A WAY OF DESCRIBING POLLUTION PROPAGATION IN RIVERS USING THE NETWORK FLOW APPROACH AND POSSIBLE EXTENSIONS

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1. Let us consider a given reach of a river. Variation of the pollutant concentration in the water with time along a given reach of the river can be described by the equation*)

$$\frac{\partial C}{\partial t} = -\frac{1}{A}\frac{\partial (QC)}{\partial x} - KC + \delta \qquad (1)$$

where

- C = the concentration of pollutant ,
- Q = the river flow in the section of the river we are considering ,
- A = the cross-section area ,
- K = the decay coefficient of the first order reaction ,
- δ = the amount of pollutant added or withdrawn from the stream per unit of time and unit of volume ,
- x = the distance along the river ,
- t = time .

In equation (1) the diffusion term $D(\partial^2 C/\partial X^2)$ is omitted (D is the diffusion coefficient for the pollutant). This represents a reasonable approximation when the mean velocity of the stream is high relative to the longitudinal dispersion. If we assume that the flow is stationary, then $(\partial C/\partial t) = 0$ and equation (1) reduces to the form

$$\frac{d(QC)}{dx} = -KAC + A\delta \qquad (2)$$

*) See, for example, reference [8].

It describes the variation of pollutant concentration along the river. We can write the equation (2) in a finite difference form for the i-th reach of the river (see fig. 1).

$$Q_{i+1}C_{i+1} - Q_iC_i = -K_iA_i\Delta x_iC_i + A_i\Delta x_i\delta_i \qquad (3)$$

Here

C_i and C_{i+1} = the concentration of the pollutant at the beginning and end cross-sections of the i-th reach of the river,

Q_i and Q_{i+1} = the river flows coming to the i-th reach per unit of time and going away, correspondingly,

K_i = the decay coefficient of the first order reaction for the i-th reach,

δ_i = the amount of pollutant added or withdrawn from the stream per unit of time and unit of volume at i-th reach,

A_i = the average cross-section area for i-th reach,

Δx_i = the length of the i-th reach.

Equation (3) describes the conservation of the pollutant mass at i-th reach of the river and could be obtained directly but not as the finite difference analog of equation (2).

Equation (3) could be rewritten in the form

$$G_{i+1} = G_i + Z_i - \sigma_i \qquad (4)$$

where

$$G_i = C_i Q_i \qquad (5)$$

$$Z_i = A_i \Delta x_i \delta_i \qquad (6)$$

$$\sigma_i = K_i A_i \Delta x_i C_i \qquad (7)$$

G_i and G_{i+1} = the amount of pollutant coming to the i-th reach per unit of time and going away, correspondingly,

Z_i = the amount of pollutant added to i-th reach of river by industries, cities, agricultural runoff during a unit of time.

σ_i = the amount of pollutant removed from i-th reach of the river per unit of time by the purification process. This is due to deposition and biochemical oxidation which convert the pollutant into another, perhaps less degrading, form.

Note that in (3) and (7) we suppose that purification of the river at i-th reach depends on the concentration of pollutant coming to this reach. But it is more natural to assume that it depends on some average concentration $\bar{C}_i$

$$\bar{C}_i = \alpha C_i + (1 - \alpha) C_{i+1} \qquad (8)$$

where α is some weighting coefficient

$$0 \leq \alpha \leq 1 \qquad (9)$$

Then formula (7) will take the form

$$\sigma_i = K_i A_i \Delta x_i \left[d C_i + (1 - \alpha) C_{i+1} \right] \qquad (10)$$

and from equation(3) we get the equation for the output concentration at i-th reach

$$C_{i+1} = \frac{G_i + Z_i - \alpha K_i A_i \Delta x_i C_i}{Q_i + R_i - S_i + (1 - \alpha) K A_i \Delta x_i} \qquad (11)$$

Here we have used the equation of conservation of the flow at i-th reach

$$Q_{i+1} = Q_i + R_i - S_i \qquad (12)$$

where

R_i = the total inflow from tributaries, surface flow, underground water and so on, coming to the i-th reach,

S_i = the water supply from i-th reach for industries, cities, irrigation systems and so on.*)

Value Z_i in equations (4) and (11) could be generally expressed as the sum

$$Z_i = Z_{R_i} + Z_{I_i} - Z_{S_i} \qquad (13)$$

where

Z_{R_i} = the amount of pollutant contributed by tributaries and surface runoff,

*) Values R_i and S_i, in general, could also include water coming from and water put in storages from the river, correspondingly.

