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ABSTRACT

The paper presents a survey of dynamic linear programming models. First, models are considered which can be referred to, rather conventionally, as a (specific) resource supply model (energy supply model, extraction and exploration of mineral resources, water management systems, manpower and educational models, agricultural models), then we describe the production or economy model. The linkage of such models into an integrated system (examples are energy-economy or manpower-economy interactions) is discussed in the final part of the paper.

MODELS OF DYNAMIC LINEAR PROGRAMMING

A. Propoi*

INTRODUCTION

The impact of linear programming (LP) (Dantzig 1963, Kantorovich 1965) models and methods in the practice of decision making is well known. However, up to now most of the LP applications are of one-stage, static nature; that is, the problem of the best allocation of limited resources is considered at some fixed stage in the development of a system. When the system to be optimized is developing -- not only in time but possibly also in space -- a one-stage approach is no more adequate. In this case decisions should be phased over time and the problem of optimization becomes a dynamic, multi-stage one. In fact, almost every static LP model has its own dynamic variant, the latter being of growing importance because of the increasing role of planning in decision making (Propoi 1976, Propoi 1979).

The purpose of this paper is to review different dynamic linear programming (DLP) models. First we consider models, which rather conventionally can be referred to as a resource supply model (Sections 2 - 7), then production or economy development models (Section 8). The linkage of such models into an integrated model is discussed in the final part of the paper (Section 9). Emphasis is put on long-term applications, though of course there are many different short- and medium-term DLP models (some of them are given in the references). Solution methods for DLP models are not considered in this paper; a survey of these methods is given in (Propoi 1979). See also (Beer 1977, Bulavskii et al. 1977, Dantzig 1963, Kantorovich 1965, Madsen 1977, Propoi 1973, 1976).

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ENERGY SUPPLY MODELS

We begin with the Energy Supply System (ESS) model because it plays a central role in many energy resources studies, is rather well known and typical enough to be generalized as some resource supply model.

The main purpose of the ESS model is to study major energy options over the next 25-50 years and longer, thus determining the optimal-feasible transition from the mix of technologies for energy production currently used (fossil), to a more progressive and, in some sense, optimal, future mixture of technologies (nuclear, coal, solar, etc.) for a given region (country).

In formulating DLP problems, it is useful to single out (Propoi 1976,1979): (i) state equations of the systems with the distinct separation of state and control variables; (ii) constraints imposed on these variables; (iii) planning period T - the number of time periods during which the system is considered and the length of each time period; (iv) performance index (objective function) which quantifies the quality of a program. We will consider these four stages separately as applied to the ESS model.

State Equations. The ESS model is broken down into two subsystems: energy production and resource consumption subsystems. Hence two sets of state equations are needed.

Energy Production and Conversion Subsystems. The subsystem consists of a certain number n of technologies for energy production (fossil, nuclear, solar, etc.). The state of the subsystem at each time period t is described by the values of the capacities in that period t for all energy production technologies.

Let $y_i(t)$ be the value of the ith energy production capacity in time period t; $v_i(t)$ be the increase of the ith capacity in time period t $(i = 1,\ldots,n)$. It is assumed that a service (life-)time of each capacity is limited and constitutes τ_i periods for the ith capacity.

Thus the state equations, which describe the development of the energy production and conversion subsystem will be the following:

$$y_i(t+1) = y_i(t) + v_i(t) - v_i(t-\tau_i) \tag{1}$$

$$(i = 1,\ldots,n; t = 0,1,\ldots,T-1)$$

with the given initial conditions

$$y_i(0) = y_i^0 \quad (i = 1,\ldots,n) \quad . \tag{2}$$

The increase of the new capacities $v_i(t)$ in preplanning period $(t < 0)$ is also assumed to be known:

$$v_i(-\tau_i) = v_i^0(-\tau_i), \ldots, v_i(-1) = v_i^0(-1) \quad . \tag{3}$$

The state equations for production capacities can also be written in the form:

$$y_i(t+1) = (1-\delta_i(t))y_i(t) + v_i(t) \quad ; \quad y_i(0) = y_i^0 \tag{1a}$$

Here $\delta_i(t)$, $0 \leq \delta_i(t) \leq 1$ is the depreciation factor for the ith production capacity.

Resources Consumption Subsystem. State equations of this subsystem describe the dynamics of cumulative amounts of extracted primary energy resources.

Let $z_j(t)$ be the cumulative amount of the jth resource extracted by the beginning of time period t $(j = 1,\ldots,m)$; m be the total number of different primary resources under consideration; $q_{ji}(t)$ be the ratio of the amount of the jth resource (primary energy input) required for loading the ith energy production capacity (secondary energy output) in time period t $(i = 1,\ldots,n; j = 1,\ldots,m)$; $q_{ji}(t)$ represents the conversion process $j \rightarrow i$.

Assuming a complete load of production capacities and that the primary energy resource extraction in time period t is proportional to the value of the energy production capacities in this period we can write the state equations in the form:

$$z_j(t+1) = z_j(t) + \sum_{i=1} q_{ji}(t)y_i(t); \quad z_j(0) = z_j^0 \quad . \tag{4}$$

If some capacities are not completely loaded, then the intensities $u_i(t)$ of production capacities are introduced with condition: $u_i(t) \leq y_i(t)$ $(i = 1,\ldots,n)$. In this case variables $y_i(t)$ should be replaced by $u_i(t)$ in equation (4).

Constraints. The state equations (1) and (4) determine dynamic (intertemporal) constraints on variables. We also have static (intratemporal) constraints on variables for each time period t.

