Application of optimal control to a biomechanics model

Andrey A. Krasovskii*

Abstract

A model of sport biomechanics describing short-distance running (sprinting) is developed by applying methods of optimal control. In the considered model the motion of a sportsman is described by a second order ordinary differential equation. Two interconnected optimal control problems are formulated and solved: the minimum energy and time-optimal control problems. Based on the comparison with real data, it is shown that the proposed approach to sprint modeling provides realistic results.

1 Introduction

In this study two interrelated problems of optimal control [1–3] are considered. The first problem consists in finding the minimum energy reserve required for a sprinter to run a 100 meter distance in a given time. The second problem is to find the fastest time in which an athlete possessing a given amount of energy can run the distance. The first problem is formalized in the paper as a minimum energy control problem, and the second one, as a time-optimal control problem.

The dynamics of an athlete in the considered model are described by an ordinary differential equation studied in papers on sprint modeling [4]. The model originated in the paper by J. Keller [5], who suggested a sprinter's dynamics as:

$$ma = F - \gamma v, \tag{1}$$

*International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria. E-mail: krasov@iiasa.ac.at

where m is the mass of a sportsman, v is the velocity, a is the acceleration, F is the applied force, and γ is a constant physical parameter responsible for the resistance force and calculated for the athlete based on the experimental data. Equation (1) has been considered in several papers [4,6–8], from which it follows that the equation fits the real data for short-distance running (100 and 200 meters). The indicated studies are devoted to the calibration of parameter γ, to the wind impact analysis, and to the choice of function $F(t)$ such that the solution of (1) satisfies the actual motions, e.g., the running records of Usain Bolt [9]. The recent paper [4] contains a review of approaches to analysis of the model. Note that so far the model has attracted the attention of physicists and biomechanic specialists.

In our view, the natural development of the model is connected with the application of methods of optimal control theory [1–3] to the problem of searching for an optimal force function. As the distance is short, the program (open-loop) control seems to be quite an appropriate tool for modeling the process of a sprinter's planning program for distributing his energy over the distance. In this case he has no time for feedback (closed-loop) control unlike athletes (stayers) competing on long-distance runs.

Based on equation (1), we consider a linear dynamical system and formalize the optimal control problem. The sprinter becomes a controlled object subject to the dynamics (1), the control action stands for the applied force, and the wind speed is introduced in the model as an external disturbance. The intensity of a sprinter's force is estimated by a functional interpreted as the energy spent by the sportsman. It is assumed that while running, the sportsman distributes his energy to minimize the cost of his energy resources. Although similar criteria ("minimum energy criterion") are considered in studies on the biomechanics of running (e.g. in [10]), the author of this paper has not come across a justification for its use in sprinting models so far. The choice of this optimality criterion can be considered as an assumption that is is later validated by a comparison of the modeling results with the data. In that way, the problem of moving the controlled object from a given initial state to a final state in a given time and with the minimum energy is formulated in the study. In case when the values of initial and final states are given completely, we obtain an analytical solution to the problem based on methods of functional analysis (namely, the moment problem) following the book [3]. In reality the sprinter's velocity at the finish line is not fixed (but bounded), and thus, the final state is given by a manifold. In the study we show that in the framework of the problem it is possible to choose the final velocity that provides a strict minimum of the functional and, hence, gives a unique optimal solution to the problem.

The developed model is applied to Usain Bolt's performance data at the Beijing Olympic Games in 2008, which was available in the literature [4,8,9]. Firstly, in the framework of the proposed model we calculate the minimum energy required to run the distance in the time that he showed (9.69 s) with the fixed final velocity, the velocity that he actually had at the finish line. We call the solution to this problem a *suboptimal solution*. Secondly, using the calculated energy, we solve the time-optimal control problem under constraints on energy and now with free velocity at the end. We find that the optimal time achieved with this amount of energy is less than the suboptimal time. According to the model, with this energy supply, under the same conditions he could have distributed his energy optimally and run the distance in 9.56 s. It is worth mentioning that this time is close to his current world record, 9.58 s, achieved at the 2009 World Championships in Berlin. In the paper we also provide modeling results for optimal (energy-efficient) running over 100 m: calculation of the minimum energy and trajectories of acceleration, velocity and distance from start.

