SOME CONDITIONS FOR OPTIMAL DETERMINISTIC SOLUTIONS TO STOCHASTIC DYNAMIC LINEAR PROGRAMS

John R. Birge

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

PREFACE

This paper represents the results of a three month study, in which several Junior Scientists from many countries took part during the summer of 1979 at IIASA. While many of these results are not fully completed, and some represent only preliminary directions of research, we feel that the documentation of the efforts of the Junior Scientists is justified.

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ABSTRACT

Many problems that require decisions made over time can be formulated as dynamic linear programs. Complications arise in solving these programs when one allows stochastic elements to alter the state to state transitions. Finding the stochastic linear programming solutions may be very difficult since their formulation often greatly increases the problem size. This paper shows that, under certain conditions, a simple deterministic solution technique obtains the same optimal controls as more complicated stochastic methods.

Key words: Dynamic linear programming, stochastic programming, large scale systems.

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I. INTRODUCTION AND PROBLEM DESCRIPTION

Dynamic linear programming problems occur in a variety of applications. They entail optimal control decisions made over time. Complications arise when some stochastic variation occurs in the transition of the process to subsequent states. In general, complicated stochastic programming methods are required to solve these problems optimally. In some instances, however, a deterministic approach involving expected values of the stochastic elements is sufficient. We will show below conditions for this result.

We write the basic dynamic linear programming problem in the following form:

$$\min \sum_{t=0}^{T-1} [c(t)x(t) + d(t)u(t)] + c(T)x(T)$$

$$\text{s.t.} \quad G(t)x(t) + D(t)u(t) = f(t) \qquad \text{(I.1)}$$

$$A(t)x(t) + B(t)u(t) = x(t+1)$$

$$\text{for } t = 0,1,\ldots, T-1 \quad ,$$

where $x(t) \in \mathbb{R}^{n(t)}$, a vector in $n(t)$-dimensional Euclidean space, $u(t) \in \mathbb{R}^{p(t)}$, and $f(t) \in \mathbb{R}^{m(t)}$, $c(t)$ and $d(t)$ are correspondingly dimensioned vectors, and $G(t)$, $D(t)$, $A(t)$ and $B(t)$ are corresponding matrices. In this problem, $x(t)$ represents the state of the system at time t and $u(t)$ represents the optimal control applied at that time. We, therefore, wish to minimize a linear cost function of these variables over time.

Problems occur in this system when we introduce a stochastic variation $v(t)$, for some $v(t) \in V(t)$, where $V(t) \subset \mathbb{R}^{q(t)}$. We consider that this error or noise term enters the state transition equation as:

$$A(t)x(t) + B(t)u(t) + E(t)v(t) = x(t+1) \quad , \qquad \text{(I.2)}$$

where $E(t)$ is a corresponding given non-stochastic matrix. The problem is then how to determine the optimal controls in order to allow for this stochastic element. The best possible solution would be to know the outcome of the stochastic variations through time. The object then is to solve the problem:

$$\begin{aligned} J_1(v(0),\ldots, v(T-1)) \equiv \min \sum_{t=0}^{T-1} [c(t)x(t)+d(t)u(t)] + c(T)x(T) \\ \text{s.t.} \quad G(t)x(t) + D(t)u(t) = f(t) \\ A(t)x(t) + B(t)u(t) + E(t)v(t) = x(t+1) \\ \text{for } t = 0,1,\ldots, T-1 \quad , \end{aligned} \qquad \text{(I.3)}$$

for every realization $(v(0),\ldots, v(T-1))$. From these solutions, one could take an expected value of the different $J(v(0),\ldots,v(T-1))$ values and find the best possible expected objective function value as

$$\bar{J}_1 \equiv \int_{V(0)\times\cdots\times V(T-1)} J_1(v(0),\ldots, v(T-1))\,dF(v(0),\ldots, v(T-1)) \quad , \text{(I.4)}$$

where $F(v(0),\ldots, v(T-1))$ is the joint distribution function of the stochastic elements.