Nonnegativity Constraints. Evidently, all variables introduced into the state equations (1) and (4) cannot be negative:

$$v_i(t) \geq 0 \quad , \quad y_i(t) \geq 0 \quad , \quad z_j(t) \geq 0 \quad . \tag{5}$$

Availability Constraints. First, the upper bounds should be imposed on the annual construction rates

$$v_i(t) \leq \bar{v}_i(t) \quad , \tag{6}$$

where $\overline{v}_i(t)$ are the given numbers. In a more general form these constraints can be written as

$$\sum_{i=1}^{n} f_{si}(t)v_i(t) \leq f_s(t) \quad , \quad (s = 1,\ldots,r) \quad , \tag{7}$$

where $\{f_s(t)\}$ is the vector of non-energy inputs which are needed for the energy production subsystem (e.g. labour). The coefficient $f_{si}(t)$ denotes the amount of the resource s required for the construction of a unit of the ith capacity in time period t. Bounds on new technology introduction rates can also be written in the form (7).

The constraints on the availability of the primary energy resources can be given in the form:

$$z_j(t) \leq \overline{z}_j(t) \quad , \tag{8}$$

where $\{\overline{z}_j(t)\}$ is the vector of all available energy resources (resources in the ground) in time period t.

Demand Constraints. The intermediate and final demands of energy are supposed to be given for all planning periods. Hence the demand constraints can be written as

$$\sum_{i=1}^{n} d_{ki}(t)u_i(t) \geq d_k(t) \quad , \tag{9}$$

where $\{d_k(t)\}$ is the given vector for all $t = 0,1,\ldots,T-1$ of energy demand, both intermediate and final (e.g., electricity and nonelectric energy for final demand); coefficient $d_{ki}(t)$ defines either intermediate consumption of the secondary energy k per unit of the secondary energy production or conversion efficiency of capacity i to produce a unit of the secondary energy k.

Planning Period. The planning period is broken down into T steps where T is given exogenously. Each step contains a certain number of years (e.g. three, five). In (Haefele and Manne 1974) the planning period equals 75 years and each step corresponds to three years, thus T = 25. Since information of the coefficients of the model becomes more inaccurate with the increasing number of steps it is useful to consider steps which have different length. For example, in (Markuse et al. 1976) the planning period is 100 years and T is equal to 10 periods (five periods six years each, the next three periods ten years each and the last two periods twenty years each.)

Objective Function. The choice of the objective function is one of the important stages in model building. Discussion of economic aspects of ESS modeling objectives comes out of the framework of this paper. Here we would like specifically to underline only two points: 1) in many cases the objective functions can be expressed as linear functions of state and control variables, thus making it possible to use LP techniques; 2) the optimization procedure should not be viewed as a final one in the planning process (yielding a "unique" optimal solution), but only as a tool for analyzing the connection between policy alternatives and system performance. Thus in practical applications the policy analysis with different objective functions is required. For our purpose it is sufficient however to limit ourselves by some typical examples of objectives.

Below we consider the objective function which expresses the total capital costs both for operation and construction, discounted over time:

$$J = \sum_{t=0}^{T-1} \beta(t) \left[\sum_{i=1}^{n} c_i^y(t) y_i(t) + \sum_{i=1}^{n} c_i^v(t) v_i(t) \right] , \qquad (10)$$

where $c_i^y(t)$ is the operating and maintenance cost for the ith capacity in time period t; $c_i^v(t)$ is the investment cost for the ith capacity in time period t; $\beta(t)$ is the discount rate.

It should be noted that the first sum in (10) expresses not only direct operating and maintenance costs at step t but also may indirectly include the cost for primary resources consumed at this step. In a more explicit way this cost can be written as $c_j(t) q_{ji}(t) y_i(t)$, where $c_j(t)$ should increase with the cumulative amount of resources being consumed. This leads to a non-linear objective function (Manne 1976, Manne et al. 1979). A reasonable approximation in this case is a step-wise function for $c_j(t)$.

Now we can formulate the model. But before let us introduce definitions.

A sequence of vectors $v = \{v_i(t)\}$, $(i = 1,\ldots,n;\ t = 0,1,\ldots, T-1)$ is control or program of the system. A sequence of vectors $y = \{y_i(t)\}$, $(i = 1,\ldots,n;\ t = 0,\ldots,T)$ determined by (1) and (2), (3) is a (capacities) trajectory of the system; a sequence of vectors $z = \{z_j(t)\}$ $(j = 1,\ldots,m;\ t = 0,\ldots,T)$ determined by (4) is a (cumulative resources) trajectory of the system. Sequences of vectors $\{v,y,z\}$, which satisfy all constraints of the problem are feasible.

Choosing a control v one can obtain by (1), (2) and (4) trajectories y and z and compute the value of objective function (10). Thus, $J = J(y(0), z(0), v) = J(v)$. A feasible control $\{v^*\}$, which minimizes the (10) (or $J(v)$) is an optimal control.

The optimization problem associated with the ESS model can be stated now as follows.

Problem 1. Given the state equations (1) and (4) with initial conditions y^0 and z^0 and known parameters (3), find control v and corresponding trajectories y,z, which satisfy the constraints (5) - (9) and minimize the objective function (10).

Problem 1 represents only a very simplified version of ESS models. For detailed discussion of these models see (Haefele and Manne 1974, Makarov and Melentjev 1973, Manne 1976, Manne et al. 1979, Markuse 1976, Propoi and Zimin 1979).

Verbally, the policy analysis in the energy supply system model, which is formalized as Problem 1, can be stated in the following way. In a country or in a region there are some initial capacities for production of energy resource and there are different ways (options) of developing these capacities during the considered period. Each of these options has its own advantages and disadvantages. The problem is to find such a mix of these options, which

-- meets the given demand in secondary energy (9);
-- satisfies the availability constraints on the primary energy and other resources (labour, etc.), which are needed for developing the ESS system (6) - (8);
-- minimizes the total operational and construction cost (10).

Clearly, this formulation is general enough in order to permit different specifications for other types of resources. We illustrate it by examples.

