An exact solution method for binary equilibrium problems with compensation and the power market uplift problem*

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Mathematical cheat sheet, March 17, 2016

This document summarizes the mathematical formulation for easier reference of the GAMS implementation provided at `http://danielhuppmann.github.io/binary_equilibrium/`. All equation numbers are identical to those in the manuscript (version as of February 4, 2016).

Theoretical formulation (Section 3.4)

$$\min_{\substack{x_i, y_i, \widetilde{y}_i^{(\overline{\mathbf{x}}_\mathbf{i})}, \widetilde{\lambda}_i^{(\overline{\mathbf{x}}_\mathbf{i})} \\ \kappa_i^{(\overline{\mathbf{x}}_\mathbf{i})}, \zeta_i^{(\overline{\mathbf{x}}_\mathbf{i})}}} \quad F\Big(\big(x_i, y_i\big)_{i\in I}\Big) + G\Big(\big(\zeta_i^{(\overline{\mathbf{x}}_\mathbf{i})}\big)_{i\in I}\Big) \tag{14a}$$

$$\text{s.t.} \quad \nabla_{y_i} f_i\Big(\mathbf{1}, \widetilde{y}_i^{(\mathbf{1})}, y_{-i}\Big) + \big(\widetilde{\lambda}_i^{(\mathbf{1})}\big)^T \nabla_{y_i} g_i\Big(\mathbf{1}, \widetilde{y}_i^{(\mathbf{1})}\Big) = 0 \tag{14b}$$

$$0 \le -g_i\Big(\mathbf{1}, \widetilde{y}_i^{(\mathbf{1})}\Big) \perp \widetilde{\lambda}_i^{(\mathbf{1})} \ge 0 \tag{14c}$$

$$\nabla_{y_i} f_i\Big(\mathbf{0}, \widetilde{y}_i^{(\mathbf{0})}, y_{-i}\Big) + \big(\widetilde{\lambda}_i^{(\mathbf{0})}\big)^T \nabla_{y_i} g_i\Big(\mathbf{0}, \widetilde{y}_i^{(\mathbf{0})}\Big) = 0 \tag{14d}$$

$$0 \le -g_i\Big(\mathbf{0}, \widetilde{y}_i^{(\mathbf{0})}\Big) \perp \widetilde{\lambda}_i^{(\mathbf{0})} \ge 0 \tag{14e}$$

$$f_i\Big(\mathbf{1}, y_i^{(\mathbf{1})}, y_{-i}\Big) + \kappa_i^{(\mathbf{1})} - \zeta_i^{(\mathbf{1})} - \kappa_i^{(\mathbf{0})} + \zeta_i^{(\mathbf{0})} = f_i\Big(\mathbf{0}, y_i^{(\mathbf{0})}, y_{-i}\Big) \tag{14f}$$

$$\kappa_i^{(\mathbf{1})} + \zeta_i^{(\mathbf{1})} \le x_i \, \widetilde{K} \tag{14g}$$

$$\kappa_i^{(\mathbf{0})} + \zeta_i^{(\mathbf{0})} \le \big(1 - x_i\big) \, \widetilde{K} \tag{14h}$$

$$\widetilde{y}_i^{(\mathbf{0})} - x_i \, \widetilde{K} \le y_i \le \widetilde{y}_i^{(\mathbf{0})} + x_i \, \widetilde{K} \tag{14i}$$

$$\widetilde{y}_i^{(\mathbf{1})} - \big(1 - x_i\big) \, \widetilde{K} \le y_i \le \widetilde{y}_i^{(\mathbf{1})} + \big(1 - x_i\big) \, \widetilde{K} \tag{14j}$$

$$x_i \in \{0,1\}, \big(y_i, \widetilde{y}_i^{(\overline{\mathbf{x}}_\mathbf{i})}\big) \in \mathbb{R}^{3m}, \big(\lambda_i^{(\overline{\mathbf{x}}_\mathbf{i})}, \kappa_i^{(\overline{\mathbf{x}}_\mathbf{i})}, \zeta_i^{(\overline{\mathbf{x}}_\mathbf{i})}\big) \in \mathbb{R}_+^{2k+4}$$

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The power market application (Section 4)