This approach of perfect information is not implementable because of our assumption that the stochastic variations cannot be observed before the control in a period has been applied. We may, however, assume that, at any stage of the process, we are able to observe the past. We can, thereby, use a backwards inductive method of solution in order to find an optimal control trajectory. We start by solving:

$$J(T,x(T)) \equiv c(T)x(T) \quad . \tag{I.5}$$

We then continue to iterate backward by solving for every t:

$$J(t-1,x(t-1)) \equiv \min c(t-1)x(t-1) + d(t-1)u(t-1) + \int_{V(t-1)} J(t,x(t))\,dF(v(t-1)) \quad , \tag{I.6}$$

$$\text{s.t.}\quad \begin{aligned} &G(t-1)x(t-1) + D(t-1)u(t-1) = f(t-1) \\ &A(t-1)x(t-1) + B(t-1)u(t-1) + E(t-1)v(t-1) = x(t) \quad . \end{aligned} \tag{I.6a}$$

In (I.6), the constraint (I.6a) implicitly enters the integral, so that $x(t)$ is a function of $v(t-1)$. This program finds the lowest expected remaining cost, given that we are in state $x(t-1)$ at $t-1$. This standard dynamic programming problem gives us the value:

$$J_2 \equiv J(0,x(0)) \quad . \tag{I.7}$$

If we consider the controls involved in solving the problem by this method for different realizations of $(v(0),\ldots, v(T-1))$, we obtain $J_2(v(0), v(1),\ldots, v(T-1))$ for every $(v(0), v(1),\ldots, v(T-1)) \in V(0) \times V(1)\cdots \times V(T-1)$. The expected value is then:

$$\bar{J}_2 \equiv \int_{V(0)\times\cdots\times V(T-1)} J_2(v(0),\ldots, v(T-1))dF(v(0),\ldots, v(T-1)) \quad . \qquad (I.8)$$

The method employed in finding J_2 yields an excellent expected solution value, but the solution of problems in the form of (I.6) are extremely difficult since x(t) depends on both v(t-1) and u(t-1). For general distributions of v(t-1), the objective value function represents a complicated integral formula. Linear programming methods cannot, therefore, be applied to this problem with a non-linear objective function. By applying a discrete distribution for each v(t), (an approximation of the actual distribution), the problem can, however, be transformed into a stochastic linear program. We assume, for this next approach, that v(t) is independent of v(τ) for all $\tau \neq t$. We also assume the following probability distribution for each t and some $x \in \mathcal{R}^{q(t)}$:

$$\mathcal{P}\{v(t) = x\} = \begin{cases} p_1(t) & \text{if } x = v_1(t) \\ p_2(t) & \quad x = v_2(t) \\ \vdots & \quad \vdots \\ p_k(t) & \quad x = v_k(t) \\ 0 & \text{all other } x \quad . \end{cases} \qquad (I.9)$$

We assume further, without loss of generality, that k is the same for all t.

(I.6) becomes, according to this distribution:

$$J^*((t-1),x(t-1)) \equiv \min c(t-1)x(t-1) + d(t-1)u(t-1) + \sum_{i=1}^{k} p_i(t-1)c(t)x_i(t)$$

$$\text{s.t.} \quad G(t-1)x(t-1) + D(t-1)u(t-1) = f(t-1) \qquad (I.10)$$

$$A(t-1)x(t-1) + B(t-1)u(t-1) + E(t-1)v_i(t-1) = x_i(t) \quad ,$$

$$\text{for } i = 1,\ldots, k \quad .$$

The solution to the problem of optimal control over the entire planning horizon is, therefore, $J^*(0,x(0))$, where $x(0)$ is some given initial state. Solution by the iterative dynamic programming technique may be quite complicated, however, because we must find $J^*(t,x(t))$ for every possible $x(t)$ at every point in time t. This is especially difficult since $x(t)$ is not even discrete. The following theorem allows us to consider instead a single linear program.