EXTRACTION AND EXPLORATION OF MINERAL RESOURCES

In this section we describe the model for analyzing different policies in extraction and exploration activities for some mineral or primary energy resource (e.g. coal, oil, etc.).

The model is literally a repetition of the above model: for a given region (country) there are known initial values of identified and hypothetical stocks of the resource, classified on n different categories (e.g. on-shore crude oil, natural gas and off-shore crude oil). There are also M different extraction and K different exploration technologies. The intensities of the technologies depend on the extraction and exploration capacities available at this time period.

The problem is to determine the optimal mix of extraction and exploration activities in a given planning horizon, which is balanced with the development of the capacity subsystem and yields the maximum output for this planning horizon.

The model is formalized as follows. For each category $i = 1,\ldots,n$ let initial stocks of identified and hypothetical resources be given:

$$x_i^1(0) = x_i^{1,0} \quad ; \quad x_i^2(0) = x_i^{2,0} \quad , \tag{11}$$

with the state equations for extraction activities

$$x_i^1(t+1) = x_i^1(t) - \sum_{m=1}^{M} u_{mi}^1(t)/\delta_{mi}^1(t) + \sum_{k=1}^{K} u_{ki}^2(t) \quad , \tag{12}$$

and exploration activities

$$x_i^2(t+1) = x_i^2(t) - \sum_{k=1}^{K} u_{ki}^2(t) + \tilde{u}_i^2(t) \quad . \tag{13}$$

Here $x_i^1(t)$ is the amount of the identified resource of category i at the beginning of time period t; $x_i^2(t)$ is the same quantity for hypothetical resource of category i; $u_{mi}^1(t)$ is the (net) amount of resource i extracted by technology m in time period t (extraction activity); $\delta_{mi}^1(t)$ is the recoverability factor of resource i by extraction activity m; $u_{ki}^2(t)$ is the (gross) amount of resource i shifted from the hypothetical to the identified category by exploration activity k; $\tilde{u}_i^2(t)$ is the exogenously given increase of the hypothetical resource of category i at time t (discovery rate).

Let also be given the initial values of extraction and exploration capacities (m = 1,...,M; k = 1,...,K)

$$z_m^1(0) = z_m^{1,0} \quad ; \quad z_k^2(0) = z_k^{2,0} \quad , \tag{14}$$

with the state equations

$$z_m^1(t+1) = z_m^1(t) + v_m^1(t) - v_m^1(t-\tau_m^1) \quad , \tag{15}$$

$$z_k^2(t+1) = z_k^2(t) + v_k^2(t) - v_k^2(t-\tau_k^2) \quad . \tag{16}$$

Here, as in (1) $v_m^1(t)$ is the increase of the m - th extraction capacity in time period t; τ_m^1 is the service time for this capacity; $v_k^2(t)$ and τ_k^2 have the same meaning for exploration capacity k.

The intensities of extraction or exploration activities cannot exceed the existing capacities

$$\sum_i u_{mi}^1(t) \leq z_m^1(t) \quad ; \quad \sum_i u_{ki}^2(t) \leq z_k^2(t) \quad . \tag{17}$$

Besides, we have to take into account budget and other resources availability constraints which are needed for operation and contraction of new capacities. These constraints are written in a form similar to (7).

Finally, the problem is to find such nonnegativity controls $\{u^1_{mi}(t)\}$, $\{u^2_{ki}(t)\}$ and $\{v^1_m(t)\}$, $\{v^2_k(t)\}$ with corresponding trajectories $\{x^1_i(t)\}$, $\{x^2_i(t)\}$ and $\{z^1_m(t)\}$, $\{z^2_k(t)\}$ which satisfy the given state equations (12), (13), and (15), (16) with initial conditions (11) and (14) and the availability constraints (17), (7) which yield a maximum total output of the resource in question:

$$J = \sum_{t=0}^{T-1} \sum_{m,i} \alpha_i u^1_{mi}(t) \quad ,$$

where α_i is the weight coefficient for the resource of category i (e.g. energy conversion factor for primary energy resource).

Another type of objective which may be of interest here is the minimization of total cost for extraction and exploration (under given demand constraints) which is similar to the objective of Problem 1. Different modifications and generalizations of this model are discussed in (Propoi and Zimin 1979).

WATER MANAGEMENT SYSTEMS

In the same way as the above models it can be formulated for a water supply model (Agarkov et al. 1957). There is one important difference, however. In comparison with the secondary energy, the expenditures for transhipment of water or many other primary resources are rather significant, therefore the model should be regionalized in most cases. Note also, that there are many other different applications of DLP in water management (alternative evaluation of a river basin, etc. (Agarkov et al. 1957, Biswas 1976, Parikh 1966).

MANPOWER AND EDUCATIONAL MODELS

In these models we deal with a special kind of resource, namely, with the human resource. Different manpower and educational DLP models are discussed in (Bartholomew 1973, Charnes et al. 1978, Grinold and Marshall 1977, Propoi 1978). Some problems of policy analysis in migration or, generally, in national settlement systems planning as well as problems of health care planning can also be considered in the framework of DLP (Propoi 1977a, Propoi and Willekens 1978). Here we describe only an educational model which can be viewed as a skilled labour supply model.

Let $x_i(t)$ be the number of specialists of type i (grade, speciality, age, etc.) at time period t ($i=1,\ldots,n$) and $u_k(t)$ be the number of entrants to the educational system of type k (school, faculty, vocational courses, etc.) at time period t ($k=1,\ldots,r$). It is assumed that τ_k time periods are required for graduating from the educational system of type k.

The vector $x(t) = \{x_i(t)\}$ represents the distribution of specialists at time t (manpower stock) and vector $u(t) = \{u_k(t)\}$ is the distribution of new enrollments at time t over different types of the educational system. Vector x(t) is the state of the system and vector u(t) is the control.