1.1 Mathematical model

We use an approach developed in the control theory for linear systems and proposed in the book [3]. We introduce coordinates: $x_1 = s$ and $x_2 = \dot{x}_1 = v$, where s is the distance from start. Differential equation (1) is now represented in the following way:

$$\begin{aligned} \dot{x}_1 &= x_2 + w_s, \\ \dot{x}_2 &= -\frac{\gamma}{m}x_2 + \frac{F}{m}. \end{aligned} \tag{2}$$

Here, we added the external disturbance w_s, the wind speed. Wind can be favorable or adverse depending on weather conditions and, in general, can be a given function $w_s(t)$. Some studies [5,8] assume the force F to be constant over the entire distance. This is not a realistic assumption [4]. We will search for the force function $F(t)$ as an optimal energy-efficient program of the sprinter. Hence, force F in the model stands for the control action u, $u = F$. In the vector notation, system (2) takes the form

$$\dot{x} = Ax + Bu + w, \tag{3}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{\gamma}{m} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}, \quad w = \begin{pmatrix} w_s \\ 0 \end{pmatrix}. \tag{4}$$

Given t_0 and $x(t_0) = x^0 = (x_1^0, x_2^0)'$, one can apply the Cauchy formula [11] in order to solve equation (3), (4). To obtain the Cauchy formula, let us consider the homogeneous equation

$$\dot{z} = Az. \tag{5}$$

The eigenvalues of the matrix A (4) are found from the characteristic equation

$$\lambda(\frac{\gamma}{m} + \lambda) = 0, \tag{6}$$

and are equal to $\lambda_1 = 0$, $\lambda_2 = -\gamma/m$. The fundamental matrix for the homogeneous equation (5) and the matrix inverse to it are calculated using the corresponding eigenvectors (see, e.g., [12]):

$$Z[t] = \begin{pmatrix} 1 & e^{-\frac{\gamma}{m}t} \\ 0 & -\frac{\gamma}{m}e^{-\frac{\gamma}{m}t} \end{pmatrix}, \quad Z^{-1}[t] = \begin{pmatrix} 1 & \frac{m}{\gamma} \\ 0 & -\frac{m}{\gamma}e^{\frac{\gamma}{m}t} \end{pmatrix}. \tag{7}$$

The fundamental matrix for the Cauchy problem (3), (4) is given by:

$$X[t,\tau] = Z[t]Z^{-1}[\tau] = \begin{pmatrix} 1 & \frac{m}{\gamma}(1 - e^{-\frac{\gamma}{m}(t-\tau)}) \\ 0 & e^{-\frac{\gamma}{m}(t-\tau)} \end{pmatrix}. \tag{8}$$

The impulse response function of the object (see [3]) is given by the relation

$$H[t,\tau] = X[t,\tau]B = \begin{pmatrix} h^{(1)}[t,\tau] \\ h^{(2)}[t,\tau] \end{pmatrix} = \begin{pmatrix} \frac{1}{\gamma}(1 - e^{-\frac{\gamma}{m}(t-\tau)}) \\ \frac{1}{m}e^{-\frac{\gamma}{m}(t-\tau)} \end{pmatrix}. \tag{9}$$

Finally, let us introduce the Cauchy formula for solving system (3), (4):

$$x(t) = X[t,t_0]x^0 + \int_{t_0}^{t} H[t,\tau]u(\tau)d\tau + \int_{t_0}^{t} X[t,\tau]w(\tau)d\tau, \tag{10}$$

which describes the dynamics of the motion of the object (3), (4) starting from the initial state $x(t_0) = x^0$ subject to control action u and external disturbance w.

1.2 The minimum energy control problem

In the model of sprinting over 100 m, it is natural to consider the following control problem. At the time moment $t_0 = 0$ the object (3), (4) starts from the initial state $x^0 = (0,0)'$. At the final time t_f he must reach the final state (finish) $x_1^f(t_f) = 100$ m. Apparently, in the problem definition only one component of the final state vector $x^f = (x_1^f, x_2^f)'$ is fixed. Indeed, in our statement the final velocity is not specified and belongs to an interval $x_2^f \in [\varepsilon, 12]$ chosen from physical considerations. Here $\varepsilon > 0$ is a small parameter close to zero, and the upper boundary, 12 m/s, corresponds to the highest possible velocity of a runner at the finish line [9]. In other words, the final state of the object is given by the manifold

$$\Theta = \{x^f : x_1^f = 100,\ \varepsilon \le x_2^f \le 12\}. \tag{11}$$

As an optimality criterion in the control problem, the energy cost of the athlete is chosen. Thus, the intensity of the control action u in the model (3), (4) is given by the integral

$$\varkappa[u] = \Big[\int_{t_0}^{t_f} u^2(t)dt\Big]^{\frac{1}{2}}. \tag{12}$$

Similar optimality criteria are considered in biomechanics with applications to running [10]. It is assumed that the processes in a sportsman's body are oriented to minimizing the energy cost. We will not investigate the physical meaning of this functional, as its validity will be verified by the modeling results provided below. Let us formulate an optimal control problem.