Theorem 1. The problem of finding $J^*(0,x(0))$ derived above is equivalent to:

$$J_3 \equiv \min c(0)x(0) + d(0)u(0) + \sum_{i_0} p_{i_0}(0)[c(1)x_{i_0}(1)+d(1)u_{i_0}(1)]$$

$$+ \sum_{i_1,i_0} p_{i_0,i_1}(1)[c(2)x_{i_0,i_1}(2)+d(2)u_{i_0,i_1}(2)]$$

$$+ \cdots$$

$$+ \sum_{i_{T-1},\ldots,i_0} p_{i_0,\ldots,i_{T-1}}(T-1)\left[c(T)x_{i_0,\ldots,i_{T-1}}(T)\right]$$

s.t.

$$G(0)x(0) + D(0)u(0) = f(0)$$

$$A(0)x(0) + B(0)u(0) + E(0)v_{i_0}(0) = x_{i_0}(1) \qquad \text{(I.11)}$$

$$i_0 = 1,\ldots, k$$

$$G(1)x_{i_0}(1) + D(1)u_{i_0}(1) = f(1)$$

$$i_0 = 1,\ldots, k$$

$$A(1)x_{i_0}(1) + B(1)u_{i_0}(1) + E(1)v_{i_1}(1) = x_{i_0,i_1}(1)$$

$$i_0 = 1,\ldots, k$$

$$i_1 = 1,\ldots, k$$

$\vdots$

$$G(T-1)x_{i_0 \cdots i_{T-2}}(T-1) + B(T-1)u_{i_0 \cdots i_{T-2}}(T-1) = f(T-1)$$

$$i_0 = 1,\ldots,k$$
$$\vdots$$
$$i_{T-2} = 1,\ldots,k$$

$$A(T-1)x_{i_0 \cdots i_{T-2}}(T-1) + B(T-1)u_{i_0 \cdots i_{T-2}}(T-1)$$
$$+E(T-1)v_{i_{T-1}}(T-1) = x_{i_0 \cdots i_{T-1}}(T)$$

$$i_0 = 1,\ldots,k$$
$$\vdots$$
$$i_{T-1} = 1,\ldots,k \quad ,$$

where $p_{ijk}(3)$ represents the probability of events $(v_i(1), v_j(2), v_k(3))$ occurring. Here, we do not necessarily assume independence. If independence is present, then we have $p_{ijk}(3) = p_i(1) \cdot p_j(2) \cdot p_k(3)$.

Proof. The proof follows directly by induction on T, the number of periods.

This characterization, because it does not require independence, is more general than the dynamic programming solution in finding $J^*(0,x(0))$. It is also more easily implemented since each state need not be specified.

Again, if we solve (I.11) and find J_3, we use the given controls and obtain different objective values for different realizations of $(v(0),\ldots, v(T-1))$. The expected value is then

$$\bar{J}_3 \equiv \int_{V(0)\times\cdots\times V(T-1)} J_3(v(0),\ldots,v(T-1))\,dF(v(0),\ldots,v(T-1)) \quad .$$

In this multistage stochastic linear program, many blocks, non-zero partitions of the program matrix corresponding to different realizations of $v(t)$, appear. The number of separate blocks increases exponentially with the number of periods. This complication makes problems with a great number of transitions very difficult to solve. One, therefore, usually requires that the blocks are aggregated or that expected values are substituted for the assumed distribution. The most simplified approach would be to consider only expected values for each of the stochastic variables, v(t).

The resultant deterministic problem can then be written simply as:

$$J_4 \equiv \min \sum_{t=0}^{T-1} [d(t)u(t)+c(t)x(t)] + c(T)x(T)$$

$$\text{s.t.} \quad G(t)x(t) + D(t)u(t) = f(t) \tag{I.12}$$

$$A(t)x(t) + B(t)u(t) + E(t)\bar{v}(t) = x(t+1)$$

$$\text{for } t = 0,1,\ldots,T-1 \quad ,$$

where $\bar{v}(t) = \int_{V(t)} v(t)dF(v(t))$.

Again, we take expected values for actual realizations of v(t) to find:

$$\bar{J}_4 \equiv \int_{V(0)\times\cdots\times V(T-1)} J_4(v(0),\ldots,v(T-1))dF(v(0),\ldots,v(T-1)) \quad .$$

A hierarchy exists among the four solutions to the stochastic linear programming problem posed here. The following theorem establishes this.