The state equations describing the development of the manpower system are

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{k=1}^{r} b_{ik}u_k(t-\tau_k) \quad . \tag{17}$$

Here a_{ij} is the coefficient which shows how many specialists of type j progress to group i between steps t and t+1; in many cases $a_{ij} = 1-\tilde{a}_{ii}$, if $i = j$ and is zero otherwise; $\tilde{a}_{ii}$ is the manpower stock attrition rate; b_{ik} is the coefficient which shows how many enrollments to educational system k (at time $t-\tau_k$) will enter the manpower stock of type i at time t. These coefficients denote the ratio of graduates of type i to the total number of enrollments of type k.

The constraints on the variables can be written in standard form of (7):

$$\sum_k f_{sk}u_k(t) \leq f_s(t) \quad ,$$

where $f(t) = \{f_s(t)\}$ is the vector of given resources required for education (teachers, buildings, equipment, etc.), coefficients f_{sk} specify the requirements in resource s per unit of education of type k.

Sometimes it is more convenient to evaluate the required resources for the total number of students at current time period t:

$$\sum_{k=1}^{r} \sum_{\tau=0}^{\tau_k - 1} f_{sk} u_k(t-\tau) \leq f_s(t) \quad .$$

In many cases it is also necessary to single out the constraints on the availability of teachers or instructors, which may be a part of the educational system. In this case

$$\sum_{k} g_{jk} u_k(t) \leq \sum_{j} h_{ij}(t) x_i(t) \quad ,$$

where $h_{ij}(t) = 1$ if the i - th type specialists are full-time teachers of type j and $0 \leq h_{ij}(t) \leq 1$ for part-time teachers; coefficients g_{jk} specify the requirements for teachers of type j by students of type k.

Usually, the objective of the educational system is to meet the given demand $\{\bar{x}_i(t)\}$ in manpower as closely as possible. This closeness can be expressed by the piece-linear objective function

$$J = \sum_{t=0}^{T-1} \sum_{i=1}^{n} \phi_i^t (x_i(t) - \bar{x}_i(t)) \quad ,$$

where $\phi_i^t = \alpha_i^1 (x_i(t) - \bar{x}_i(t))$ for surplus of specialists and $\phi_i^t = \alpha_i^2 (\bar{x}_i(t) - x_i(t))$ for their shortage. This objective function can be easily reduced to linear

$$J = \sum_{t=0}^{T-1} \sum_{i=1}^{n} (\alpha_i^1 \xi_i(t) + \alpha_i^2 \eta_i(t)) \quad , \tag{18}$$

with additional constraints

$$x_i(t) + \xi_i(t) - \eta_i(t) = \bar{x}_i(t); \quad \xi_i(t), \quad \eta_i(t) \leq 0 .$$

Other objectives are also of interest, for example, to develop a special program for training the maximum feasible number of specialists of the eligible group of specialities by the end of this program, etc. (Propoi 1978).

As mentioned above, the separation of the models into supply and production type is rather conventional. In fact, for example, the educational model considered above can be viewed as either a labour supply model or as a model for planning the "production" of specialists. This is also true for different agricultural models, which are described in the next section.

AGRICULTURAL MODELS

First, we describe a model for livestock breeding (Propoi 1979, Swart 1975).

Livestock Breeding. Let $x_{ia}(t)$ be the number of animals of type i (calf, heifer, dairy cow, sow, etc.) and age group a at time period t (e.g. year), $u^+_{ia}(t)$ be the number of animals of type i and age group a purchased at time period t, and $u^-_{ia}(t)$ is the same number for sold animals. Then in matrix form the state equations which describe the development of livestock will be

$$x(t) = Gx(t) + B^+u^+(t) - B^-u^-(t) \quad ; \quad x(0) = x^0 \quad . \tag{19}$$

Here vectors $x(t) = \{x_{ia}(t)\}$; $u^+(t) = \{u^+_{ia}(t)\}$; $u^-(t) = \{u^-_{ia}(t)\}$ $(i=1,\ldots,n;\ a=1,\ldots,N)$ specify respectively the state and controls of the system at time t; G is a so called growth matrix:

$$G = \begin{bmatrix} 0 & 0 & \cdots & B(N_1) & \cdots & B(N) \\ S(1) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & S(2) & \ddots & & & \vdots \\ \vdots & \vdots & & S(N_1) & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & S(N) \end{bmatrix}$$

Here B(a) is a birth matrix for age group a; the element $b_{ij}(a)$ of B(a) shows what number of animals of type i is "produced" (born) by one animal of type j and age group a. It is assumed that the reproductive age begins with group N_1 and ends with group N. S(a) is a survival matrix which shows what proportion of animals of group a progresses to group a + 1 for one time period (I-S(a) is an attrition matrix). Diagonal matrices B^+ and B^- specify the purchasing and selling activities in the system (e.g. there is 1 on main diagonal if we have such activity for given type i and age group a and 0 otherwise).

Along with evident nonnegativity constraints both for control and state variables, it is necessary to take into account constraints associated with care and feeding of animals. They can be written again in the form of (7).

This model can be considered as a resource supply model for a food production system (milk, meat, cheese, etc.). Or, it can also be viewed itself as a production system, which requires its own resource supply system, for example, forage production (hay, corn, haylage, etc.) for feeding animals (see Section 9).

In spite of its simplicity (or maybe due to its generality) this basic model can be used in many fields of population control (cattle, pig, and sheep breeding, poultry farming, optimal control of fish breeding, fur farming, etc.). Similar problems also arise when planning the migration of wild animals or when the suppression of pests is necessary.