Minimum energy control problem. Let the control system be described by the linear equation (3), (4). The time interval $t_0 \le t \le t_f$, the initial state x^0, and the manifold Θ (11) of final states x^f of the object are given. To solve the minimum energy problem, one needs to find an optimal control $u^0(t)$ (among the admissible controls $u(t)$) that brings system (3) from the given state $x(t_0) = x^0$ to a point on the manifold Θ of final states $x(t_f) = x^f$ and has the lowest possible intensity (12).

Remark. The admissible control u belongs to the functional space $\mathscr{L}^{(2)}_{[0,t_f]}$ (see Appendix A).

1.3 Solution to the optimal control problem

The problem is solved using the method proposed in the book [3]. The solution consists of the following steps:

1. The problem is solved for a fixed final state x^f, i.e., for a fixed coordinate x_2^f. An interpretation of the problem as a minimum norm problem in functional analysis is given in Appendix A. An analytical solution is obtained in Appendix B with a method based on the moment problem.

2. The optimal result (minimum energy) as a function of the final velocity $x_2^f \in [\varepsilon, 12]$ is constructed numerically.

3. The final velocity $\tilde{x}_2^f$ that provides the lowest minimum intensity (12) is chosen, and the corresponding trajectories of optimal motion are constructed.

Let us note that the solution satisfies the necessary and sufficient optimality conditions. This means that one can solve the corresponding time-optimal control problem: find the shortest time t_f that can be achieved on the distance by a control whose intensity does not exceed a given level.

2 Modeling results

The computational procedure when implementing the solution described above is coded in the R software environment.[1] In our study we use the parameters calibrated for Usain Bolt in the paper [8], which analyzes his world record race at the 2008 Summer Olympic Games in Beijing: $m = 86$ kg and $\gamma = 59$. The wind speed is taken to be the constant function $w_s(t) = w_s = 0.5$ m/s, and the parameter ε (11) is chosen at the level $\varepsilon = 0.01$ m/s.

2.1 Optimal model trajectories

To begin with, let us show the solution to the control problem formulated in Subsection 1.2 for the fixed time $t_f = 10$ s. In Fig. 1 the plot of the minimum energy function versus the final velocity is shown. One can see in the plot that there exists a lowest minimum energy $\varkappa[u^0] = 2004.14$ achieved at the

[1]http://www.r-project.org/

final velocity $\tilde{x}_2^f = 6.06$ m/s. The optimal control, acceleration, velocity, and distance from start are depicted in Fig. 2. The maximum force $F^0 = 715.9$ N is applied to the first part of the distance and then declines as depicted in Fig. 2. The maximum acceleration $a^0_{\max} = 8.32\text{m/s}^2$ is made during the first seconds and drops to a negative level in the end. The velocity reaches its maximum $v^0_{\max}(t_{\max}) = 11.59$ m/s at the time moment $t_{\max} = 5.5$ s (when $a^0(t_{\max}) = 0$). The optimal program indicated in Fig. 2 meets the actual goals of sprinters, who are trying to apply maximum force during the first meters of the distance in order to maximally speed up, and finish the distance with the remaining energy. In Table 1 the modeled optimal indicators are provided for the results of 11 and 12 seconds. One can see that the higher the energy reserve, the stronger and faster the runner speeds up, and, hence, the faster he is.

