k*} + H_x^{k\hat{x}^k} = - \sum_{i \in \bar{K}^k} g_{ix}^{k*} (y_i^k + \hat{y}_i^k) \\ - \rho \sum_{i \in S^k} g_{ix}^{k*} (g_i^k + g_{ix}^k \hat{x}^k)$$

where the last sum expresses the influence of convexifying terms. If  $S^k \subset \bar{K}^k$ , then the last sum can be simply omitted in the necessary condition, since  $g_i^k + g_{ix}^k \hat{x}^k = 0$  for  $i \in \bar{K}^k$ , but then the necessary condition would not be sufficient. Conversely, if  $S^k = \emptyset$  originally (which corresponds to the use of a quadratic approximation to the normal Lagrange function), then  $S^k$  can be increased up to  $S^k = \bar{K}^k$  without influencing the solution of (57a,b).

It follows that if the quadratic approximation method (15), (16) based on the normal Lagrange function happens to generate  $(y^k, x^k)$  and  $(y^{k\hat{y}}, x^{k\hat{x}})$  sufficiently close to  $(\hat{y}, \hat{x})$  for a non-convex problem, then it can converge, since it could be convexified by increasing  $S^k$  from  $\emptyset$  to  $\bar{K}^k$  in (60) and all results of this paper would be applicable. Under the assumption of strict complementarity,  $\hat{W}\hat{A} = \phi$ , a similar result was obtained in [9] by choosing for  $(\hat{y}^k, \hat{x}^k)$  not really the solutions of the quadratic approximation (16) of the normal Lagrange function, but the points which satisfy the necessary conditions of optimality and are closest to  $(0,0)$ . The strict complementarity assumption was relaxed in [22], but only for convex problems with positive definite  $L_{xx}(\hat{y}, \hat{x})$ . There are also practical examples of a successful application of the method (15), (16) to nonconvex problems, but the success in those cases is clearly due to luck.

Another possibility implied by relation (60) is to use more constraints than are actually needed to convexify the problem, to be on the safe side. If, for example,  $i \in A^k$  were used instead of  $i \in S^k$  in the last sum of (60), then after solving the corresponding quadratic programming problem,  $\bar{K}^k = A^k$  should be checked.

If  $\bar{K}^k \neq A^k$ , then the constraints with  $i \in A^k \setminus \bar{K}^k$  would have to be deleted from the last sum and the quadratic programming repeated, since the solution could have been falsified by unnecessary convexifying terms. It is probably possible to develop a special quadratic programming code with an automatic deletion of unnecessary convexifying terms; the only question is how to provide for finite termination without cycling. This possibility will not be investigated further in this paper.

For further analysis, it is assumed that the set of convexifying constraints in (60) is defined by  $i \in S^k$  determined, for example, by (46b) so that  $S^k = \hat{S} \cap K^k$  can be assumed for  $(y^k, x^k)$  sufficiently close to  $(\hat{y}, \hat{x})$ . In this case, the following lemma holds.

*Lemma 4. If the problem (51)  $\Leftrightarrow$  (52)  $\Leftrightarrow$  (54) has a solution  $\hat{x}^k$  with a corresponding vector of Lagrange multipliers  $y^k + \hat{y}^k$  and if  $S^k \subset \bar{K}^k$ , where  $\bar{K}^k$  is the set of indexes of active constraints (59b) then*

$$(61a) \quad \|\Lambda_x^{k+1}\| = \|\Lambda_x(y^k + \hat{y}^k, \rho, x^k + \hat{x}^k)\| < \|(H^k - L_{xx}^k) \hat{x}^k\| + o(\hat{y}^k, \hat{x}^k)$$

$$(61b) \quad \|\Lambda_y^{k+1}\| = \|\Lambda_y(y^k + \hat{y}^k, \rho, x^k + \hat{x}^k)\| = o(\hat{y}^k, \hat{x}^k)$$

where  $o(z)$  denotes an arbitrary function such that  $\lim_{\|z\| \rightarrow 0} \frac{o(z)}{\|z\|} = 0$  (various functions of this type are not distinguished by indexes here).

*Proof.* Denote  $y^{k+1} = y^k + \hat{y}^k$ ,  $x^{k+1} = x^k + \hat{x}^k$ ,  $f_x^{k+1} = f_x(x^{k+1})$ ,  $g_i^{k+1} = g_i(x^{k+1})$ , etc. Observe that:

$$\begin{aligned} (62) \quad \Lambda_x^{(k+1)*} &= \Lambda_x^*(y^{k+1}, \rho, x^{k+1}) = f_x^{(k+1)*} + \\ &+ \sum_{i \in I} \left( y_i^{k+1} + \rho g_i^{k+1} \right)_+ g_{ix}^{(k+1)*} = \\ &= f_x^{k*} + f_{xx}^k \hat{x}^k + \sum_{i \in I} (y_i^{k+1})_+ (g_{ix}^{k*} + g_{ixx}^k \hat{x}^k) + o(\hat{x}^k) = \end{aligned}$$

$$= f_x^k + L_{xx}^k \hat{x}^k + \sum_{i \in I} \left( (y_i^{k+1} + \rho g_i^{k+1})_+ - (y_i^k + \rho g_i^k)_+ \right) g_{ixx}^k \hat{x}^k + \sum_{i \in SA^{k+1}} (y_i^{k+1} + \rho g_i^{k+1}) g_{ix}^{k*} + o(\hat{x}^k)$$

where  $(y_i^{k+1} + \rho g_i^{k+1})_+ = \max(0, y_i^{k+1} + \rho g_i^{k+1}) \neq 0$  for  $i \in SA^{k+1}$  only and  $L_{xx}^k = f_{xx}^k + \sum_{i \in I} (y_i^k + \rho g_i^k)_+ g_{ixx}^k$ . But it is necessary for optimality of  $\hat{x}^k$  that (60) holds; since  $g_i^k + g_i^k \hat{x}^k = 0$  for  $i \in \bar{K}^k$  and  $(g_i^k + g_{ix}^k \hat{x}^k) g_{ix}^{k*} = g_i^{k+1} g_{ix}^{k*} + o(\hat{x}^k)$ , hence the relation (60) can be rewritten as:

$$(63) \quad f_x^{k*} + H^k \hat{x}^k + \sum_{i \in \bar{K}^k} (y_i^{k+1} + \rho g_i^{k+1}) g_{ix}^{k*} + o(\hat{x}^k) = 0$$