Perennial Crop Production (Carter et al. 1977). It is interesting to note that we will obtain practically the same equations when the perennial crop (grapes, apricots, apples, alfalfa, etc.) production model is considered. To illustrate we will describe only the model for the production of one type of perennial. Let $x_i(t)$ be the number of hectares used for a perennial crop of age group i (i = 1,...,N) at time period t, and $u^+(t)$ be the number of hectares used for new planting at time t. Then the state equations will be in coordinate form:

$$x_1(t+1) = b_0u^+(t)$$

$$x_2(t+1) = a_{21}x_1(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_N(t+1) = {}_{NN}x_N(t) + {}_{N,N-1}x_{N-1}(t)$$

Or, in matrix form

$$x(t+1) = Ax(t) + bu^+(t) \quad , \quad x(0) = x^0 \quad ,$$

where

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{21} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{N,N-1} & a_{N,N} \end{bmatrix} ; \quad b = \begin{bmatrix} b_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

In more general form, when we have both planting $u^+(t)$ and harvesting $u^-(t)$ activities, the state equations acquire the form of (19).

Note, that the same equations are also applied for forest management models (Kallio et al. 1979). Different types of DLP models for separate agricultural activities are considered in (Carter et al. 1977), Csaki 1977, Propoi 1979, Swart 1975).

ECONOMY DEVELOPMENT MODEL

Several examples of DLP models were considered above which can be referred to as a (specific) resource supply model under the given demand constraints for this specific resource and the availability constraints for other resources which are required for development of the system. A counterpart of such models is a production or, generally, an economy development model (Aganbegian et al. 1972, Dantzig 1955,1963, Dukalov et al. 1974, Ivanilov and Petrov 1970, Kantorovich 1965, Koehler et al. 1975, Pegels 1976, Propoi and Zimin 1979). Below we describe a simple DLP model of an economy (see also Ivanilov and Petrov 1970, Propoi and Zimin 1979).

State equations. The system under consideration is broken down into two subsystems: production and capacities development (or capital stock accumulation).

Production subsystem. The operation of industry is described in terms of n production sectors. Let $x_i(t)$ be the stock of production in sector i $(i=1,\ldots,n)$ accumulated up to a time period t, $u_i(t)$ be the gross output (production level) of sector i in time period t, $v_i(t)$ be the additional capital stock constructed in time period t, and $a_{ij}(t)$ be the input-output coefficients. We assume also, that τ_j is the time (number of time periods) required to construct and to put into operation additional capacity in sector j; $b_{ij}(\tau)$ are capital coefficients; $b_{ij}(\tau)$ shows the requirements in good i to build a unit capacity in sector j which will be available τ time periods later; $w_i(t)$ is the final consumption of sector i products, and $s_i(t)$ is the net export. Then the state equations describing the production subsystem can be written as follows:

$$\begin{aligned} x_i(t+1) = x_i(t) + u_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_j(t) - \\ - \sum_{j=1}^{n} \sum_{\tau=0}^{\tau_j} b_{ij}(\tau)v_j(t-\tau) - w_i(t) - s_i(t) \quad . \end{aligned} \tag{20}$$

Initial inventories and preplanning controls are assumed to be given $(i=1,\ldots,n; t=0,\ldots,\tau_i-1)$:

$$x_i(0) = x_i^0 \quad , \quad v_i(t-\tau_i) = v_i^0(t-\tau_i) \quad . \tag{21}$$

Capacities Development Subsystem. Let $y_i(t)$ be the value of the production capacities of type i and $\delta_i(t)$ be the depreciation factor in sector i at time period t. Then the dynamics of production capacities is written as follows (cf.(1a)):

$$y_i(t+1) = (1-\delta_i(t))y_i(t) + v_i(t-\tau_i) \; ; \quad y_i(0) = y_i^0 \quad . \tag{22}$$

Constraints. Evidently any economic system is operating within certain constraints which imply physical, economic, institutional and other limits to the choice of controls.

Resources Availability Constraints. The production system requires certain external resource inputs for its operation. First of all, these are labor and primary resources. Both constraints can be written in a similar way: for labor resources:

$$\sum_{j=1}^{n} \ell_{kj}(t)u_j(t) \leq \ell_k(t) \quad , \quad (k=1,\ldots,K) \quad , \tag{23}$$

where $\ell_k(t)$ is the total labor of category k available in time period t; $\ell_{kj}(t)$ are the labor output ratios for sector j; for other resources (e.g. land, water, etc.):

$$\sum_{j=1}^{n} r_{mj}(t)u_j(t) \leq r_m(t) \quad , \quad (m=1,\ldots,M) \quad , \tag{24}$$

where $r_m(t)$ is the total amount of resource category m, available in time period t; $r_{mj}(t)$ are resource requirements per unit of sector i production in time period t.

Production Capacities Constraints. The gross output of each sector is limited by the available production capacity

$$u_i(t) \leq y_i(t) \quad (i=1,\ldots,n) \quad . \tag{25}$$

Inventory Constraints. These constraints relate to the possibility of accumulating limited amounts of good stocks. For storable goods:

$$0 \leq x_i(t) \leq \bar{x}_i(t) \quad , \tag{26}$$

where $\bar{x}_i(t)$ are the given stock capacities, and $x_i(t)$ are calculated from (20).

For nonstorable goods we have instead of (26):

$$u_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_j(t) - \sum_{j=1}^{n} \sum_{\tau=0}^{\tau_j} b_{ij}(\tau)v_j(t-\tau)$$
$$- w_i(t) - s_i(t) \geq 0 \quad . \tag{27}$$

It should be stressed that in many practical cases, the accumulation of goods stocks in large amounts is unreasonable or too expensive. Hence, $\{x_i(t)\}$ are small in comparison to the outputs of the system. Therefore we can consider the balance equation (bill of goods) in the form of inequality (which is the matrix form of (27)):

$$(I-A(t))u(t) \geq \sum_{\tau=0}^{\bar{\tau}} B(\tau)v(t-\tau) + w(t) + s(t) \tag{28}$$

or equality

$$(I-A(t))u(t) = \sum_{\tau=0}^{\bar{\tau}} B(\tau)v(t-\tau) + w(t) + s(t) \tag{28a}$$

both for storable and nonstorable goods. (In (28) and (28a) it is assumed that $\tau = \tau_j$).