Theorem 2. The optimal values for solutions to problems such as (I.1) with stochastic transition equation (I.2) are ordered as:

$$\bar{J}_1 \leq \bar{J}_2 \leq \bar{J}_3 \leq \bar{J}_4 \quad , \tag{I.13}$$

where V(t) is assumed convex for all t.

Proof. The inequalities follow by observing that the successive complications from $\bar{J}_4$ to $\bar{J}_1$ involve inclusions of the previous solution. The first inequality, $\bar{J}_1 \leq \bar{J}_2$, follows from our use of the optimal solution for any realization, $(v'(0),\ldots,v'(T-1))$ in J_1. Hence, $J_1(v'(0),\ldots,v'(T-1)) \leq J_2(v'(0),\ldots,v'(T-1))$ for any $(v'(0),\ldots,v'(T-1))$. Integration preserves the inequality, so $\bar{J}_1 \leq \bar{J}_2$.

Since V(t) is assumed convex, in J_3, the discrete approach is, at best, an approximation. By definition, therefore, the solutions by $\bar{J}_2$ are always better. Hence, $\bar{J}_2 \leq \bar{J}_3$.

For the remaining inequality, observe that $\bar{v}(t)$ is included in any $\bar{J}_3$ solution because it is in V(t) and is a member of the discrete approximation for $\bar{J}_3$. If $\bar{v}(t)$ is realized, one optimizes in $\bar{J}_3$. This is the only realization, for which, the solution in $\bar{J}_4$ must be optimal. For all other v'(t), we have $J_3(v'(0),\ldots,v'(T-1)) \leq J_4(v'(0),\ldots,v'(T-1))$. Again, integration yields $\bar{J}_3 \leq \bar{J}_4$. We then have $\bar{J}_1 \leq \bar{J}_2 \leq \bar{J}_3 \leq \bar{J}_4$. ||

The question of choosing which of the above four solutions to use depends on the complexity of the problem, the difficulty of using the various techniques, and the actual differences that may occur in the inequalities. If, for example, one considered a problem, for which, $\bar{J}_1 = \bar{J}_4$, the value of perfect information is zero and a deterministic solution technique is adequate and recommended.

II. CONDITIONS FOR OPTIMAL DETERMINISTIC SOLUTIONS

The solution to dynamic linear programming problems usually seeks an optimal control for the entire planning horizon, [0,T]. This solution can, however, usually be altered after a certain period of time. By following this procedure, one can observe the behavior of stochastic variables in this first period and use the information to make better projections for the future. The problem in this framework becomes one of finding the optimal first period control, given future controls and future uncertainties. An entire optimal control trajectory is found, but only the first period control must be implemented before one allows for a changing environment. This method appears well adapted to real world applications of optimal decision making over time.

Within this repeated solution technique, one may still have difficulties in finding the first period control because of the large number of possibilities for future controls and the first period's dependence on this future. We will give conditions, under which, the first period controls can be found optimally by a deterministic approach as in (I.12). In other words, we have the same u(0) controls for J_1 and J_4, and need only solve deterministic problems over time in order to find the best possible control trajectory.

The following lemma will be used in finding these conditions for a deterministic optimal control solution. It follows from sensitivity analysis on the standard primal linear program:

$$\begin{aligned} &\min \quad cx \\ &\text{s.t.}\ Ax = b \\ &\qquad x \geq 0 \ . \end{aligned} \tag{II.1}$$

Lemma 1. If B is an optimal basis for (II.1) and if B remains feasible for all possible right hand side variations, then B will remain an optimal basis.

Proof. We partition the matrix A and cost vector c into basic and non-basic parts. (II.1) becomes:

$$\begin{aligned} &\min c_B x_B + c_N x_N \\ &\text{s.t.} \\ &\quad Bx_B + Nx_N = b \\ &\quad x_B \geq 0,\ x_N \geq 0 \ . \end{aligned} \tag{II.2}$$

Now, if B is an optimal feasible basis for some $\bar{b}$, then we have associated prices, π, such that

$$\pi B = c_B$$
$$\pi N \leq c_N \quad , \qquad \text{(II.2)}$$

and we have

$$x_B = B^{-1}\bar{b} \geq 0 \quad , \quad x_N = 0 \quad . \qquad \text{(II.3)}$$

If x_B remains feasible for $\bar{b} + \Delta b$, i.e., if $B^{-1}\bar{b}+B^{-1}\Delta b \geq 0$, then the prices π remain unchanged and the optimality conditions (II.2) and (II.3) remain also. B is, therefore, still an optimal basis.||

This lemma leads to a theorem for the optimal basis in a stochastic linear program. For this general program, we let b in (II.1) be b (ξ), a random variable, where $\xi \in \Xi$.