2.2 Optimal and suboptimal solutions

In the previous section the problem in our statement is solved completely: the optimal final velocity corresponding to the minimum possible intensity (12) is chosen from (11). Let us illustrate the important possibility to choose the final velocity using the available data for Usain Bolt [9]. Firstly, let us solve the minimum energy problem with the fixed velocity $v(t_f) = 9$ m/s, $t_f = 9.69$ s, taken from the data and, thus, obtain the theoretical suboptimal trajectory. In Table 2 we provide the minimum energy in this case, $\varkappa_1 = 2067.82$ units, and the suboptimal indicators corresponding to it. Let us point out that they are in agreement with the real data presented in [9]. Secondly, we take the final velocity as a free parameter and apply our approach in order to solve the problem for the same $t_f = 9.69$ s. In the Table 2 is is shown that in this case the sprinter needs $\varkappa_2 = 2048.33 < \varkappa_1$ units of energy to run the distance under the same conditions. In the optimal running with the energy supply of $\varkappa_2$ he could achieve the higher maximum velocity and finish with the lower final velocity $v_2(t_f) = 6.32$ m/s, while showing the same time $t_f = 9.69$ s.

2.3 Application of the time-optimal control problem

The natural question arises: If Bolt used more energy and ran suboptimaly, how fast could he have run in the optimal regime? The answer is given by the solution to the time-optimal control problem. Under the same conditions with the energy reserve of $\varkappa_2 = 2067.82$ units, he could have run the distance in $t_f^0 = 9.56$ s. In Fig. 3 we compare the suboptimal trajectory (see Table 2) and the time-optimal trajectory of the sprinter with energy supply of 2067.82 units. One can see in the plot that the faster program (dashed line) is associated with achieving the higher maximum velocity. Finally, let us note, that the obtained time, $t_f^0 = 9.56$ s, is close to the current world record of Usain Bolt (9.58 s), which was set one year later (2009) at the World Championships in Berlin (see [7]). To a certain extent, this indicates the validity of the chosen approach based on the minimum energy control problem in the framework of the considered model of sprinting.

3 Conclusions

The biomechanical model of sprinting [4] is developed in the paper by means of applying methods of optimal control. The optimal trajectories of the force, acceleration, and velocity are modeled on the basis of parameters of a sprinter and weather conditions (wind). The modeling results with the application of methods of the mathematical control theory generate realistic trajectories of optimal running over 100 m. It is shown that the optimality of a distribution of energy over the distance can be characterized by the minimum energy criterion. As a future extension of the model, it would be interesting to consider a hybrid control system. For example, such a system could be a model of long jump that combines a horizontal run-up and a vertical (impulse) take-off.

A Formulation of the minimum norm problem

For a fixed final state x^f, using the Cauchy formula (10), we obtain a system of integral equations

$$\begin{aligned} \int_0^{t^f} h^{(1)}[t,\tau]u(\tau)d\tau = c_1, \\ \int_0^{t^f} h^{(2)}[t,\tau]u(\tau)d\tau = c_2, \end{aligned} \tag{13}$$

where

$$c_1 = x_1^f - \int_0^{t_f} w_s(\tau)d\tau, \quad c_2 = x_2^f. \tag{14}$$

Let us consider the functions $h^{(1)}[t,\tau]$ and $h^{(2)}[t,\tau]$ of the argument $\tau(0 \le \tau \le t_f)$ as elements $h(\tau)$ of the functional space $\mathscr{L}^{(2)}_{[0,t_f]}$, where the norm $\rho[h(\tau)]$ is given by the relation

$$\rho[h(\tau)] = \Big[\int_0^{t_f} h^2(\tau)d\tau\Big]^{\frac{1}{2}}. \tag{15}$$

It is known that every function $u(\tau)$ that satisfies the condition

$$\int_0^{t_f} u^2(\tau)d\tau < \infty, \tag{16}$$

generates a linear operation in the space $\mathscr{L}^{(2)}_{[0,t_f]}$:

$$\phi_u[h(\tau)] = \int_0^{t_f} h(\tau)u(\tau)d\tau. \tag{17}$$

Here the norm $\rho^*[\phi_u]$ of the operation $\phi_u[h(\tau)]$ is in turn given by the relation

$$\rho^*[\phi_u] = \rho^*[u] = \Big[\int_0^{t_f} u^2(\tau)d\tau\Big]^{\frac{1}{2}}. \tag{18}$$

Since the possible control $u(\tau)$ obviously satisfies the condition (16), based on (13) we conclude that the values c_1 and c_2 at the time moment $t = t_f$ can be considered as the result of the linear operation $\phi_u[h]$ generated by the function $u(\tau)$ to the functions $h^{(1)}(\tau) = \gamma^{-1}(1 - e^{-\frac{\gamma}{m}(t-\tau)})$ and $h^{(2)}(\tau) = m^{-1}e^{-\frac{\gamma}{m}(t-\tau)}$ (9). Moreover, we see that in this case the intensity $\varkappa[u]$ (12) chosen from physical considerations is precisely the norm $\rho^*[\phi_u]$ (18) of the operation $\phi_u[h(\tau)]$. The functions $u(\tau)$ that constitute the space $\mathscr{L}^{(2)}_{[0,t_f]}\{u(\tau)\}$ adjoint to $\mathscr{L}^{(2)}_{[0,t_f]}\{h(\tau)\}$ are called *admissible controls*.