Z_{I_i} = the amount of pollutant from industries,

Z_{S_i} = the amount of pollutant removed by water supply.

Each of these components can be presented in more detailed form. For example, for Z_{S_i} we have

$$Z_{S_i} = [\alpha C_i + (1 - \alpha) C_{i+1}] \cdot S_i \qquad (14)$$

and then equation (11) can be rewritten as follows

$$C_{i+1} = \frac{G_i + Z_{R_i} + Z_{I_i} - \alpha C_i (S_i + K_i A_i \Delta x_i)}{Q_i + R_i - S_i + (1 - \alpha)(S_i + K_i A_i \Delta x_i)} \qquad (15)$$

As was indicated in equation (9) the coefficient in the above formulae can take any value from 0 to 1 and can be estimated on the basis of additional physical conditions or in connection with a desired finite-difference approximation.*) The relation (8) could be rewritten in the following form

$$\bar{C}_i = C_i + (1 - \alpha)\Delta C_i \qquad (8')$$

where $\Delta C_i = C_{i+1} - C_i$ is a small value in comparison with C_i. That means that even if we take $\bar{C}_i = C_i$ in the corresponding formulae as $\alpha = 0$, the error will be small enough.

Therefore let us consider the case where σ_i is defined by formula (7) which after employing equation (5) will take the form

$$\sigma_i = K_i A_i \Delta x_i \frac{G_i}{Q_i} \qquad (16)$$

*) Some recommendations concerning the choice of the coefficient α have been discussed in many papers. See, for example, [4] (p. 9), [2] (p. 128) and [8] (p. 134)

Using (16) it is possible to prepare the table (matrix) for σ, which depends on two parameters: G and $\beta = KA\Delta x/Q$ (see Table 1). In addition, we can prepare a table (matrix) describing the concentration C, which also depends on two parameters: C and Q (see Table 2).

Tables 1 and 2 allow us to solve the problem of pollution distribution along the river as a network flow problem, taking into account the discharge of pollutant into the river as well as the process of self-purification. Table 1 supplies σ for a given G_i and Q_i when we solve the systems (4) and (12) and Table 2 allows us to go from constraints on concentration

$$C_i \leq C_i^*, \; C_i \geq 0, \qquad i = 1,\ldots,N \tag{17}$$

to some "matrix" constraints on G_i for each given Q_i. We will not touch now on the problem of the accuracy of the tables nor the possible ways of interpolation since this has no fundamental significance.

2. Some problems of optimal control can be formulated for pollution distribution in the river. It seems interesting, for example, to consider the following problem:

Find the maximum possible discharge of a given pollutant along the N reaches of the river

$$\max \{R = \sum_{i=0}^{N-1} Z_i\}, \; Z_i \geq 0, \; i = 0,1,\ldots,N-1 \tag{18}$$

under the condition that concentration at those reaches does not exceed the given standard values.

Value Z_i is related to the concentration by the equation

$$Z_i = -(Q_i - K_i A_i \Delta x_i) C_i + Q_{i+1} C_{i+1}, \qquad (19)$$

which follows from (3) and (6) or from (11) and (12) at $\alpha = 1$. The system of equations (19) for all reaches of the river ($i = 0, 1, \ldots, N-1$) could be presented in matrix form

$$Z = MC \qquad (20)$$

where C and Z are vectors of concentration and discharge (or removal) of pollutant, correspondingly, and M is a square matrix which relates the Z's and C's. The matrix M has an interesting form. It is shown in Table 3. Note that this is a square diagonal matrix with elements on the main diagonal and on the diagonal directly above the main one. All other elements are zero.