By subtracting (63) from (62) the following relation is obtained:

$$(64) \quad \Lambda_x^{(k+1)*} = (L_{xx}^k - H^k) \hat{x}^k + \sum_{i \in I} ((y_i^{k+1} + \rho g_i^{k+1})_+ - (y_i^k + \rho g_i^k)_+) g_{ixx}^k \hat{x}^k + \sum_{i \in SA^{k+1} \setminus \bar{K}^k} (y_i^{k+1} + \rho g_i^{k+1}) g_{ix}^{k*} - \sum_{i \in \bar{K}^k \setminus SA^{k+1}} (y_i^{k+1} + \rho g_i^{k+1}) g_{ix}^{k*} + o(\hat{x}^k)$$

Since the operation  $(\cdot)_+$  is Lipschitz continuous with coefficient one, hence there exist constants  $\delta_i > 0$  such that

$$| (y_i^{k+1} + \rho g_i^{k+1})_+ - (y_i^k + \rho g_i^k)_+ | \leq | y_i^{k+1} + \rho (g_i^{k+1} - g_i^k) | \leq \delta_i \| (\hat{y}^k, \hat{x}^k) \|$$

and there is  $\delta_0 > 0$  such that :

$$(64b) \quad \left\| \sum_{i \in I} \left( \left( y_i^{k+1} + \rho g_i^{k+1} \right)_+ - \left( y_i^k + \rho g_i^k \right)_+ \right) g_{ix}^k \right\| \leq \\ \leq \delta_0 \|(\hat{y}^k, \hat{x}^k)\| \|g_{xx}^k\| \|\hat{x}^k\| = o(\hat{y}^k, \hat{x}^k)$$

Observe, moreover, that if  $i \in SA^{k+1} \setminus \bar{K}^k$ , then  $y_i^{k+1} + \rho g_i^{k+1} = y_i^k + \hat{y}_i^k + \rho(g_i^k + g_{ix}^k \hat{x}^k) + o(\hat{x}^k) > 0$ , whereas  $y_i^k + \hat{y}_i^k = 0$ ,  $g_i^k + g_{ix}^k \hat{x}^k < 0$ ; this is possible only if  $y_i^{k+1} + \rho g_i^{k+1} = o(\hat{x}^k)$ . If  $i \in \bar{K}^k \setminus SA^{k+1}$ , then  $y_i^k + \hat{y}_i^k \geq 0$ ,  $g_i^k + g_{ix}^k \hat{x}^k = 0$  and  $y_i^{k+1} + \rho g_i^{k+1} = \hat{y}_i^k + \hat{y}_i^k + \rho(g_i^k + g_{ix}^k \hat{x}^k) + o(\hat{x}^k) < 0$ ; this is again possible only if  $y_i^{k+1} + \rho g_i^{k+1} = o(\hat{x}^k)$ . Since  $o(\hat{x}^k)$  is clearly of order  $o(\hat{y}^k, \hat{x}^k)$ , all sums in (64a) are of order  $o(\hat{y}^k, \hat{x}^k)$  and

$$(65) \quad \Lambda_x^{(k+1)*} = \left( L_{xx}^k - H^k \right) \hat{x}^k + o(\hat{y}^k, \hat{x}^k)$$

which implies (61a). To prove (61b), observe that

$$(66) \quad \Lambda_i^{(k+1)} = \Lambda_{yi} \left( y^{k+1}, \rho, x^{k+1} \right) = \left( \frac{y_i^{k+1}}{\rho} + g_i^{k+1} \right)_+ - \frac{y_i^{k+1}}{\rho} = \\ = \begin{cases} g_i^{k+1}, & i \in SA^{k+1} \\ \frac{y_i^{k+1}}{\rho}, & i \in SA^{k+1} \end{cases}$$

and that  $y_i^{k+1} = 0$  if  $i \notin K^k$ . But if  $i \in SA^{k+1} \cap K^k$ , then  $g_i^{k+1} = g_i^{k+1} + g_{ix}^k \hat{x}^k + o(\hat{x}^k) = o(\hat{x}^k)$ ; if  $i \in SA^{k+1} \setminus K^k$ , then  $y_i^{k+1} = 0$ ,  $g_i^k + g_{ix}^k \hat{x}^k + o(\hat{x}^k) > 0$  whereas  $g_i^k + g_{ix}^k \hat{x}^k \leq 0$ , which is possible only if  $g_i^{k+1} = o(\hat{x}^k)$ ; if  $i \in K^k \setminus SA^{k+1}$ , then  $y_i^{k+1} = y_i^k + \hat{y}_i^k > 0$ ,  $g_i^{k+1} = g_i^k + g_{ix}^k \hat{x}^k + o(\hat{x}^k)$  whereas  $y_i^{k+1} + \rho g_i^{k+1} < 0$ , which is again possible only if  $y_i^{k+1} = o(\hat{x}^k)$ . Thus,  $\Lambda_{yi}^{k+1} = o(\hat{x}^k)$  for  $i \in SA^{k+1} \cup K^k$  and  $\Lambda_{yi}^{k+1} = 0$  for  $i \in I \setminus (SA^{k+1} \cup K^k)$ , which implies (61b). It is easy to check that a possible redefinition of  $y_i^{k+1}$  to  $y_i^{k+1} = 0$  if  $i \notin SA^{k+1}$  does not change the conclusions of the lemma.