<u>Theorem 3</u>. If B is a feasible basis for (II.1) for any $b(\xi)$, $\xi \in \Xi$, then B is an optimal basis for all $b(\xi)$, $\xi \in \Xi$.

<u>Proof</u>. Apply Lemma 1 directly to the problem (II.1) with constraints

$$Ax = b(\xi)$$
$$x \geq 0 \quad .$$

Now, by the assumption, no variation in ξ will make $Bx^B = b(\xi)$ infeasible. Therefore, by Lemma 1, B is an optimal basis for all $b(\xi)$, $\xi \in \Xi$.||

This last result gives conditions under which the optimal basis for every realization $(v(0),\ldots, v(T-1))$ in (I.3) will be the same. We write the optimal basis for (I.3) as

$$
\begin{array}{ccccc}
u^B(0) & x^B(1) & u^B(1) & \cdots & x^B(T) \\
\end{array}
$$

$$
\left[\begin{array}{ccccc}
D^B(0) & & & & \\
B^B(0) & -I^B(2) & & & \\
 & G^B(0) & D^B(1) & & \\
 & A^B(1) & B^B(1) & -I^B(2) & \\
 & & & \ddots & -I^B(T)
\end{array}\right]
\begin{array}{l}
= f(0)-G(0)x(0) \\
= -A(0)x(0)-E(0)v(0) \\
= f(1) \\
 \\
= -E(T-1)v(T-1)
\end{array}
\qquad (II.4)
$$

Inverting this matrix gives unique values for the basic variables for each realization $(v(0),\ldots, v(T-1))$. The objective value is then

$$J_1 = \alpha(0)v(0)+\cdots+\alpha(T-1)v(T-1) + k \quad ,$$

where $\alpha(0),\ldots, \alpha(T-1)$ and k are constant over ranges of $(v(0),\ldots, v(T-1))$ if the same basis remains. Therefore, if the basis remains unchanged, from integration in (I.4),

$$\bar{J}_1 = \alpha(0)\bar{v}(0)+\cdots+ \alpha(T-1)\bar{v}(T-1) + k \quad . \qquad (II.5)$$

Now, if $\bar{v}(t) \in V(t)$ for all t, then an optimal solution of (I.12) gives us the same value for J_4 as (II.5), since the optimal basis is the same, implying the same weights $\alpha(0),\ldots, \alpha(T-1)$ and constant k. We then have the following corollary to Theorem 3.

<u>Corollary 1</u>. If B is a feasible basis for every $(v(0),\ldots, v(T-1)) \in (V(0),\ldots, V(T-1))$ in (I.3) and if B is optimal for some $(\bar{v}(0),\ldots, \bar{v}(T-1)) \in (V(0),\ldots, x(T-1))$, then $\bar{J}_1 = J_4$.

The equality would imply that using the expected values of stochastic variables and a deterministic solution would be optimal. We note, however, that implementation of the entire deterministic control program may be infeasible. Different realizations of the stochastic elements may lead to this infeasibility. We would therefore like to find conditions for which the optimal controls are independent of the stochastic elements. Otherwise, an infeasible value may result from a control that is a function of an expected value, i.e., when $u_B(0)(\bar{v}(0)) \neq u_B(0)(\bar{v}(0) + \varepsilon)$.

To this end, we have another corollary:

Corollary 2. If B is a feasible basis for every $(v(0),\ldots, v(T-1)) \in (V(0),\ldots, V(T-1))$ in (I.3), and, if B is optimal for some $(\bar{v}(0),\ldots, \bar{v}(T-1)) \in (V(0),\ldots, V(T-1))$, then a set of optimal first period controls $\bar{u}_B(0)$ does not depend on $(v(0),\ldots, v(T-1))$.