The generator's optimization problem

Each generator $i \in I$ seeks to maximize her profits from generating and selling electricity over the time horizon $t \in T$:

$$\min_{x_{ti}, y_{ti}, z_{ti}^{\text{on}}, z_{ti}^{\text{off}}} \quad -p_{tn(i)} y_{ti} + c_i^G y_{ti} + c_i^{\text{on}} z_{ti}^{\text{on}} + c_i^{\text{off}} z_{ti}^{\text{off}} \tag{15a}$$

$$\text{s.t.} \quad x_{ti} g_i^{min} \leq y_{ti} \leq x_{ti} g_i^{max} \quad \left(\alpha_{ti}^{\text{on}}, \beta_{ti}^{\text{on}}\right) \tag{15b}$$

$$x_{ti} - x_{(t-1)i} = z_{ti}^{\text{on}} - z_{ti}^{\text{off}} \tag{15c}$$

$$x_{ti} \in \{0,1\}, \quad y_{ti}, z_{ti}^{\text{on}}, z_{ti}^{\text{off}} \in \mathbb{R}_+$$

Demand for electricity and network feasibility (market clearing)

The spot market is cleared by a player seeking to maximize the welfare (utility) of consumers while guaranteeing feasibility of the transmission system. She takes the unit commitment at time t and the dispatch of each power plant as given, written here as y_{ti}, and she assigns locational prices p_{tn}, which the generators consider in their optimization problems:

$$\min_{d_{tj}, \delta_{tn}} \quad \sum_{j \in J} -u_{tj}^D d_{tj} \tag{16a}$$

$$\text{s.t.} \quad \sum_{j \in J_n} d_{tj} - \sum_{i \in I_n} y_{ti} + \sum_{m \in N} B_{nm} \delta_{tm} = 0 \quad (p_{tn}) \tag{16b}$$

$$d_{tj}^{max} - d_{tj} \geq 0 \quad (\nu_{tj}) \tag{16c}$$

$$f_l^{max} - \sum_{n \in N} H_{ln} \delta_{tn} \geq 0 \quad (\mu_{tl}^+) \tag{16d}$$

$$f_l^{max} + \sum_{n \in N} H_{ln} \delta_{tn} \geq 0 \quad (\mu_{tl}^-) \tag{16e}$$

$$\pi - \delta_{tn} \geq 0 \quad (\xi_{tn}^+) \tag{16f}$$

$$\pi + \delta_{tn} \geq 0 \quad (\xi_{tn}^-) \tag{16g}$$

$$\delta_{t\hat{n}} = 0 \quad (\gamma_t) \tag{16h}$$

Sets & Mappings		
	$n, m \in N$	... nodes
	$t \in T$	... time step, hours
	$i \in I$	... generators, power plant units
	$j \in J$	... load, demand units
	$l \in L$	... power lines
	$i \in I_n, j \in J_n$	... generator/load unit mapping to node n
	$n(i), n(j)$	... node mapping to generator i/load unit j
	$\phi \in \Phi$	... set of dispatch options (schedules) for each generator
	$t \in T_\phi$	... hours in which a generator is active in dispatch option ϕ
Primal variables		
	x_{ti}	... on/off decision for generator i in hour t
	$z_{ti}^{\text{on}}, z_{ti}^{\text{off}}$	... inter-temporal start-up/shut-down decision
	y_{ti}	... actual generation by generator i in hour t
	y_{ti}^{on}	... generation if binary variable is fixed at $\overline{\mathbf{x}}_\mathbf{i}$
	d_{tj}	... demand by unit j in hour t
	δ_{tn}	... voltage angle
Dual variables		
	$\alpha_{ti}^{\text{on}}, \beta_{ti}^{\text{on}}$	... dual to minimum activity/maximum generation capacity
	p_{tn}	... locational marginal price
	ν_{tj}	... dual to maximum load constraint
	$\mu_{tl}^{+}, \mu_{tl}^{-}$	... dual to voltage angle band constraints
	$\xi_{tn}^{+}, \xi_{tn}^{-}$	... dual to thermal line capacity constraints
	γ_t	... dual to slack bus constraints
Switch and compensation variables		
	$\kappa_{ti}^{\text{on}}, \kappa_{ti}^{\text{off}}$	... switch value (defined per time step)
	ζ_i	... compensation payment (defined over entire time horizon)
Parameters		
	c_i^G	... linear generation costs
	$c_i^{\text{on}}, c_i^{\text{off}}$	... start-up/shut-down costs
	c_ϕ^D	... total commitment costs in dispatch option ϕ (start-up, shut-down)
	g_i^{min}	... minimum activity level if power plant is online
	g_i^{max}	... maximum generation capacity
	x_i^{init}	... power plant status at start of model horizon ($t = 0$)
	u_{tj}^D	... utility of demand unit j for using electricity
	d_{tj}^{max}	... maximum load of unit j
	f_l^{max}	... thermal capacity of power line l
	B_{nk}, H_{lk}	... line/node susceptance/network transfer matrices