Once Lemmas 2,3,4 are proved, a theorem on the convergence of the quadratic approximation methods can be formulated and proved in a natural way, typical for Newton-like optimization methods. This, however, is done after presenting the complete algorithm first.

#### 4. A DOUBLE-PHASE ALGORITHM FOR NONCONVEX NONLINEAR PROGRAMMING

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The algorithm finds a minimum (not necessarily global) of a function  $f : R^n \rightarrow R^1$  in a set  $X_0 = \{x \in R^n : g_i(x) \leq 0, i \in I, g_i(x) = 0, i \in J\}$  where  $(g_i(x))_{i \in I \cup J} = g(x) \in R^m$ . The function  $f, g_i$  are assumed to be twice differentiable but not necessarily convex; it is assumed only that the full-rank postulate, that is, linear independence of the gradients of active constraints, and the second-order sufficient conditions of the type (9) are satisfied at the optimal point  $\hat{x}$ .

The algorithm makes use of three subroutines not described here in detail; a subroutine of finding an unconstrained minimum (preferably of variable metric type), a subroutine of approximating a square matrix  $H : R^n \rightarrow R^n$  which satisfies  $r^k \approx Hs^k$  for  $k = 1, 2, \dots$  by the use of data  $\{r^j, s^j\}_{j=k-n+1}^k$  (preferably a modified rank-one variable metric routine) and a subroutine of solving quadratic programming problems.

Beside the numbers  $n, m$ , the functions  $f$  and  $g_i$  and their gradients  $f_x, g_{ix}$  for  $i = 1, \dots, m$ , the sets  $I$  and  $J$ , the following data is to be specified by the user of the algorithm:

$\epsilon_x^f$  - the required final accuracy of the norms of gradients in  $x$ .

$\epsilon_y^f$  - the required final accuracy of constraint violations (or equivalently, the norms of gradients in  $y$ )

$\gamma \in (0; 1)$  - the desired ratio of convergence of the norms of gradients (suggested value  $\gamma = 0.1$ )

$\kappa > 1$  - the ratio of increasing penalty coefficients (suggested values  $\kappa = 5.0$  or  $\kappa = 25.0$  in case the user estimates that the problem might not be regular).



$\rho^1 > 0$  - the starting value of penalty coefficient (suggested values  $\rho^1 = 0.2$  or  $\rho^1 = 4.0$  in the case of possible irregularity).

$x^0 \in R^n$  - The starting point for optimization ( $x^0 \in X_0$  is not required although the better the starting guess, the more effective the algorithm).

$y^1 \in R^m$  with  $y_i^1 \geq 0$  for  $i \in I$  - the starting point for Lagrange multipliers (suggested value  $y^1 = 0$ , if no better guess is available).

The user has also the option of specifying starting accuracy parameters  $\varepsilon_x^1$  and  $\varepsilon_y^1$  (reasonable bounds on the norms of gradients in  $x$  and constraint violations after first iteration). If he does not use this option,  $\varepsilon_x^1 = (\gamma)^{-4} \varepsilon_y^f$  and  $\varepsilon_y^1 = (\gamma)^{-4} \varepsilon_y^f$  are used which implies that the algorithm will usually need five iterations (one large in the first phase and four in the second) to achieve the required final accuracy.

The following functions, sets and parameters are used in the algorithm. The augmented Lagrangian function (used also as a penalty function) is defined by:

$$(67a) \quad \Lambda(y, \rho, x) = f(x) + \frac{1}{2\rho} \left( \sum_{i \in J} \left( g_i(x) + \frac{y_i}{\rho} \right)^2 + \sum_{i \in I} \left( g_i(x) + \frac{y_i}{\rho} \right)_+^2 - \sum_{i \in J \cup I} \left( \frac{y_i}{\rho} \right)^2 \right)$$

where  $(g_i(x) + \frac{y_i}{\rho})_+ = \max(0, g_i(x) + \frac{y_i}{\rho})$ . The first-order derivatives of this function are:

$$(67b) \quad \Lambda_x(y, \rho, x) = f_x(x) + \sum_{i \in J} (\rho g_i(x) + y_i) g_{ix}(x) + \sum_{i \in I} (\rho g_i(x) + y_i)_+ g_{ix}(x)$$

$$(67c) \quad \Lambda_{yi}(y, \rho, x) = g_i(x), i \in J; \Lambda_{yi}(y, \rho, x) = (g_i(x) + \frac{y_i}{\rho})_+ - \frac{y_i}{\rho}, i \in I$$

In further description, various values of functions, gradients

and sets at a point  $(y^k, \rho^k, x^k)$  are denoted by an upper index  $k$ , e.g.,  $f(x^k) = f^k$ ,  $\Lambda_x(y^k, \rho^k, x^k) = \Lambda_x^k$ , etc. The set of currently strongly active constraints is :

$$(67d) \quad SA^k = \{i \in I : g_i^k + \frac{y_i^k}{\rho} > 0\} \cup J$$

The sets of probably active constraints and of probably strongly active constraints are :

$$(67e) \quad A^k = \{i \in I : g_i^k + \frac{y_i^k}{\rho} > -\eta_g^k\} \cup J$$

$$(67f) \quad S^k = \{i \in I : \frac{y_i^k}{\rho} > \eta_y^k\} \cup J$$

with the variable bounds  $\eta_g^k = \zeta_g(\epsilon_Y^k)^{\frac{1}{2}}$ ,  $\zeta_g = (\gamma^{-1} \epsilon_Y^1)^{\frac{1}{2}}$ ,  $\eta_y^k = \zeta_y(\epsilon_Y^k)^{\frac{1}{2}}$ ,  $\zeta_y = \frac{1}{\rho^k} (\gamma \epsilon_Y^1)^{\frac{1}{2}}$  (this results, if  $\gamma=0.1$ , in  $\eta_g^1 = 10^{\frac{1}{2}} \epsilon_Y^1$ ,  $\eta_y^1 = \frac{1}{\rho^k} 10^{-\frac{1}{2}} \epsilon_Y^1$