Thus, to solve the minimum energy problem formulated in Subsection 1.2, one needs to find an optimal control $u^0(t)$ (among the admissible controls $u(t)$) that moves system (3) from the initial state $x(t_0)$ to the final state $x(t_f) = x^f$ and has the lowest possible norm $\rho^*[u^0]$ (18).

B Analytical solution to the control problem

To solve the problem of finding $\rho^*[u^0]$ (18), we follow the approach proposed in [3] and based on the moment problem. First, we need to solve the minimum problem

$$\rho^0 = \min \rho[h] = \min_l \rho\big[l_1 h^{(1)} + l_2 h^{(2)}\big] = \rho\big[l_1^0 h^{(1)} + l_2^0 h^{(2)}\big] = \rho[h^0(\tau)] \quad (19)$$

subject to constraints

$$c_1 l_1 + c_2 l_2 = 1. \quad (20)$$

That is, in our case we search for the minimum of the function

$$\min_l \int_0^{t_f} \{l_1 \frac{1}{\gamma}(1 - e^{-\frac{\gamma}{m}(t_f-\tau)}) + l_2 \frac{1}{m} e^{-\frac{\gamma}{m}(t_f-\tau)}\}^2 d\tau, \quad (21)$$

subject to (20).

The Lagrangian for this problem has the form

$$\begin{aligned} \mathcal{L} &= t_f(\frac{l_1}{\gamma})^2 + 2\frac{l_1 m}{\gamma^2}(\frac{l_2}{m} - \frac{l_1}{\gamma})(1 - e^{-t_f \frac{\gamma}{m}}) && (22)\\ &+ \frac{m}{2\gamma}(\frac{l_2}{m} - \frac{l_1}{\gamma})^2(1 - e^{-2t_f \frac{\gamma}{m}}) + \lambda(c_1 l_1 + c_2 l_2 - 1), && (23) \end{aligned}$$

where λ is the Lagrange multiplier.

The necessary conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial l_1} &= G_1 l_1 + G_2 l_2 + c_1 \lambda = 0 && (24)\\ \frac{\partial \mathcal{L}}{\partial l_2} &= G_2 l_1 + G_3 l_2 + c_2 \lambda = 0, && (25) \end{aligned}$$

where the parameters are calculated by the formulas

$$\begin{aligned} G_1 &= \big(\frac{2t_f}{\gamma^2} - 4\frac{m}{\gamma^3}(1 - e^{-\frac{\gamma}{m}t_f}) + \frac{m}{\gamma^3}(1 - e^{-\frac{2\gamma}{m}t_f})\big), && (26)\\ G_2 &= \big(\frac{2}{\gamma^2}(1 - e^{-\frac{\gamma}{m}t_f}) - \frac{1}{\gamma^2}(1 - e^{-\frac{2\gamma}{m}t_f})\big), && (27)\\ G_3 &= \frac{1}{\gamma m}(1 - e^{-\frac{2\gamma}{m}t_f}), && (28) \end{aligned}$$

together with (20) form a system of linear equations with respect to l_1, l_2, and λ. The solution is given by the formulas

$$l_1^0 = \frac{c_1 G_3 - c_2 G_2}{c_2^2 G_1 - 2c_1 c_2 G_2 + c_1^2 G_3}, \quad l_2^0 = \frac{1 - c_1 l_1^0}{c_2}. \tag{29}$$

Thus, we have found the desired function

$$h^0[\tau] = l_1^0 \frac{1}{\gamma}(1 - e^{-\frac{\gamma}{m}(t_f - \tau)}) + l_2^0 \frac{1}{m} e^{-\frac{\gamma}{m}(t_f - \tau)}. \tag{30}$$

that provides the minimum norm in problem (19), (20):

$$(\rho^0)^2 = \int_0^{t_f} \left[h^0(\tau)\right]^2 d\tau. \tag{31}$$

The optimal control can be found as the solution to the corresponding maximum problem (see [3]):

$$\int_0^{t_f} h^0(\tau) u^0(\tau) d\tau = \max_u \int_0^{tf} h^0(\tau) u(\tau) d\tau = 1, \tag{32}$$

subject to constraint

$$\int_0^{t_f} u^2(\tau) d\tau = (\frac{1}{\rho^0})^2. \tag{33}$$

Finally, the optimal control is given by the formula

$$u^0(t) = (\frac{1}{\rho^0})^2 h^0(t), \tag{34}$$

which is calculated analytically using relations (26)-(31).