The value $(Q_0 - K_0 A_0 \Delta x_0) C_0$ which must be specified for the "0"-th reach according to boundary conditions can be included in the first element of vector Z

$$Z_0' = Z_0 + (Q_0 - K_0 A_0 \Delta x_0) C_0. \qquad (21)$$

Equation (19) at $i = 0$ then takes the form

$$Z_0' = Q_1 C_1 . \qquad (19')$$

There could be a few approaches to the water quality problem. We may, for instance, solve the problem of quantitative distribution of the water first, without consideration of its quality. Then, on the basis of that solution, we could solve the problems of water quality.

Following this approach, the problem of maximum possible discharge of the pollutant along the river, as indicated above, can be formulated as follows

$$\max \{R = \sum_{i=0}^{N-1} z_i\} \qquad (18)$$

subjected to

$$Z - MC = 0 \qquad (20')$$

$$C \leq C^* \qquad (17')$$

$$Z \geq 0 \qquad (22)$$

Up to this point we have not mentioned the possibility of artificial purification of the polluted water. The inclusion of treatment plants in the model seems necessary when we formulate the benefit-cost problem for the use of water in a given region.

If we assume that a given vector Y is the amount of pollutant discharged into the river after treatment, then the benefit-cost problem for treatment to achieve a given water quality "standards" can be formulated as follows:

$$\min \{\mu(Y - Z)\} \qquad (23)$$

subjected to

$$Z - MC = 0 \qquad (20')$$

$$C \leq C^* \qquad (17')$$

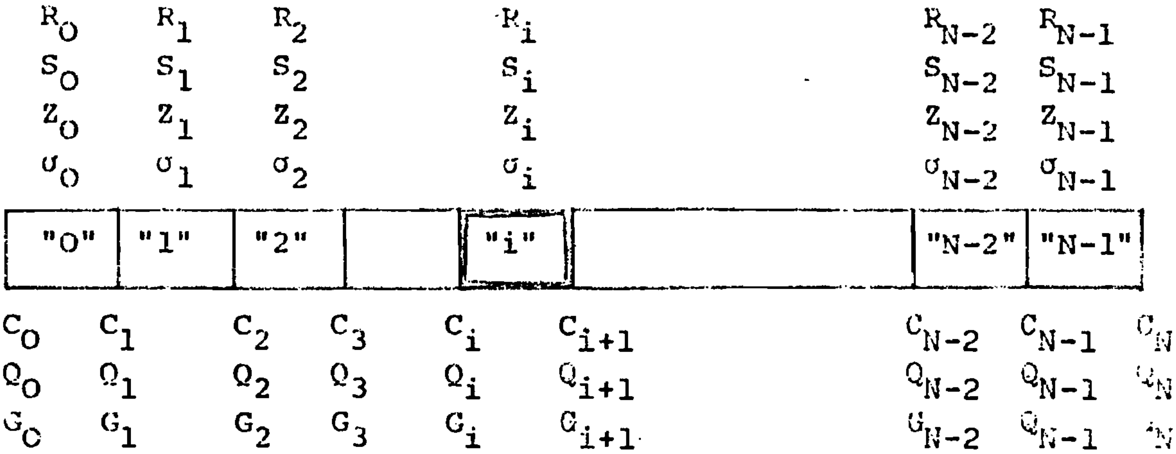
$$Z \leq Y \qquad (24)$$

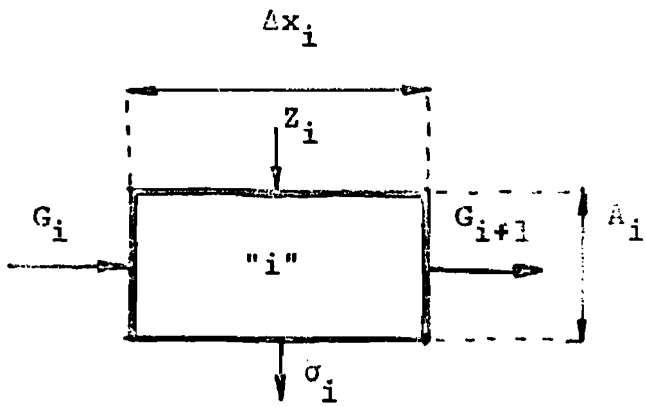
$$Z \geq 0 \qquad (22)$$

where μ is a vector of waste treatment cost coefficients for different reaches of the river. We assume here for simplicity that treatment cost T_i for each reach is a linear function of $(Y_i - Z_i)$. In a more general case, when $T_i = f_i(Z_i)$ is a concave function, it could be reduced to a linear function by piece-wise linearization of the concave function f_i.