in the first iteration ; many inactive constraints are counted as probably active at the beginning ; in the fourth iteration,  $\eta_g^4 = 10^{-1} \epsilon_Y^1$ ,  $\eta_y^4 = \frac{1}{\rho^4} 10^{-2} \epsilon_Y^1$ , whereas  $\epsilon_Y^4 = 10^{-3} \epsilon_Y^1$  ; if the scaling of the problem is reasonable, then, for the fifth iteration, all constraints with  $g_i(\hat{x}) > -\bar{\omega} = -1.1 \cdot 10^{-1} \epsilon_Y^1$  are counted as active and all constraints with  $\hat{y}_i > \bar{\tau} = 1.1 \cdot 10^{-2} \epsilon_Y^1$  are counted as strongly active).

The current approximation of the Hessian matrix of second-order derivatives of normal Lagrangian function  $L(\bar{y}, x) = f(x) + \sum_{i \in I \cup J} \bar{y}_i g_i(x)$  is estimated at  $\bar{y} = (y^{k+\rho^k} g^k)_+$  by  $H^k$  :

$$(67g) \quad H^k \approx L_{xx}((y^{k+\rho^k} g^k)_+, x^k) = f_{xx}^k + \sum_{i \in J} (y_i^{k+\rho^k} g_i^k) g_{ixx}^k + \\ + \sum_{i \in I} (y_i^{k+\rho^k} g_i^k)_+ g_{ixx}^k$$

and the approximation  $H^k$  is constructed, without using second-order derivatives, by computing finite differences of gradients and solutions  $\{r^j, s^j\}_{j=k-n+1}^k$ :

$$(67h) \quad s^j = x^{j+1} - x^j$$

$$\begin{aligned} (67i) \quad r^j &= L_x^*((y^j + g^k g^j)_+, x^{j+1}) - L_x^*((y^j + \rho^k g^j)_+, x^j) = \\ &= f_x^{(j+1)*} - f_x^{j*} + \sum_{i \in J} (y_i^j + \rho^k g_i^j) (g_{ix}^{(j+1)*} - g_{ix}^{j*}) + \\ &+ \sum_{i \in I} (y_i^j + \rho^k g_i^j)_+ (g_{ix}^{(j+1)*} - g_{ix}^{j*}) \end{aligned}$$

The approximation  $H^k$  is obtained mainly due to the data from the first phase of the algorithm, where  $j$  denotes small iterations in unconstrained minimization of  $\Lambda(y^k, \rho^k, x^j)$ ; thus,  $y^j = y^k$  and  $\rho^k$  are constant in small iterations. Observe that all data needed to compute  $s^j$ ,  $r^j$  are available when determining  $\Lambda_x^{j+1}$ ,  $\Lambda_x^j$ , but  $r^j$  differs slightly from  $\Lambda_x^{j+1} - \Lambda_x^j$ .

The maximal number of small iterations of unconstrained minimization in a large iteration is  $\bar{K} = \frac{(20+n)n}{1+n}$ ; this is usually sufficient for a variable metric algorithm to achieve a good approximation of the minimum.

The augmenting matrix  $G^{k*} G^k : R^n \rightarrow R^n$  for the matrix  $H^k$  is:

$$(67j) \quad G^k = (g_{ix}^k)_{i \in S^k} \quad ; \quad G^{k*} G^k = \sum_{i \in S^k} g_{ix}^{k*} g_{ix}^k \quad ,$$

where  $g_{ix}^k$  are row-vectors and  $G^k : R^n \rightarrow R^{\tilde{m}^k}$ ,  $\tilde{m}^k$  being the number of probably strongly active constraints.

The algorithm proceeds as follows:

Set  $x = x^0$ ,  $y = y^1$ ,  $\rho = \rho^1$ ,  $k = 1$ ,  $j = 0$ ,  $H^k = I$ , specify other parameters. Compute  $\Lambda_x^0 = \Lambda_x(y^1, \rho^1, x^0)$ ,  $\Lambda_y^0 = \Lambda_y(y^1, \rho^1, x^0)$ .

If  $\|\Lambda_x^0\| \leq \epsilon_x^f$ ,  $\|\Lambda_y^0\| \leq \epsilon_y^f$ , stop. Otherwise:

### A. First Phase

Generate a sequence  $\{x^j\}_0^K$  starting with  $x^0 := x^{k-1}$  by minimizing the function  $\Lambda(y^k, \rho^k, x)$  in  $x$ , applying the subroutine of unconstrained minimization; compute additionally  $s^j, r^j(67h, i)$  and update  $H^k$  by the subroutine of variable metric approximation. Stop function minimization at  $x^j = x^K$  when  $\|\Lambda_x^K\| \leq \epsilon_x^k$ . If such a point cannot be reached for  $j = \bar{K}$ , then set  $\rho^k := \kappa \rho^k$  and repeat minimization. Otherwise, set  $x^k = x^K$ , compute  $\Lambda_y^k = \Lambda_y(y^k, \rho^k, x^k)$ ,  $\Lambda_x^k = \Lambda_x^K$ . If  $\|\Lambda_y^k\| > \epsilon_y^k$ , set  $y^{k+1} = y^1$ ,  $\rho^{k+1} = \kappa \rho^k$ ,  $\epsilon_y^{k+1} = \epsilon_y^k$ ,  $\epsilon_x^{k+1} = \epsilon_x^k$ ,  $k := k+1$  and repeat A. If  $\|\Lambda_x^k\| \leq \epsilon_x^f$ ,  $\|\Lambda_y^k\| \leq \epsilon_y^f$ , stop. If  $\|\Lambda_y^k\| \leq \epsilon_y^k$ , set  $y^{k+1} = (y^k + \rho^k g^k)_+$ ,  $(y_i^{k+1} = (y_i^k + \rho^k g_i^k)$  for  $i \in J$ ,  $y_i^{k+1} = (y_i^k + \rho^k g_i^k)_+$  for  $i \in I$ ),  $\rho^{k+1} = \rho^k$ ,  $x^{k+1} := x^k$ ,  $H^{k+1} = H^K$ ,  $\epsilon_y^{k+1} = \gamma \epsilon_y^k$ ,  $\epsilon_x^{k+1} = \gamma \epsilon_x^k$ ,  $\eta_g^{k+1} = \xi_g(\epsilon_y^{k+1})^{\frac{1}{2}}$ ,  $\eta_y^{k+1} = \xi_y(\epsilon_y^{k+1})^{\frac{1}{2}}$ , determine the sets  $A^{k+1}$ ,  $S^{k+1}$  and the matrix  $G^{k+1}(67e, f, j)$ , set  $k := k+1$  and go to B.

