SOME APPROACHES TO DETERMINING THE HEIGHT OF DIKES ALONG RIVERS

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Some Approaches to Determining the Height of Dikes Along Rivers

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1. Suppose we have a river with dikes on both sides of it (see Figure 1, showing the cross-section of the possible flows in the river and explaining the variables). For simplicity, we consider a case where the river is a channel with a rectangular cross-section of constant width. The side walls of the channel, the height of the dikes, could be some function $D(x)$ of distance, x, along the river.

The flow and depth (stage) in the river can be described by the equations *

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = g(\Theta - \chi) - g \frac{\partial h}{\partial x} \tag{1}$$

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = \frac{1}{B} q(x,t) \tag{2}$$

where

$$\chi = \frac{u|u|}{\gamma R^n} \quad . \tag{3}$$

and where

u and h = the average velocity and water depth in any cross-section of channel, respectively ,

Θ = the slope of the channel bed ,

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* See, for example, J. J. Stoker, Water Waves, Interscience publichers, Inc., New York, 1957.

χ = the "frictional slope" of the channel ,

R = the hydraulic radius equal to A(h)/P(h) ,

A(h) = the cross-sectional area of the water ,

P(h) = the wetted perimeters ,

γ and n = positive empirical parameter ,

g = the acceleration of gravity ,

B = the width of the channel ,

q = the lateral outflow (overflow) per unit lenght of the channel , and

x and t = space and time variables .

The total overflow through the dikes along the length X during the period of time T will be

$$W(T,X) = -\int_0^T \int_0^X q(x,t)\, dx\, dt \quad . \qquad (4)$$

We assume that in the interval $[0,X]$ at $0 \leq t \leq T$ the function q(x,t) is finite and piece-wise continuous; it could be positive, negative or equal to zero.

We could have the following regimes of the flow when we change the initial and boundary conditions and the physical parameters of the flow such as the slope of the channel, friction on its bottom and walls, and the height of the dikes.

a) Regular flow (see Figure 1a): the water depth in the channel is below the height of the dikes, so there is no overflow:

$$h(x,t) \leq D(x) \qquad q(x,t) = 0 \quad . \qquad (5)$$

b) The first type of overflow (see Figure 1b): the water depth in the channel is above the height of the dikes and the level of water outside the channel is below the height of the dikes, so there is overflow:

$$h(x,t) \geq D(x) \geq h^*(x,t) \qquad q(x,t) = -\alpha[h(x,t) - D(x)]^{\beta} \tag{6}$$

where α and β are empirical constants ($\alpha > 0$, $\beta > 0$) and $h^*(x,t)$ is the level of the water outside the channel.

For practical purposes, there is an engineering formula for this overflow (see, for example, I.I. Agroskin, G.B. Dmitriev, and F.I. Pikalov, Hydraulics, Pub. "Energy", Moscow, Leningrad, 1964, pp. 256-257) as follows

$$Q_\delta = m_\delta l \sqrt{2g}\, H_2^{3/2} \tag{7}$$

where

Q_δ = the overflow through the side wall per unit of time ,

m_δ = the empirical coefficient ,

l = the spill length , and

$H_2 - h - p$ = the head of the water spill .

The height of the water spill p in our consideration (see Eq. (6)) is the height of the dikes. Equation (7) is successfully used in many practical applications.

c) The second type of overflow (see Figure 1c): the water depth in the channel is above the level of the water outside the channel, but both of them are above the height of the dikes, so there is overflow:

$$h(x,t) \geq h^*(x,t) \geq D(x)$$

$$q(x,t) = -\alpha_1[h(x,t) - h^*(x,t)]^{\beta_1} \quad , \tag{8}$$

This case is more difficult for practical analysis, because the value $h^*(x,t)$ is unknown and may be found only after solution of the problems for the flow inside and outside of the channel, and matching their solutions by condition (8).

d) The first type of inflow (see Figure 1d): the water depth in the channel is below the height of the dikes and the level of water outside the channel is above the dikes, so there is return inflow into the channel:

$$h(x,t) \leq D(x) \leq h^*(x,t) \quad ,$$

$$q\ x,t) = \alpha^2 \left[h^*(x,t) - D(x)\right]^{\beta_2} \quad , \tag{9}$$

where α_2 and β_2 are empirical constants ($\alpha_2 > 0$, $\beta_2 > 0$).