<u>Consumption Constraints</u>. Final consumption is usually bounded for each sector i. In many cases it can be represented in the form:

$$w_i(t) \geq g_i(t)\omega(t) \quad , \tag{29}$$

where $\omega(t)$ is the total final consumption, $g_i(t)$ is the share of total consumption provided by sector i. Exogenously given vector $g(t) = \{g_i(t)\}$ predefines the profile of a final consumption over time. The introduction of a consumption profile allows one to use a scalar control $\omega(t)$ instead of control vector $w(t) = \{\omega_i(t)\}$.

Objective Function. Above, $\{u,v,w,\} = \{u_i(t), v_i(t), w_i(t)\}$ are control variables, $\{x,y\} = \{x_i(t), y_i(t)\}$ are state variables. The choice of optimal control depends on the choice of the objective of the model. In the following we consider typical examples of the objective functions.

Maximization of the cumulative discounted goods supply (final consumption). In this case the objective function is

$$J = \sum_{t=0}^{T-1} \beta(t)\omega(t) \quad , \tag{30}$$

where $\beta(t)$ is the discount factor.

For the last step, the objective function will be

$$J = \sum_{i=1}^{n} h_i^w(T) w_i(T) \quad ,$$

where $h_i(t)$ is the weight coefficient for sector i products.

Maximization of the final stock of goods.

$$J = \sum_{i=1}^{n} h_i^x(T) x_i(T) \quad ,$$

$h_i^x(T)$ is the weight coefficient ("cost") for production stock $x_i(T)$ in sector i.

Maximization of the terminal values of production capacities.

$$J = \sum_{i=1}^{n} h_i^y(T) y_i(T) \quad ,$$

$h_i^y(T)$ is the weight coefficient for production capacity $y_i(T)$ in sector i.

Minimization of total expenses. This criterion is similar to the objective functions, considered in Sections 1 and 2:

$$J = \sum_{t=0}^{T-1} \sum_{i=1}^{n} \beta(t) \left[c_i^u(t) u_i(t) + c_i^v(t) v_i(t) + c_i^y(t) y_i(t) \right] \quad ,$$

where $c_i^u(t)$, $c_i^y(t)$ are operating and maintenance costs, $c_i^v(t)$ is the investment cost, $\beta(t)$ is the discounting factor.

Other objective functions are also possible. It should be noted that goals of control can also be expressed by additional constraints, such as $w(T) \geq \bar{w}(T)$; $x(T) \geq \bar{x}(T)$; $y(T) \geq \bar{y}(T)$. For example, one wishes to maximize the total expenses under the given level of final consumption $\bar{w}(T)$.

Using the above conditions one can specify different economy models. As an example we formulate the following model.

Problem 2. Given the state equations (20), (22) with initial conditions (21). Find control $\{u,v,w\}$ and the corresponding trajectories $\{x,y\}$ which satisfy conditions (23)-(29) with nonnegativity constraints and minimize the objective function (30).

INTEGRATED MODELS

Above some separate DLP models were described which can be used individually for different purposes. However, this approach of separate analysis is limited in its possibilities because many important features of systems are missing due to their interactions. Therefore integrated models are needed which describe resource supply economy or production interrelations. Below we describe three such models: energy supply economy; skilled labour supply economy and an integrated model for agricultural production.

Energy-Economy Model. (Dantzig 1976). Considering the ESS and economy models one can see that there are two main links between them: final demand for energy which is the output of the economy model and nonenergy resources supply for which the requirements are outputs of the ESS model. We shall combine the ESS model (Problem 1) and the economy model (Problem 2) into a whole system, using subscripts E for the energy sector and NE for the nonenergy sectors. For a uniform representation we assume that both the industrial processes of economy and energy sectors are described in terms of physical flows. Besides, we omit, for simplicity, time lags in construction and putting into operation production capacities.

Production Subsystem is described by a combination of state equations (1) and (2) for energy and nonenergy sectors respectively in their simplified form (we describe depreciation of the capacities in the same way for both equations):

$$y_E(t+1) = (I - \Delta_E(t))y_E(t) + v_E(t) \quad ; \quad y_E(0) = y_E^0 \qquad (31)$$

$$y_{NE}(t+1) = (I - \Delta_{NE}(t))y_{NE}(t) + v_{NE}(t); \; y_{NE}(0) = y_{NE}^0 \; . \qquad (32)$$

Here $y_E(t)$ and $y_{NE}(t)$ are vectors of production capacities for energy and nonenergy sectors, $v_E(t)$ and $v_{NE}(t)$ are the increases of these capacities in time period t.

To describe the accumulation consumption of primary energy resources we can use the equation (4), the total load of capacities is assumed hereafter.

$$z_E(t+1) = z_E(t) + Q_E(t)y_E(t) \quad ; \quad z_E(0) = z_E^0 \qquad (33)$$

$$0 \leq z_E(t) \leq \overline{z}_E(t) \quad . \qquad (34)$$

The most important constraint in the model is the balance between the production of goods and their consumption (Bill-of-Goods). Here we neglect the possibility to stock goods, thus considering the static form of these conditions: for energy output (upper index "E" for matrices):

$$\begin{aligned} -A_{NE}^{E}(t)y_{NE}(t) + (I - A_E^E(t))y_E(t) &= \\ = B_{NE}^{E}(t)v_{NE}(t) + B_E^E(t)v_E(t) + w_E(t) + s_E(t) & \end{aligned} \qquad (35)$$

for nonenergy products (upper index "NE" for matrices):

$$\begin{aligned} (I - A_{NE}^{NE}(t))y_{NE}(t) - A_E^{NE}y_E(t) &= \\ = B_{NE}^{NE}(t)v_{NE}(t) + B_E^{NE}(t)v_E(t) + w_{NE}(t) + s_{NE}(t) & \end{aligned} \qquad (36)$$

Labor availability constraints (23) are written in the form:

$$L_{NE}(t)y_{NE}(t) + L_E(t)y_E(t) \leq \ell(t) \quad . \qquad (37)$$

Similarly it can be written constraints (24) for the availability of other resources, which are external to the system (e.g. land, water, etc.)