### B. Second Phase

Minimize  $(f_x^k \bar{x} + \sum_{i \in S^k} \rho^k g_i^k g_{ix}^k \bar{x} + \frac{1}{2} \langle \bar{x}, (H^k + \rho^k G^{k*} G^k) \bar{x} \rangle$  subject to  $g_i^k + g_{ix}^k \bar{x} = 0$  for  $i \in J$ ,  $g_i^k + g_{ix}^k \bar{x} \leq 0$  for  $i \in S^k/J$  to obtain  $\hat{x}^k$  and  $\hat{y}^k = y^k + \hat{x}^k$  with the help of the subroutine of quadratic programming. Compute  $\Lambda_x^{k+1} = \Lambda_x(\hat{y}^k, \rho^k, x^k + \hat{x}^k)$ ,  $\Lambda_y^{k+1} = \Lambda_y(\hat{y}^k, \rho^k, x^k + \hat{x}^k)$ . If  $\|\Lambda_x^{k+1}\| > \epsilon_x^k$  or  $\|\Lambda_y^{k+1}\| > \epsilon_y^k$ , set  $y^{k+1} = (y^k + \rho^k g^k)_+$ ,  $\rho^{k+1} = \rho^k$ ,  $x^{k+1} = x^k$ ,  $\epsilon_x^{k+1} = \epsilon_x^k$ ,  $\epsilon_y^{k+1} = \epsilon_y^k$ ,  $k := k+1$ , go to A. If  $\|\Lambda_x^{k+1}\| \leq \epsilon_x^f$ ,  $\|\Lambda_y^{k+1}\| \leq \epsilon_y^f$ , stop with  $\hat{y}^k$ ,  $\hat{x}^k = x^k + \hat{x}^k$ . If  $\|\Lambda_x^{k+1}\| \leq \epsilon_x^k$ ,  $\|\Lambda_y^{k+1}\| \leq \epsilon_y^k$ , set  $x^{k+1} = x^k + \hat{x}^k$ , determine the set  $SA^{k+1}(67d)$  with  $y^{k+1} = \hat{y}^k$ , readjust  $y_i^{k+1} = 0$  if  $i \notin SA^{k+1}$ , compute  $s^j, r^j(67h, i)$  with  $j = k$ , update  $H^k$  to  $H^{k+1}$ , set  $\epsilon_y^{k+1} = \gamma \epsilon_y^k$ ,  $\epsilon_x^{k+1} = \gamma \epsilon_x^k$ ,  $\eta_g^{k+1} = \xi_g(\epsilon_y^{k+1})^{\frac{1}{2}}$ ,  $\eta_y^{k+1} = \xi_y(\epsilon_y^{k+1})^{\frac{1}{2}}$ ,

determine the sets  $A^{k+1}$ ,  $S^{k+1}$  and the matrix  $G^{k+1}$  (67e,f,j), set  $k : k+1$  and repeat B.

Comments. The indices  $k, j$  denote large and small iterations, the latter used only in the first phase and related to unconstrained function minimization. If a large iteration in the first phase is not successful, that is, does not end with  $\|\Lambda_y^k\| \leq \epsilon_y^k$ , the penalty coefficient  $\rho^k$  is increased and the first phase is repeated. If the iteration is successful by ending with  $\|\Lambda_x^k\| \leq \epsilon_x^k$ ,  $\|\Lambda_y^k\| \leq \epsilon_y^k$ , the penalty coefficient is not increased and the algorithm proceeds to the second phase with higher accuracy requirements. An iteration of the second phase is not successful if it does not end with  $\|\Lambda_x^{k+1}\| \leq \epsilon_x^k$ ,  $\|\Lambda_y^{k+1}\| \leq \epsilon_y^k$  (the change of the iteration index is due to the fact that  $\Lambda_x, \Lambda_y$  are computed in the first phase at old  $y^k$  and actually new  $x^k$  which becomes  $x^{k+1}$ , whereas in the second phase the gradients are computed at new  $y^{k+1}, x^{k+1}$ ); in this case, new  $y^{k+1}, x^{k+1}$  are not accepted and the algorithm returns to the first phase with  $y^{k+1} = (\rho^k g^k + y^k)_+$  and  $x^{k+1} = x^k$ , without higher accuracy requirements and without increasing  $\rho^k$ . Successful iterations of the second phase are equivalent to successful iterations of the first phase since they also end with  $\|\Lambda_x^{k+1}\| \leq \epsilon_x^k$ ,  $\|\Lambda_y^{k+1}\| \leq \epsilon_y^k$ . Therefore, a switch to the first phase even after some successful iterations in the second phase followed by an unsuccessful one can be considered as an unperturbed continuation of the first phase without increasing penalty coefficients in the case of success. The first phase is thus responsible for the general convergence of the algorithm; the second phase is used in order to speed up the convergence, if possible.