Table 2: Notation for the nodal power market problem

The bi-level multi-objective program

$$\min \sum_{t\in T}\left[\sum_{i\in I} c_i^G y_{ti} + c_i^{\text{on}} z_{ti}^{\text{on}} + c_i^{\text{off}} z_{ti}^{\text{off}} - \sum_{j\in J} u_{tj}^D d_{tj}\right] + \sum_{i\in I}\zeta_i \tag{20}$$

$$\text{s.t.} \qquad 0 \le -u_{tj}^D + p_{tn(j)} + \nu_{tj} \quad \perp \quad d_{tj} \ge 0 \tag{19a}$$

$$0 = \sum_{m\in N} B_{mn} p_{tm} + \sum_{l\in L} H_{ln}\big(\mu_{tl}^+ - \mu_{tl}^-\big) + \xi_{tn}^+ - \xi_{tn}^- - \begin{Bmatrix} \gamma_t & \text{if } n = \hat{n} \\ 0 & \text{else}\end{Bmatrix} \quad , \quad \delta_{tn} \text{ (free)} \tag{19b}$$

$$0 = \sum_{j\in J_n} d_{tj} - \sum_{i\in I_n} y_{ti} + \sum_{m\in N} B_{nm}\delta_{tm} \quad , \quad p_{tn} \text{ (free)} \tag{19c}$$

$$0 \le d_{tj}^{max} - d_{tj} \quad \perp \quad \nu_{tj} \ge 0 \tag{19d}$$

$$0 \le f_l^{max} - \sum_{n\in N} H_{ln}\delta_{tn} \quad \perp \quad \mu_{tl}^+ \ge 0 \tag{19e}$$

$$0 \le f_l^{max} + \sum_{n\in N} H_{ln}\delta_{tn} \quad \perp \quad \mu_{tl}^- \ge 0 \tag{19f}$$

$$0 \le \pi - \delta_{tn} \quad \perp \quad \xi_{tn}^+ \ge 0 \tag{19g}$$

$$0 \le \pi + \delta_{tn} \quad \perp \quad \xi_{tn}^- \ge 0 \tag{19h}$$

$$0 = \delta_{t\hat{n}} \quad , \quad \gamma_t \text{ (free)} \tag{19i}$$

$$0 = c_i^G - p_{tn(i)} + \beta_{ti}^{\text{on}} - \alpha_{ti}^{\text{on}} \quad , \quad y_{ti}^{\text{on}} \text{ (free)} \tag{17a}$$

$$0 \le -g_{ti}^{min} + y_{ti}^{\text{on}} \quad \perp \quad \alpha_{ti}^{\text{on}} \ge 0 \tag{17b}$$

$$0 \le g_{ti}^{max} - y_{ti}^{\text{on}} \quad \perp \quad \beta_{ti}^{\text{on}} \ge 0 \tag{17c}$$

$$x_{(t-1)i} + z_{ti}^{\text{on}} - z_{ti}^{\text{off}} = x_{ti} \tag{21a}$$

$$\beta_{ti}^{\text{on}} g_{ti}^{max} - \alpha_{ti}^{\text{on}} g_{ti}^{min} - \kappa_{ti}^{\text{on}} + \kappa_{ti}^{\text{off}} = 0 \tag{21b}$$

$$|\kappa_{ti}^{\text{on}}| \le x_{ti}\,\widetilde{K} \tag{21c}$$

$$|\kappa_{ti}^{\text{off}}| \le (1 - x_{ti})\,\widetilde{K} \tag{21d}$$

$$\sum_{t\in T}\left[\kappa_{ti}^{\text{on}} - c_i^{\text{on}} z_{ti}^{\text{on}} - c_i^{\text{off}} z_{ti}^{\text{off}}\right] + \zeta_i \ge \sum_{t\in T_\phi}\left[\beta_{ti}^{\text{on}} g_{ti}^{max} - \alpha_{ti}^{\text{on}} g_{ti}^{min}\right] - c_{\phi i}^D \qquad \forall\, \phi \in \Phi \tag{21e}$$

$$0 \le y_{ti} \le x_{ti} g_{ti}^{max} \tag{21f}$$

$$y_{ti}^{\text{on}} - (1 - x_{ti})\, g_{ti}^{max} \le y_{ti} \le y_{ti}^{\text{on}} + (1 - x_{ti})\, g_{ti}^{max} \tag{21g}$$