The optimal treatment problem could be solved for different values of the vector Y, i.e. different alternatives of waste-source distribution in a given region. This means that it is possible to compare different locations and output of waste-sources and find the best one from the point of minimal cost of polluted water treatment. It is clear that water quality requirements can be met in at least two ways: (1) by decreasing the discharge of pollutant Z_i and (2) by increasing the flow of the river Q_i. Combining the water supply and the water quality problem is more difficult than the former problems for several reasons, among them: increasing the number of variables and equations relating them; transforming some "constant" coefficients in the relationships, such as K_i and A_i, into functions of river flow Q_i. As a first step, we might be able to neglect the variations in K_i and A_i as a function of Q_i. But we still have to deal with the nonlinear relationship among Q, Z and C.

One approach to the nonlinear water supply and water quality management problem was suggested by W.O. Spofford (see [7]). He showed how it is possible to make problems linear by removing nonlinear constraint sets and substituting them in the objective function as a specially constructed penalty function for exceeding water quality standards.





$$C_i = \frac{G_i}{Q_i} \qquad C_{i+1} = \frac{G_{i+1}}{Q_{i+1}}$$

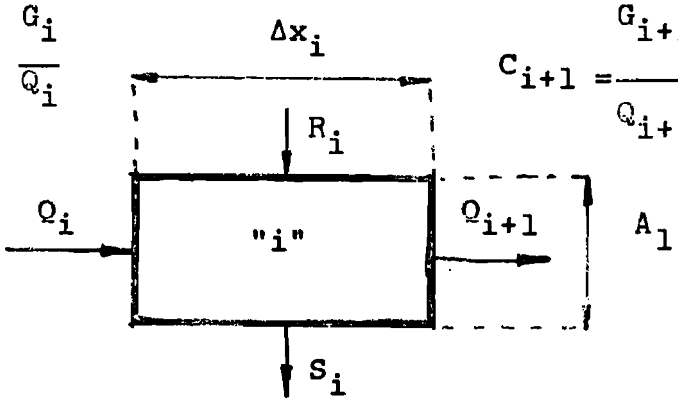


TABLE 1

G^1 / $\beta'' = \frac{Q}{A\Delta x}$	G^1	G^2	G^3	G^4	G^5	G^6	....
β^1	σ^{11}	σ^{12}	σ^{13}	σ^{14}	σ^{15}	σ^{16}	....
β^2	σ^{21}	σ^{22}	$\sigma\sigma^{23}$	σ^{24}	σ^{25}	σ^{26}	....
β^3	σ^{31}	σ^{32}	σ^{33}	σ^{34}	σ^{35}	σ^{36}	....
β^4	σ^{41}	σ^{42}	σ^{43}	σ^{44}	σ^{45}	σ^{46}	....
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Fig. 2

TABLE 2

G^1 \ Q^m	G^1	G^2	G^3	C^4	G^5	G^6	G^7	....
Q^1	c^{11}	c^{12}	c^{13}	c^{14}	c^{15}	c^{16}	c^{17}	....
Q^2	c^{21}	c^{22}	c^{23}	c^{24}	c^{25}	c^{26}	c^{27}	....
Q^3	c^{31}	c^{32}	c^{33}	c^{34}	c^{35}	c^{36}	c^{37}	....
Q^4	c^{41}	c^{42}	c^{43}	c^{44}	c^{45}	c^{46}	c^{47}	....
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Fig. 3

TABLE 3

Water Quality Dispersion "M" Matrix

	C_1	C_2	C_3	····	C_{N-2}	C_{N-1}	C_N
z_0	Q_1	0	0	······	0	0	0
z_1	$-(Q_1-K_1A_1\Delta x_1)$	Q_2	0	·······	0	0	0
z_2	0	$-(Q_2-K_2A_2\Delta x_2)$	Q_3	······	0	0	0
⋮	⋮	⋮	⋮		⋮	⋮	⋮
z_{N-2}	0	0	0	·	$-(Q_{N-2}-K_{N-2}A_{N-2}\Delta x_{N-2})$	Q_{N-1}	0
z_{N-1}	0	0	0	··········		$-(Q_{N-1}-K_{N-1}A_{N-1}\Delta x_{N-1})$	Q_N