If the functions  $f, g_i$  are reasonably scaled and satisfy basic assumptions, then only one large iteration of the first phase is sufficient to provide for a crude estimation of  $(\hat{y}, \hat{x})$  and an approximation  $H^k$  such that the second phase converges rapidly and stops after a few iterations. Therefore, the approximation  $H^k$  cannot be much improved in the second phase and statements related to the nearly quadratic convergence of the algorithm (see next section) are rather of theoretical value; they

might be practically relevant only if a very high accuracy is required. More important practically are the robustness of the first phase and the superlinear convergence of the second phase which is displayed even when starting from a crude approximation of  $(\hat{y}, \hat{x})$ .

## 5. CONVERGENCE PROPERTIES OF THE ALGORITHM

The problem with inequality constraints only is considered in the convergence analysis, since equality constraints are included in the set of strongly active constraints and an explicit account of them requires only technical changes of reasoning. The convergence of the first phase of the algorithm was analyzed in many papers; an excellent general analysis of the convergence properties of a class of algorithms including the first phase algorithm is given in [3], and a general theorem implying the convergence of the first phase algorithm under weaker assumptions than postulated here can be found in [21], Theorem 11. Therefore, the convergence of the first phase algorithm is not discussed here in detail and the speed of convergence of the second phase is of primary interest.

The following theorem summarizes the convergence properties of the algorithm.

*Theorem 1.* Suppose the functions  $f, g_i, i \in I$ , are twice differentiable and let  $f$  be bounded from below. Let there exist numbers  $\alpha_1$  and  $\delta_1 > 0$  such that for each  $\delta \in [0; \delta_1]$  the sets  $X_{\alpha_1 \delta} = \{x \in \mathbb{R}^n : f(x) \leq \alpha_1, \quad \|(g(x))_+\| \leq \delta\}$  are bounded and nonempty. Let  $\hat{x}$  be the (globally unique) solution of the problem (1) and  $\hat{y}$  the vector of corresponding Lagrange multipliers; let the full-rank postulate hold at  $\hat{x}$ ,  $g_{i_x}(\hat{x})$  be linearly independent for  $i \in \hat{S} \cup \hat{W} = \{i \in I : g_i(\hat{x}) = 0\}$ . Let the second-order sufficiency conditions (9) be satisfied at  $(\hat{y}, \hat{x})$ . Then:

- a) The algorithm converges,  $\lim_{k \rightarrow \infty} \|(y^k - \hat{y}, x^k - \hat{x})\| = 0$ .

b) For any desired convergence rate  $\gamma \in (0; 1)$  there exists a number  $\beta = \beta(\gamma) > 0$  and a number  $k_1$  such that, if  $k \geq k_1$  and  $\|(L_{xx}^k - H^k)\bar{x}\| \leq \beta d^k$  where  $d^k = \|(\Lambda_y^k, \Lambda_x^k)\|$  (or, in particular, if  $\|L_{xx}^k - H^k\| \leq \beta_1$ ,  $\beta_1 = \frac{\beta}{\alpha}$ , cf. Lemma 3), then only the second phase of the algorithm is used and the algorithm converges with the desired convergence rate.

c) If  $\lim_{k \rightarrow \infty} \frac{\|(L_{xx}^k - H^k)\hat{x}^k\|}{d^k} = 0$  (or, in particular, if  $\lim_{k \rightarrow \infty} \|L_{xx}^k - H^k\| = 0$ ) then the algorithm converges superlinearly.

d) If the second-order derivatives  $f_{xx}(\cdot)$ ,  $g_{ixx}(\cdot)$  for  $i \in \hat{S} \cup \hat{W}$  are Lipschitz-continuous and there exists a number  $\alpha_2 > 0$  such that  $\|(L_{xx}^k - H^k)\hat{x}^k\| \leq \alpha_2 d^{k-n+1} d^k$  where  $d^{k-n+1} = \|(\Lambda_y^{k-n+1}, \Lambda_x^{k-n+1})\|$  (or, in particular,  $\|L_{xx}^k - H^k\| \leq \alpha_2^1 d^{k-n+1}$  with  $\alpha_2^1 = \frac{\alpha_2}{\alpha}$ ), then the algorithm converges quadratically in each  $n$ -th iteration, that is, there exists  $\alpha_3 > 0$  such that  $d^{k+1} \leq \alpha_3 (d^{k-n+1})^2$  for sufficiently large  $k$ .

e) If, additionally,  $L_{xx}^k = H^k$  for all  $k$ , then the algorithm converges quadratically, that is there exists  $\alpha_4 > 0$  such that  $d^{k+1} \leq \alpha_4 (d^k)^2$  for sufficiently large  $k$ .

Before proving the theorem, some comments are relevant. The assumption of the global uniqueness of  $\hat{x}$  can be relaxed by modifying the first phase of the algorithm in such a way that it converges to a locally unique solution. The assumptions in b), c), d), related to  $\|(L_{xx}^k - H^k)\hat{x}^k\|$  are only slightly weaker than the particular assumptions related to  $\|L_{xx}^k - H^k\|$ . But many variable metric approximation methods satisfy only the first version of these assumptions and not the second. The assumption in d) that  $\|L_{xx}^k - H^k\| \leq \alpha_2^1 d^{k-n+1}$ , or its relaxed form, is related to the fact that data from at least  $n$  past iterations are necessary in order to obtain a good approximation of  $L_{xx}^k$  by  $H^k$ ; this assumption is rarely satisfied in practical applications since the second phase algorithms usually need less than  $n$  iterations to

obtain a satisfactory accuracy. The assumption in e) that  $L_{xx}^k = H^k$  is related to a possible modification of the algorithm when the user decides to program all second-order derivatives.