Proof. We consider a set $\{v_i(0),\ldots, v_i(T-1)\}$ for $i = 1,\ldots,k$, as realizations of $v(t)$ in problem (I.10). Since B is feasible for all $(v(0),\ldots, v(T-1))$, we obtain a feasible basis for (I.11) as

$$\begin{array}{c|cccc}
D^B(0) & 0 & & & \\
\hline
B^B(0) & G_B(1) & 0 & & \\
0 & & & & \\
B^B(0) & 0 & G_B(2) & 0 & \\
0 & & & & \\
\vdots & & & \ddots & \\
B^B(0) & & & & G_B(k) \\
0 & & & &
\end{array} \qquad \text{(II.6)}$$

$$\text{where} \quad \begin{array}{c|c}
D^B(0) & 0 \\
\hline
B^B(0) & G_B(i) \\
0 &
\end{array} = \boxed{B} \quad \text{for all } i .$$

The result is equivalent to showing that $D^B(0)$ in (II.6) is square $(m(0) \times m(0))$. We let $D^B(0)$ be $(m(0) \times m)$, where $m \geq m(0)$ in order to satisfy all inequalities for $f(0) - G(0)x(0)$ in (II.4).

Since B is square, we assume it is $[m(0) + n] \times [m(0) + n]$. The basis in (II.6) is $[m(0) + kn] \times [m(0) + kn]$. $G_B(i)$ is $n \times n(0)$. If $m = m(0) + \ell$, for $\ell > 0$, then $n(0) = n - \ell$. This would mean the basis in (II.6) is $[m(0) + kn] \times [m(0)+\ell+kn-k\ell]$, a contradiction for $k > 1$. Therefore, $D^B(0)$ is m by m, and $u^B(0) =$ $= [D^B(0)]^{-1}(f(0) - G(0)x(0))$ is independent of $(v(0),\ldots, v(T-1))$. By Corollary 1, these are optimal.||

We had to specify that the entire basis B was feasible above. Below, we only fix the first period controls and consider feasibility from there. This seems more realistic, given that we do not know what we will do in the future.

We consider the 2-stage stochastic linear program, (I.11) with T = 1. We do not consider any fixed distribution in writing (I.11). Any discrete approximation is allowed. In other words, if the solutions for J_1 and J_2 are impossible to find, then we let (I.11) be the best possible solution.

The following theorem shows the 2-stage equivalence of a stochastic and deterministic program:

Theorem 4. If the basic control values $\bar{u}(0)$ are feasible for all $v(0) \in V(0)$ in (I.3) where T = 1 and x(0) is fixed, and if $\bar{u}^B(0)$ are optimal basic values for some $\bar{v}(0) \in V(0)$, then $\bar{u}(0)$ are optimal basic values for any characterization of the 2-stage stochastic linear program in (I.11). [Here, "characterization" refers to any discrete approximation of the distribution of v(0)].

Proof. We assume (I.11) has the form

$$\min\ (c(0)x(0)) + d(0)u(0) + \sum_{i=1}^{k} p_i c(1)x_i(1)$$

$$\text{s.t.}\quad D(0)u(0) \qquad\qquad = f(0)-G(0)x(0)$$

$$\begin{bmatrix} B(0)u(0) & - Ix_1(1) & & \\ B(0)u(0) & & -Ix_2(1) & \\ \vdots & & & \ddots \\ B(0)u(0) & & & & -Ix_k(1) \end{bmatrix} \begin{array}{l} = -E(0)v_1(0) \\ \quad -A(0)x(0) \\ = -E(0)v_2(0) \\ \quad -A(0)x(0) \\ \quad \vdots \\ = -E(0)v_k(0) \\ \quad -A(0)x(0) \end{array} \qquad (II.7)$$

where $x(0)$ is given.