References

[1] L. Pontryagin, V. Boltyansky, R. Gamkrelidze, and E. Mishchenko, *The Mathematical Theory of Optimal Processes.* Interscience, New York, 1962.

[2] R. Bellman, *Dynamic Programming.* Princeton, NJ, USA: Princeton University Press, 1 ed., 1957.

[3] N. Krasovskii, *Theory of motion control.* Nauka, Moscow, 1968.

[4] S. Gaudet, "A physical model of sprinting," *Journal of biomechanics*, vol. 47, no. 12, pp. 2933–2940, 2014.

[5] J. B. Keller, "A theory of competitive running," *Physics today*, vol. 26, no. 9, pp. 43–47, 1973.

[6] J. Mureika, "A realistic quasi-physical model of the 100 m dash," *Canadian Journal of Physics*, vol. 79, no. 4, pp. 697–713, 2001.

[7] O. Helene and M. Yamashita, "The force, power, and energy of the 100 meter sprint," *American Journal of Physics*, vol. 78, no. 3, pp. 307–309, 2010.

[8] J. H. Gómez, V. Marquina, and R. Gómez, "On the performance of Usain Bolt in the 100 m sprint," *European Journal of Physics*, vol. 34, no. 5, pp. 1227–1233, 2013.

[9] H. Eriksen, J. Kristiansen, Ø. Langangen, and I. Wehus, "How fast could Usain Bolt have run? A dynamical study," *American Journal of Physics*, vol. 77, no. 3, pp. 224–228, 2009.

[10] R. H. Miller, B. R. Umberger, J. Hamill, and G. E. Caldwell, "Evaluation of the minimum energy hypothesis and other potential optimality criteria for human running," *Proceedings of the Royal Society B: Biological Sciences*, vol. 279, no. 1733, pp. 1498–1505, 2012.

[11] P. Hartman, *Ordinary differential equations.* New York, USA: John Wiley and Sons, 1964.

[12] L. Pontryagin, *Ordinary Differential Equations.* Adiwes international series in mathematics, Pergamon Press, 1962.

Table 1: Optimal indicators for several times t_f: $v^0(t_f)$ is the optimal velocity at finish, $\varkappa^0$ is the minimum energy, $a^0_{\max}$ is the maximum acceleration, $v^0_{\max}$ is the maximum velocity, and $t_{\max}$ is the time of maximum velocity.

t_f, s	$v^0(t_f) = \tilde{x}_2^f$, m/s	$\varkappa^0 = \varkappa[u^0]$	$a^0_{\max}$, m/s^2	$v^0_{\max}$, m/s	$t_{\max}$, s
10	6.06	2004.14	8.32	11.59	5.5
11	5.35	1877.31	7.35	10.37	6.00
12	4.78	1769.77	6.57	9.35	6.49

Table 2: Modeling results of Usain Bolt's record perfromance at the 2008 Olympic Games in Beijing.

$t_f = 9.69$ s	$v(t_f)$, m/s	$\varkappa$	$a_{\max}$, m/s^2	$v_{\max}$, m/s	$t_{\max}$, s
Suboptimal solution	9	2067.82	8.29	11.66	5.83
Optimal solution	6.32	2048.33	8.68	12.03	5.35