*Proof.* Under the assumptions of the theorem, there exists  $\bar{\rho} > 0$  such that  $\Lambda(y, \rho, x)$  has a minimum in  $x$  for each  $\rho \geq \bar{\rho}$  and each bounded  $y$  [21]. Hence the first phase of the algorithm is well defined: if  $\rho^k$  is too small, it is increased until  $\rho^k \geq \bar{\rho}$ . If the algorithm converges, then also the second phase of the algorithm is well defined, since the approximative quadratic programming problem has a solution if  $(y^k, x^k)$  is sufficiently close to  $(\hat{y}, \hat{x})$ , see Section 3.

To prove a), assume that b) does not hold (since b) clearly implies a)). Then the sequence  $\{(y^k, x^k)\}_0^\infty$  consists of two subsequences. Subsequence  $\{(y^{k'}, x^{k'})\}_0^\infty$  is generated by the first phase algorithm: either  $\|\Lambda_x^{k'}\| \leq \epsilon_x^{k'}, \|\Lambda_y^{k'}\| \leq \epsilon_y^{k'}$  with strictly decreasing  $\epsilon_x^{k'}, \epsilon_y^{k'}$ , in which case  $\rho^{k'}$  is not increased, or  $\|\Lambda_x^{k'}\| \leq \epsilon_x^{k'}, \|\Lambda_y^{k'}\| > \epsilon_y^{k'}$ , in which case  $\epsilon_x^{k'}, \epsilon_y^{k'}$  are not decreased and  $\rho^{k'}$  is increased until  $\|\Lambda_x^{k'}\| \leq \epsilon_x^{k'}, \|\Lambda_y^{k'}\| \leq \epsilon_y^{k'}$  for some large  $\rho^{k'}$ ; in both cases,  $y^{k'+1} = (y^{k'} + \rho^{k'} g^{k'})_+$ . Subsequence  $\{(y^{k''}, x^{k''})\}_0^\infty$  is generated by the successful iterations of the second phase algorithm, with  $\|\Lambda_x^{k''+1}\| \leq \epsilon_x^{k''}, \|\Lambda_y^{k''+1}\| \leq \epsilon_y^{k''}$  and strictly decreasing  $\epsilon_x^{k''}, \epsilon_y^{k''}$ ; the unsuccessful iterations of the second phase are annulated by resetting to the last  $(y^{k''}, x^{k''})$  defining  $y^{k''+1} = (y^{k''} + \rho^{k''} g^{k''})_+$  and returning to the first phase. Therefore, the convergence of the first subsequence  $\{(y^{k'}, x^{k'})\}$  is sufficient for the convergence of the entire sequence, since the second subsequence converges if the first one does.

The convergence of the first subsequence could be deduced from [3] where, however,  $\hat{W}A = \emptyset$  was assumed. Therefore, it is better to use Theorem 11 from [21] which implies that  $\lim_{k \rightarrow \infty} \|x^{k'} - \hat{x}\| = 0$  under actually weaker assumptions. To show that  $\lim_{k \rightarrow \infty} \|y^{k'} - \hat{y}\| = 0$  observe that  $\lim_{k' \rightarrow \infty} \|\Lambda_x^{k'}\| = 0, \lim_{k' \rightarrow \infty} \|\Lambda_y^{k'}\| = 0$  with some



constant  $\rho^{k'}$  imply  $\lim_{k' \rightarrow \infty} \|y^{k'} - \hat{y}\| = 0$  by Lemma 2. Therefore, consider the case when  $\lim_{k' \rightarrow \infty} \|\Lambda_x^{k'}\| = 0$  and  $\lim_{k' \rightarrow \infty} \rho^{k'} = +\infty$ . In this case, the iteration  $y_i^{k'+1} = (y_i^{k'} + \rho^{k'} g_i^{k'})_+$  implies that there exists such  $k'$ , that  $y_i^{k'+1} > 0$  only for  $i \in \hat{S} \cup \hat{W}$ , since for  $i \in \hat{I} \cup \hat{N}$ ,  $g_i^{k'} < 0$  for all  $x^{k'}$  in a sufficiently small neighborhood of  $\hat{x}$ . Moreover,  $\Lambda_x^{k'} = f_x^{k'} +$   
 $+ \sum_{i \in \hat{S} \cup \hat{W}} (y_i^{k'} + \rho^{k'} g_i^{k'}) g_{ix}^{k'} = f_x^{k'} + \sum_{i \in \hat{S} \cup \hat{W}} y_i^{k'+1} g_{ix}^{k'}.$

Hence :

$$(68a) \Lambda_x^{k'} = \Lambda_x^{k'} - \Lambda_x(y, \rho^{k'}, \hat{x}) = (f_x^{k'} - f_x(\hat{x})) +$$

$$+ \sum_{i \in \hat{S} \cup \hat{W}} y_i (g_{ix}^{k'} - g_{ix}(\hat{x}))$$

$$+ \sum_{i \in \hat{S} \cup \hat{W}} (y_i^{k'+1} - y_i) g_{ix}^{k'}$$

Both the left-hand side and the two first terms of right-hand side converge to zero as  $k \rightarrow \infty$ . Therefore, the last term of right-hand side also converges to zero. Since  $g_{ix}^{k'}$  are linearly independent for  $i \in \hat{S} \cup \hat{W}$  and  $x^{k'}$  sufficiently close to  $x$ , the coefficients  $y_i^{k'} - \hat{y}_i$  must all converge to zero for  $i \in \hat{S} \cup \hat{W}$ . Thus,  $\lim_{k' \rightarrow \infty} \|y^{k'} - \hat{y}\| = 0$  and the algorithm converges.