Final consumption constraints (29) are written as:

$$w_E(t) \geq g_E(t)\omega(t) \tag{38}$$

$$w_{NE}(t) \geq g_{NE}(t)\omega(t) \quad , \tag{39}$$

where the given vectors $g_{NE}(t)$ and $g_E(t)$ specify profiles of final consumption for nonenergy and energy products, respectively. Evidently, all the variables are nonnegative.

Let's denote the optimization problem associated with conditions (31)-(39) and the objective function (30) as Problem 3.

In the integrated model there is an important feature which cannot be explicitly seen from the matrix notations of Problem 3. Practically all individual models which are to be incorporated into a system may have different levels of aggregation. In fact, when we investigate the influence of a resource supply system on economy development, the resource supply model should be presented in more detail than the economy model. In this case, a special model is to be developed which shows the influence (impact) of the resource supply system upon the economy as a whole (Kononov and Por 1979).

Considering the integrated model (Problem 3) one can see that it is basically the economy model (Problem 2) partitioned in energy (E) and nonenergy (NE) sectors. On the other hand, it can be reformulated in such a way that it will incorporate exactly the ESS model and the model describing the rest of the economy system plus linking coupling constraints.

In fact, the conditions (31), (33) are the same as in Problem 1. Besides, let us partition the balance equations (35) and (36) as follows:

$$A_{NE}^{E}(t)y_{NE}(t) + B_{NE}^{E}(t)v_{NE}(t) + w_E(t) + s_E(t) = d_E(t) \ ; \tag{40}$$

$$(I - A_E^E(t))y_E(t) = d_E(t) + B_E^E(t)v_E(t) \tag{41}$$

$$B_E^{NE}(t)v_E(t) + A_E^{NE}(t)y_E(t) = f_E^{NE}(t) \tag{42}$$

$$B_{NE}^{NE}(t)v_{NE}(t) + w_{NE}(t) + s_{NE}(t) = f_{NE}^{NE}(t) \tag{43}$$

$$(I - A_{NE}^{NE}(t))y_{NE}(t) = f_{NE}^{NE}(t) + f_E^{NE}(t) \tag{44}$$

Equation (41) expresses the supply of energy by the energy sector and in fact is equivalent to the demand constraints (9) with $I - A_E^E(t) = D_E(t)$ and $d_E(t)$ fixed, while the constraints (42) represent the requirements of ESS for nonenergy products with the fixed $f_E^{NE}(t)$, and, taking into account the comparative smallness of the second left-side term in (42), are equivalent to (7). Equations (40) and (43) represent the demands for energy and nonenergy products of the rest of the economy while equation (44) expresses the supply of goods by nonenergy sectors (with $d_E(t)$ and $f_E^{NE}(t)$ being fixed). Besides, we can rewrite constraints (37) in the form

$$L_E(t)y_E(t) = \ell_E(t) \qquad (45a)$$

$$L_{NE}(t)y_{NE}(t) = \ell_{NE}(t) \quad ; \qquad (45b)$$

$$\ell_{NE}(t) + \ell_E(t) \leq \ell(t) \quad . \qquad (46)$$

Finally, we find that equations (31),(33),(34),(41),(42) and (45a) with variables $d_E(t)$, $f_E^{NE}(t)$ and $\ell_E(t)$ to be given exogenously represent the complete description of the ESS model, while equations (32),(40),(43),(44) and (45b) with exogenously given $d_E(t)$, $f_E^{NE}(t)$ and $\ell_{NE}(t)$ describe the rest of the economy.

In the integrated model, variables $d_E(t)$, $f_E^{NE}(t)$, $f_{NE}^{NE}(t)$, $\ell_E(t)$ and $\ell_{NE}(t)$ should be considered as endogenous; in this case constraints (44) and (46) are coupling constraints and variable $d_E(t)$ is a coupling variable.

Manpower-Economy Model. (Propoi 1978). In the model considered in Section 5 the demand for manpower and resource constraints for education are given exogenously. Of large interest is the analysis of interrelations between manpower and economy development models. When this interaction is analyzed, two major options should be taken into account: development of some sectors in an economy in order to absorb the projected surplus in manpower of certain types and development of educational facilities in order to fill up possible shortages in manpower for other sectors of an economy. Besides, we have to add possibilities of labor force migration into and out of the system. The problem should be disaggregated on major economic activities (e.g., various industrial sectors, agriculture, construction, transportation, public administration and other services) and on the levels of education (e.g. primary, secondary, higher.)

One can see that this problem is quite similar to the analysis of energy-economy interaction. Therefore, below we describe only briefly a manpower-economy model.

Let $x(t)$ be the vector of skilled manpower at time t, $u(t)$ be the vector (of the same dimension) of manpower increase during time period t, and $A(t)$ be the transition matrix. Then the state equations for the manpower/educational subsystem will be the following (cf. (17)):

$$x(t+1) = A(t)x(t) + u(t) \quad .$$

The training of people requires resources; first of all, teachers: $u(t) \leq \phi x(t)$ and second, buildings and equipment: $u(t) \leq y_e(t)$ where $y_e(t)$ is the vector of the capital stock for the educational subsystem. The development of this subsystem can be expressed in the same terms as development of the production system:

$$y_e(t+1) = (I - \Delta_e(t))y_e(t) + v_e(t)$$

$$y(t+1) = (I - \Delta(t))y(t) + v(t)$$

where subscript e refers to the educational subsystem, $\Delta_e(t)$; $\Delta(t)$ are depreciation matrices. The balance of goods production and their consumption for the whole system is:

$$(I - A(t))z(t) = B(t)v(t) + B_e(t)v_e(t) + w(t)$$

with constraints $z(t) \leq y(t)$ and $L(t)z(t) \leq x(t)$ where the matrix $L(t)$ specifies requirement in skilled labour for each sector of the economy, $z(t)$ is the vector of gross outputs.