In this case, $q(x,t)$ is independent of the solution of the problem for the flow inside the channel, but depends on the solution of the problem for the flow outside the channel. Formula (9) is similar to (6), but in the first $q(x,t) \geq 0$ and in the second $q(x,t) \leq 0$.

e) The second type of inflow (see Figure 1e): the water depth in the channel is below the level of the water outside the channel, but both of them are above the height of the dikes, so there is a return inflow into the channel:

$$h^*(x,t) \geq h(x,t) \geq D(x) \quad ,$$

$$q(x,t) = \alpha_3 \left[h^*(x,t) - h(x,t)\right]^{\beta_3} \tag{10}$$

where α_3 and β_3 are empirical constants ($\alpha_3 > 0$, $\beta_3 > 0$).

In this case, as in case C, it is necessary to solve the problems for the flow inside and outside of the channel and match their solutions by conditions (10).

2. As a first approximation approach to the problem, let us consider the steady state flow in the channel ($\partial/\partial t = 0$) and suppose that there could be regular flow or first type overflow along some reaches of the river ($q = q(x)$). Then equations (1) and (2) take the form

$$u \frac{du}{dx} = g(\Theta - \chi) - g \frac{dh}{dx} \tag{11}$$

$$u \frac{dh}{dx} + h \frac{du}{dx} = \frac{1}{B} q(x) \tag{12}$$

where in connection with (5) and (6)

$$-q(x) = \begin{cases} 0 & \text{if } h(x) \leq D(x) \\ \alpha\left[h(x) - D(x)\right]^{\beta} & \text{if } h(x) \geq D(x) \geq h^*(x) \end{cases} \tag{13}$$

Suppose we could choose the height of the dikes $D(x)$ so that there would be some given overflow $q(x)$ along the whole distance $[0,X]$. Then, we could get $u = u(x)$ and $h = h(x)$ after solving the equations (11) and (12), and by reversing (13) we could solve for $D(x)$

$$D(x) = \begin{cases} h(x) & \text{if } q(x) = 0 \\ h(x) - \left[\frac{-q(x)}{\alpha}\right]^{1/\beta} & \text{if } q(x) < 0 \end{cases} \tag{14}$$

In the case where $q(x)$ is given, equation (12) could be solved and as a result we have

$$u = \frac{1}{h}\left[A + \frac{1}{B} Q(x)\right] \tag{15}$$

where

$$A = (u\,h)\Big|_{x=o} = u_o h_o = \text{const.} \tag{16}$$

$$Q(x) = \int_o^x q(x)\,dx \tag{17}$$

We could find

$$\frac{du}{dx} = -\frac{1}{h}\left[A + \frac{1}{B}\,Q(x)\right]\frac{dh}{dx} + \frac{1}{h}\,\frac{q(x)}{B} \tag{18}$$

and

$$\chi = \frac{\left[A + \frac{1}{B}\,Q(x)\right]^2}{h^2\,\gamma\left(\frac{Bh}{B+2h}\right)^n} \tag{19}$$

and after substitution into (11), we get

$$\frac{dh}{dx} = \frac{g\Theta - g\,\dfrac{\tilde{A}^2(x)}{\gamma h^2\left(\frac{Bh}{B+2h}\right)^n} - \dfrac{q(x)\,\tilde{A}(x)}{h^2 B}}{g - \dfrac{\tilde{A}^2(x)}{h^3}} \tag{20}$$

where

$$\tilde{A}(x) = A + \frac{1}{B}\,Q(x) \tag{21}$$

The solution of the equation (20) gives us the relation

$$h = h(x\,;\,h_o\,,\,u_o\,,\,B\,,\,q\,,\,\gamma\,,\,n)\quad . \tag{22}$$

The water depth h depends on the distance x, the boundary conditions h_o and u_o, the width of the channel B, overflow $q(x)$ and empirical constants which describe the friction on the bottom and walls of the channel.