To prove b), it is sufficient to examine a close neighborhood of  $(\hat{y}, \hat{x})$  such that the conclusions of Lemmas 3,4 are valid. Lemma 3 implies that there exist an  $\alpha > 0$  such that :

$$(68b) \|(\hat{y}^k, \hat{x}^k)\| \leq \alpha d^k$$

and the conclusions of Lemma 4 can be written together as

$$(68c) d^{k+1} \leq \|(L_{xx}^k - H^k) \hat{x}^k\| + o(\hat{y}^k, \hat{x}^k)$$

Since the function  $o(\hat{\bar{y}}^k, \hat{\bar{x}}^k)$  in (68c) can be assumed to be non-negative which together with (68b) implies that :

$$(68d) \quad 0 \leq \frac{o(\bar{y}, \bar{x})}{d^k} \leq \alpha \frac{o(\hat{\bar{y}}, \hat{\bar{x}})}{\|(\hat{\bar{y}}, \hat{\bar{x}})\|} ; \lim_{d^k \rightarrow \infty} \frac{o(\hat{\bar{y}}, \hat{\bar{x}})}{\|(\hat{\bar{y}}, \hat{\bar{x}})\|} = 0 ,$$

therefore,  $o(\hat{\bar{y}}, \hat{\bar{x}}) = o(d^k)$  where  $o(d^k)$  is another function of the of the same property  $\lim_{d^k \rightarrow \infty} \frac{o(d^k)}{d^k} = 0$ . Thus, (68c) implies

$$(68e) \quad \frac{d^{k+1}}{d^k} \leq \frac{\|(L_{xx}^k - H^k) \hat{\bar{x}}^k\|}{d^k} + \frac{o(d^k)}{d^k}$$

For any  $\eta \in (0 ; \gamma)$ , an  $\varepsilon^k > 0$  can be chosen such that  $\frac{o(d^k)}{d^k} \leq \eta$  for all  $d^k \in [0 ; \varepsilon^k]$ . Thus, if  $\varepsilon_x^k + \varepsilon_y^k \leq \varepsilon^k$  and if  $\beta = \gamma - \eta$  is chosen, that is, if :

$$(68f) \quad \|(L_{xx}^k - H^k) \hat{\bar{x}}^k\| \leq \beta d^k = (\gamma - \eta) d^k \leq \gamma d^k - o(d^k)$$

then :

$$(68e) \quad \frac{d^{k+1}}{d^k} \leq \beta + \frac{o(d^k)}{d^k} \leq \gamma$$

Clearly, if  $d^k \in [0 ; \varepsilon^k]$  and  $\gamma \in (0 ; 1)$ , then, by induction,  $\frac{d^{k''+1}}{d^{k''}} \leq \gamma$  for all  $k'' \geq k$ , which proves b) ; the strengthened assumption  $\|L_{xx}^k - H^k\| \leq \beta_1$  with  $\beta_1 = \frac{\beta}{\alpha}$  and with (68b) imply the relaxed assumption  $\|(L_{xx}^k - H^k) \hat{\bar{x}}^k\| \leq \beta d^k$ .

To prove c) it is sufficient to note that (68e) under the assumptions  $\lim_{k \rightarrow \infty} \frac{\|(L_{xx}^k - H^k) \hat{\bar{x}}^k\|}{d^k} = 0$ ,  $\lim_{k \rightarrow \infty} d^k = 0$  implies  $\lim_{k \rightarrow 0} \frac{d^{k+1}}{d^k} = 0$ .

To prove d) it is necessary to have a stronger estimation of  $o(d^k)$ . If  $f_{xx}(\cdot)$  and  $g_{ixx}(\cdot)$  are Lipschitz-continuous, then it can be shown as a corollary to Lemma 4 that there exists a number

$\lambda > 0$  such that  $o(\hat{y}^k, \hat{x}^k) \leq \lambda \|(\hat{y}^k, \hat{x}^k)\|^2$ . It follows from Lemma 3 that  $o(d^k) \leq \lambda \alpha^2 (d^k)^2$  in (68e). Hence, under the assumptions of point d) in the theorem :

$$(69a) \quad d^{k+1} \leq \alpha_2 d^{k-n+1} d^k + \lambda \alpha^2 (d^k)^2$$

This inequality suggests actually a higher order of convergence than the number  $2^{\frac{1}{n}}$  related to quadratic convergence in each n-th iteration. However, without examining (69a) more closely, observe that  $d^k \leq d^{k-n+1}$  and

$$(69b) \quad d^{k+1} \leq \alpha_3 (d^{k-n+1})^2 ; \alpha_3 = \alpha_2 + \lambda \alpha^2$$

If  $L_{xx}^k = H^k$  and  $o(d^k) \leq \lambda \alpha^2 (d^k)^2$ , then

$$(70) \quad d^{k+1} \leq \alpha_4 (d^k)^2 ; \alpha_4 = \lambda \alpha^2 ,$$

which proves the last point of the theorem.

It should be noted that the points b), c), d), e) of Theorem 1 are typical for quasi-Newton methods of optimization and, once Lemmas 2, 3, 4 are proved, the conclusions and proof of Theorem 1 are natural.

## 6. CONCLUSIONS

The algorithm presented in this paper will be coded and tested in collaboration with the Institute of Automatic Control, Technical University of Warsaw; the resulting programs and their description will be included in the library of optimization algorithms at IIASA computer center. Because the algorithm combines and extends application domain of two known robust and efficient algorithms of nonlinear programming, it is hoped that the resulting programs will be widely applicable and reliable. Applications to several problems in health care, management and technology

and other areas in IIASA are also planned.

From the theoretical point of view, the main results of this paper are the analysis of second-order approximations to the augmented Lagrange function in the case when this function is not twice differentiable, that is, without the assumption of strict complementarity, and the explanation why a quadratic approximation method can work even if the optimization problem is not convex and the assumption of strict complementarity at the solution is not satisfied. Crucial role, however, is played by the full rank postulate which guarantees the uniqueness of Lagrange multipliers. It is an interesting question whether and how this postulate can be relaxed.

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