The connection between consumption vector $w(t)$ and manpower vector $x(t)$ is assumed to be given as $w(t) = g(t) + F(t)x(t)$ where $g(t)$ is the exogenously given vector of governmental consumption, and matrix $F(t)$ specifies the consumption profile for different categories of manpower. The last two conditions describe the linkage between educational and economy submodels.

With the above model, optimal policies with different objective functions can be analyzed.

Agricultural Model. For illustration, only a very simple model of livestock-crop production interaction will be considered here (see also Propoi 1979 and Swart 1975). The DLP model of a whole agro-industrial complex development is represented in (Carter 1977).

The livestock subsystem was described in Section 7. Let the consumption of the forage of different types (corn, hay,etc.) be represented as

$$Dx(t) = d(t) \tag{47}$$

where vector x(t) of type/age distribution of animals is calculated from the state equations (19) and the vector d(t) is given exogenously in the livestock model. If a crop production subsystem is to be linked with a livestock subsystem, then condition (47) becomes a coupling constraint (with endogenous d(t)) between these subsystems. Let z(t) be the vector of different types of forage (the same dimension as vector d(t)) at the beginning of time period (year) t; vector y(t) represents the number of hectares for different crops, and vectors $w^+(t)$, $w^-(t)$ be purchasing and selling activities. Then the state equations for the crop production subsystem will be as follows (assuming that there is a possibility to stock forage):

$$z(t+1) = z(t+1) + Cy(t) - Dx(t) + w^+(t) - w^-(t) \quad ; \qquad z(0) = z^0 \tag{48}$$

where matrix $C = \{c_{sj}\}$ represents the output of forage s per one hectar of the lot j.

If there is no possibility (or necessity) to stock the forage s, then the above equation is replaced by

$$\sum_j c_{sj} y_j(t) + w_s^+(t) = \sum_{a,i} d_{ai}^s x_{ai}(t) + w_s^-(t) \quad . \tag{49}$$

There can be other constraints on variables. For example, the total area of all lots is upper bound $\sum_j y_j(t) \leq Y$ or the possibility to stock products is limited by stock capacities $z_s(t) \leq \bar{z}_s(t)$. The objective of this model can be, for example, maximization of discounted net returns of the given planning horizon.

Two basic approaches can be singled out when separate submodels are incorporated into a whole system. The first approach is the integration of separate models into an optimization problem with a corresponding objective function. The example for the energy-economy model is Problem 3. The second approach is the investigation of linkage between submodels considering these submodels on an independent basis, each with its own objective function. For the energy-economy model the links between ESS and economy models are given by the coupling constraints (44), (46) and the coupling variable $d_E(t)$.

Both approaches naturally have their own advantages and disadvantages. The major advantage of the first, "machine" approach is that it allows one to take into account all the constraints and interactions between many factors influencing the decision and combining them into some optimal mix. However, building an integrated model evidently leads to a very large optimization problem, which though it is sometimes possible to solve, is always difficult to interpret. The "manual" approach -- when information obtained from one submodel is interpreted by an analyst and is supplied as an input to another submodel -- is more attractive, but is more time consuming and sometimes may lead to an uncertainty whether the "true optimal" solution for a whole system has been obtained. The discussion of different approaches to the analysis of energy-economy interaction can be found for example in (Dantzig 1976, Hitch 1977, Makarov and Melentjev 1973, Manne 1976, Propoi 1979).

CONCLUSION

Considering the models described above one can see that they can be reduced to a canonical form:

Problem P. Given the state equations

$$x(t+1) = A(t)x(t) + \sum_{\tau=0}^{\bar{\tau}} B(\tau)u(t-\tau)$$

with initial condition

$$x(0) = x^0$$

and constraints

$$G(t)x(t) + \sum_{\tau=0}^{\bar{\tau}} D(\tau)u(t-\tau) \leq f(t)$$

$$x(t) \geq 0 \quad , \qquad u(t) \geq 0 \quad .$$

Find control $u = \{u(t)\}$ and the corresponding trajectory $x = \{x(t)\}$ which maximize the objective function

$$J(u) = a(T)x(T) + \sum_{t=0}^{T-1} (a(t)x(t) + b(t)u(t)) \quad .$$

The state variable $x(t)$ is generally associated with production capacities, the control vector $u(t)$ with activities for construction of new capacities; vector $f(t)$ represents the exogenously given resources required for construction and maintenance of the capacities.

Clearly, not only the above models can be formulated as DLP Problem P. As examples we mention here multistage structural design problems (Ho 1975) and congested urban traffic control (Tamura 1977).

The economic interpretation for dual to Problem P (which is also formulated in DLP format (Propoi 1977)) is given in (Ivanilov and Propoi 1973). The DLP Problem P can be considered as a static LP problem (with a staircase constraint matrix), therefore standard LP-packages can be (and have been) used for its solution. However, the development of special DLP methods which take into account the dynamic properties of the problem is clearly more perspective. Surveys of such methods are given in (Propoi 1976, 1979).

It should also be noted that not all dynamic optimization problems can be kept within the framework of DLP (for example, "crop" coefficients c_{sj} in (48), (49) or transition coefficients a_{ij} in manpower models (Bartolomew 1973) are in fact random variables). Therefore the extension of DLP methods to nonlinear, stochastic, multistage maxmin problems is also of large practical interest.

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