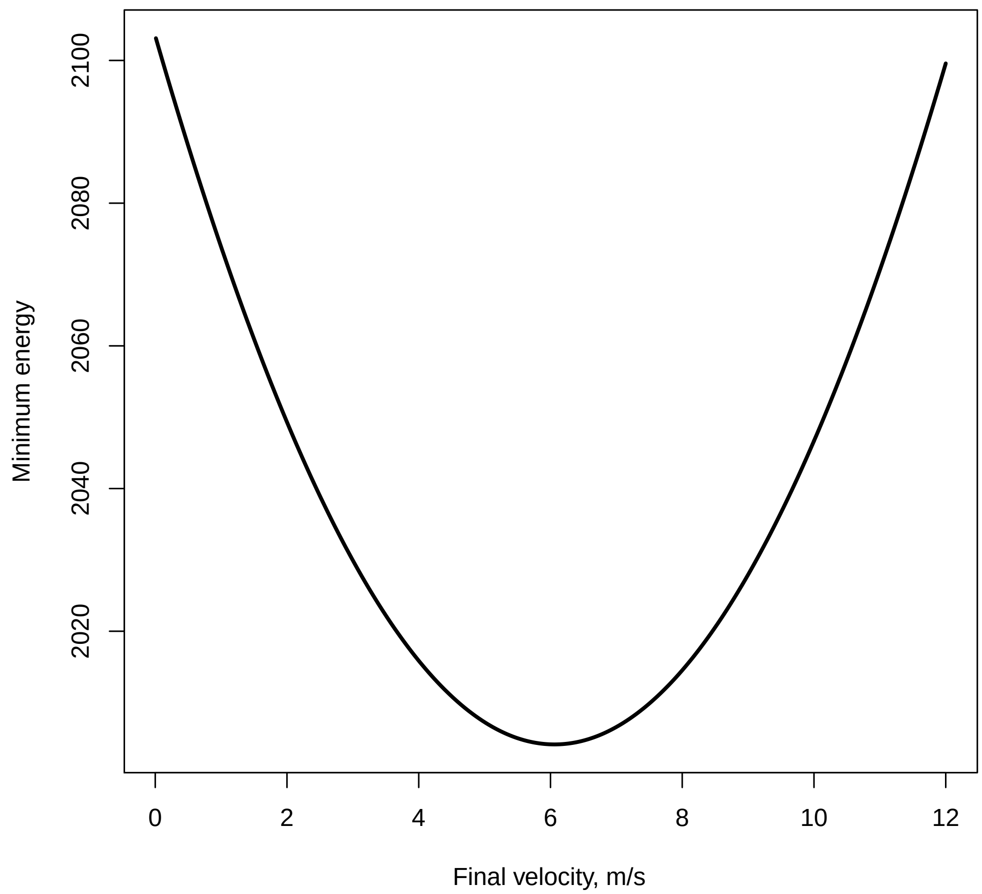

Figure 1: Plot of the minimum energy function versus the final velocity for $t_f = 10$.