Next, we assume $u'(0)$ is optimal for (II.7). For any $x_i(1)$, we have

$$\begin{aligned} D(0)u'(0) &= f(0) - G(0)x(0) \\ B(0)u'(0) - Ix_i(0) &= -E(0)v_i(0) \\ &\quad -A(0)x(0) \end{aligned} \qquad (II.8)$$

as a feasibility condition. Now, since $\bar{v}(1) = v_i(1)$ for some i, (II.8) is true for $v_i(1) = \bar{v}(1)$. But $\bar{u}(0)$ is optimal here, so

$$d(0)\bar{u}(0) + c(1)\bar{x}(1) \le d(0)u'(0) + c(1)\bar{x}'(1) \quad , \qquad (II.9)$$

where $\bar{x}(1) = E(0)\bar{v}(0) + A(0)x(0) + B(0)\bar{u}(0)$ and $\bar{x}'(1) = E(0)\bar{v}(0) + A(0)x(0) + B(0)u'(0)$. (II.9) is, therefore, equivalent to

$$d(0)\bar{u}(0) + c(1)B(0)\bar{u}(0) \le d(0)u'(0) + c(1)B(0)u'(0) \quad . \qquad (II.10)$$

From (II.10) we have

$$\begin{aligned} & p_i(d(0)\bar{u}(0)+c(1)[B(0)\bar{u}(0)+A(0)x(0)+E(0)v_i(0)]) \\ & \le p_i(d(0)u'(0)+c(1)[B(0)u'(0)+A(0)x(0)+E(0)v_i(0)]) \quad , \end{aligned} \qquad (II.11)$$

and summation and the substitution, $x_i(1)=E(0)v_i(0)+A(0)x(0)+B(0)u(0)$, yields

$$d(0)\overline{u}(0) + \sum_{i=1}^{k} p_i c(1)\overline{x}_i(1) \leq d(0)u'(0) + \sum_{i=1}^{k} p_i c(1)x'_i(1), \quad \text{(II.12)}$$

where $\overline{x}_i$ and x'_i are feasible by assumption for (II.7).

Therefore, $\overline{u}(0)$ are optimal basic values for any distribution approximation in (I.11).||

The significance of this theorem is that, if one knows that a given solution will not give infeasible results in the next periods, then one need only solve a deterministic problem, in which, the stochastic element has been replaced by an expected value. The solution found in this manner will then be as good as any stochastic programming solution in finding the best first period controls. Problems, of course, arise if the first period controls do lead to future infeasibilities.

It would also be beneficial to know what characteristics a basis for (I.11) must have, if one set of first period controls is optimal for all characterizations in (I.11). We show this in the following theorem.

Theorem 5. If basic controls $\overline{u}(0)$ are optimal and constant for all characterizations of (I.11) (for $T = 1$), then $\overline{u}(0)$ is feasible for all $v(0) \in V(0)$ in (I.3) and optimal for some $\overline{v}(0) \in V(0)$.

Proof. We again have the form (II.7) and for $v_i(0)$ arbitrary in (II.8), for $\overline{u}(0)$ feasible in (I.11), we must have $\overline{u}(0)$ feasible for (I.3) and any $v(0) \in V(0)$.

Optimality for some $\overline{v}(0)$ is trivial, since we can take our problem (I.11) to be the case of $k = 1$, where only $\overline{v}(0)$ is assumed in a degenerate distribution.||

Theorems 4 and 5 lead directly to the following Corollary.

Corollary 3. Basic controls, $\overline{u}(0)$ are optimal for all characterizations in (I.11) for $T = 1$, if and only if the $\overline{u}(0)$ values are feasible for all $v(0) \in V(0)$ and optimal for some $\overline{v}(0) \in V(0)$.

This result gives necessary and sufficient conditions for J_1 and J_4 to lead to the same optimal first period controls. It should be noted, however, that $\bar{v}(0)$ must belong to the set of possible V(0). This is always true if $\bar{v}(0)$ is an expected value and the distribution of v(0) is continuous. If v(0) has a discrete distribution, the mean may not belong to the domain of the variable and the result will not necessarily hold.

This result may be useful in solving problems where future uncertainties are involved. If one can formulate these problems so that infeasibilities are removed, then one may be assured that a deterministic approach in which the mean value is in the domain of the stochastic elements is best. The problem of dealing with infeasibilities necessitates a stochastic approach and a more complicated solution procedure in the form of Problem (I.11).

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