Fig. 4

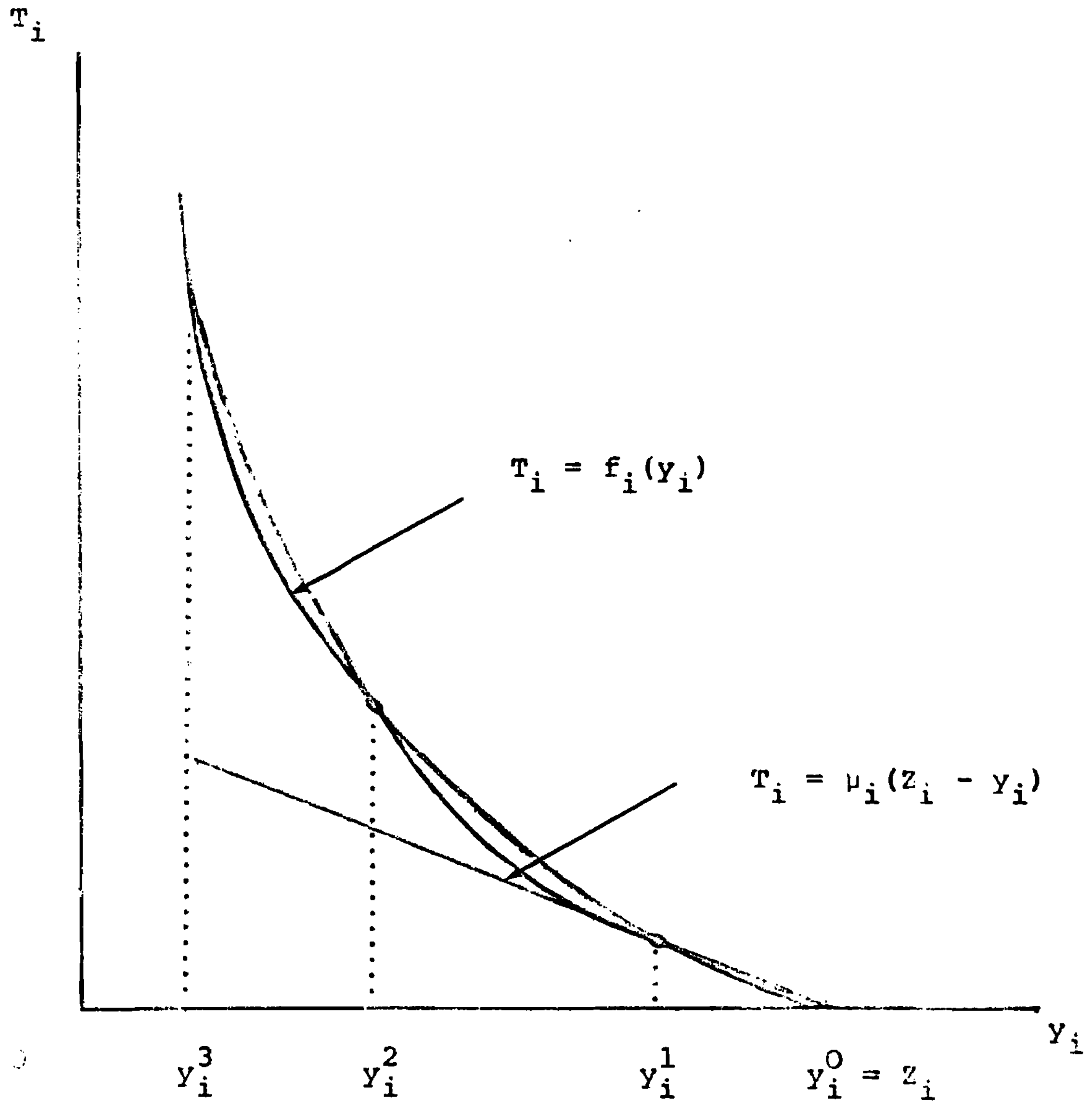


Fig. 5

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