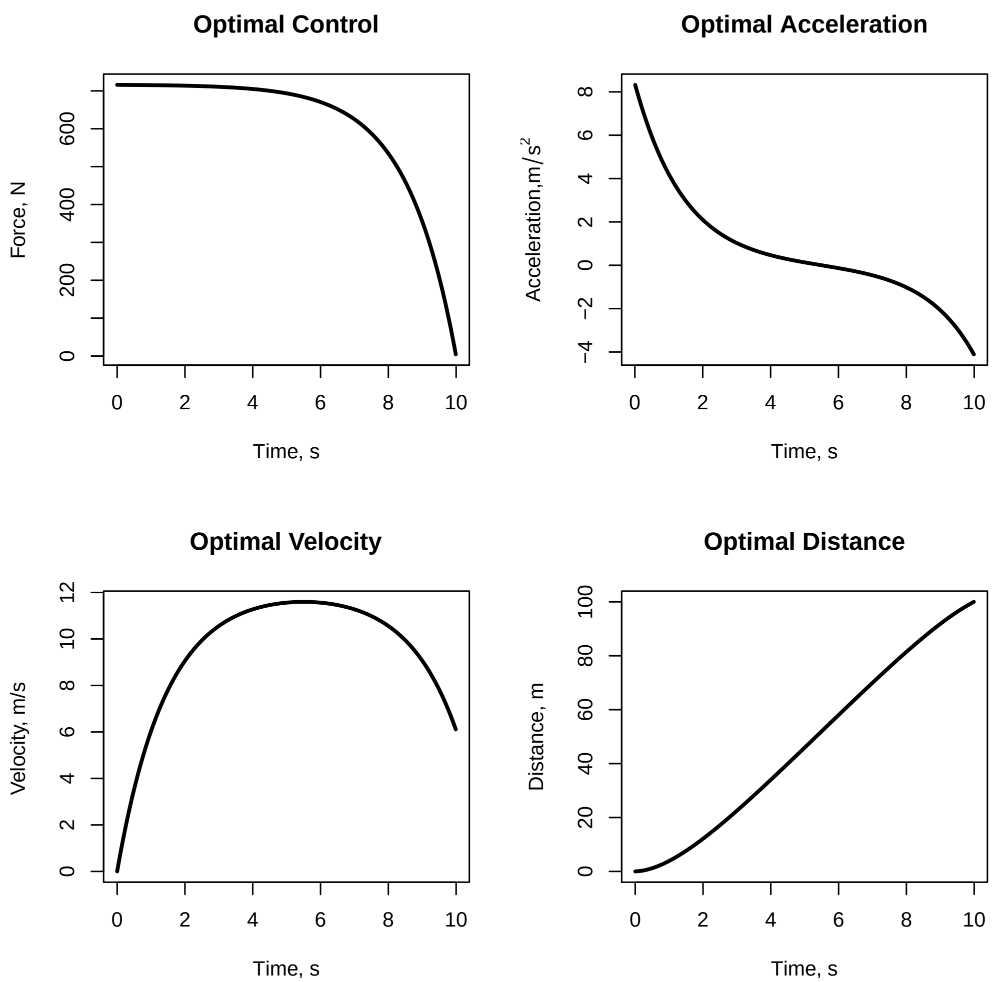

Figure 2: Plots of optimal trajectories for $t_f = 10$ s.

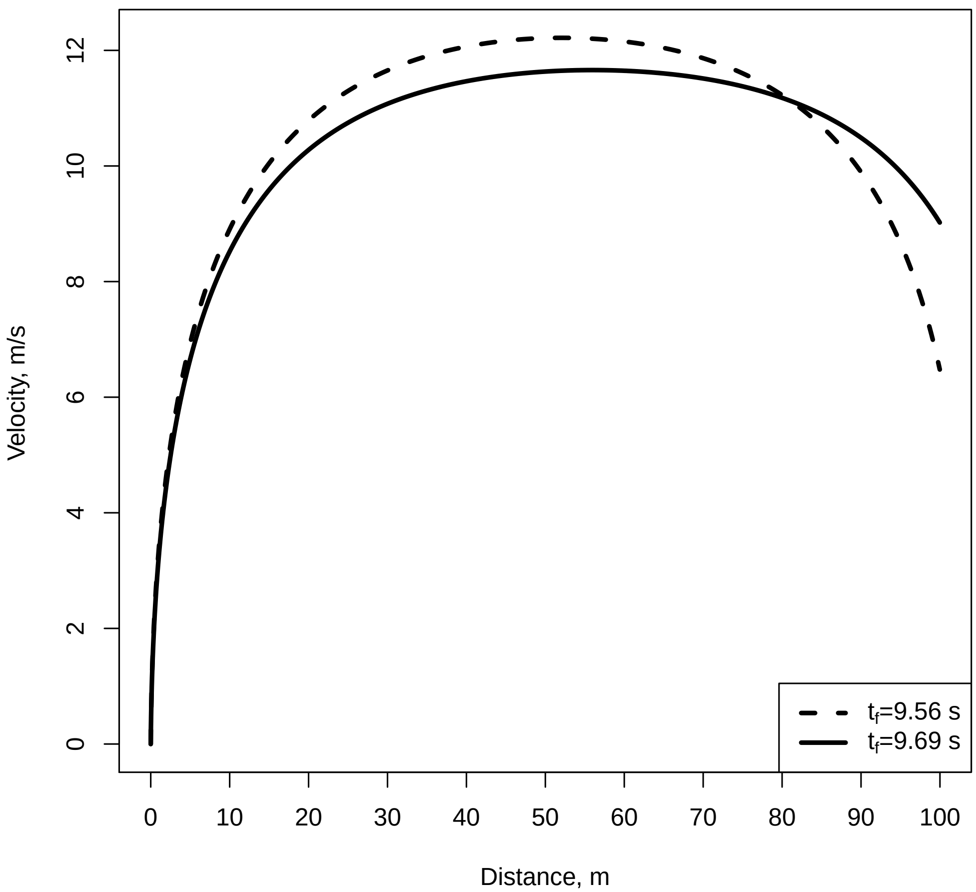

Figure 3: Comparison of the time-optimal trajectory (dashed line) with the suboptimal trajectory (solid line) in the model of Usain Bolt's running at the 2008 Olympic Games in Beijing.