Equation (20) could be simplified for some special cases. For example, when there is no overflow ($q(x) = 0$) we have

$$\tilde{A}(x) = A = \text{const.} \tag{23}$$

and equation (20) takes the well known (classical) form*

$$\frac{dh}{dx} = \frac{g\left(\Theta - \dfrac{A^2}{\gamma h^2 \left(\frac{Bh}{B+2h}\right)^n}\right)}{g - \dfrac{A^2}{h^3}} \tag{24}$$

The function $h = h(x)$ is usually called the energy curve.

In case there is a constant overflow ($q(x) = q_o = \text{const}$) we have

$$\tilde{A}(x) = A + \frac{q_o}{B}\, x \tag{25}$$

and the equation (20) takes the form of

$$\frac{dh}{dx} = \frac{g\Theta - g\dfrac{\left(A + \frac{q_o}{B}\, x\right)^2}{\gamma h^2 \left(\frac{Bh}{B+2h}\right)^n} - \dfrac{q_o\left(A + \frac{q_o}{B}\, x\right)}{h^2 B}}{g + \dfrac{\left(A + \frac{q_o}{B}\, x\right)^2}{h^3}} \; . \tag{26}$$

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* Ibid

Thus we also could find the head of the water curve $h = h(x)$ for the constant overflow.

3. Two approaches could be suggested for finding the height of the dikes $D(x)$ optimal in respect to the cost of building them (investment) and the damage from the water overflow in the steady state flow case.

First, we could take overflow $q(x)$ as the main parameter of the problem and express through it, damage from overflow S, height of dikes $D(x)$, and the cost of building the dikes I.

Indeed, if we suppose that the damage from overflow in an adjacent territory along the distance $0,X$ per unit of time is the single value function of the total amount of water overflow W

$$S = f(W) \tag{27}$$

where

$$W(X) = -\int_0^X q(x)\, dx \tag{28}$$

then it is evident that some unique value of S will correspond to each function $q(x)$ given on $[0,X]$

$$S = f_1(q) \tag{29}$$

On the other hand, to each function $q(x)$ there is a corresponding, unique function $D(x)$ at fixed values of the parameters in (14) and (20) (see Figure 2).

If we also assume that the cost of building or reconstruction of dikes could be estimated approximately as follows:

$$I = \int_{0}^{x} k(x)\ \Psi\left[D(x) - D^{o}(x)\right]\ dx \qquad (30)$$

where

$\Psi\left[D(x) - D^{o}(x)\right]$ = some function which is known from engineering practices

$D^{o}(x)$ = the height of the existing dikes (we assume for simplicity that $D^{o}(x) \le D(x)$).

$k(x)$ = the weight function, which depends on the cost of the building materials in different places along the river, its transportation expences, and other economic factors of the reconstruction,

then we could find

$$I = f_2(q) \qquad (31)$$

It is possible to change q(x) in two ways: to keep the integral (28) constant and by multiplication of q(x) with some constant value δ which leads to the increase of integral (28) in δ times. The first way of changing gives in the plane (S,I) (see Figure 3) the set of the points along the straight line S = const; the second way of changing gives the set of the curves $S_i = F_i(I)$, going through those points.

All curves $S_i = F_i(I)$ go out from one point on the OI axis, which corresponds to $W = 0$.

Second, we could take the height of the dikes $D(x)$ as the main parameter of the problem. Then, after the substitution of (13) into (12), we have $u = u(x)$ and $h = h(x)$. Using the given function $D(x)$ and the solution $h = h(x)$, we could find $q(x)$ from equation (13). Then from (29) and (30) we have

$$S = f_3(D) \quad , \tag{32}$$

$$I = f_4(D) \quad . \tag{33}$$

It is also possible to change $D(x)$ in two ways: to keep the integral (30) constant and by multiplying $D(x)$ with some constant value σ. If the function Ψ becomes equal to zero as $D(x) = D^o(x)$ and increases monotonically when $D(x) > D^o(x)$, then the curves $S_i = G_i(I)$ will have the form shown in Figure 4. Each curve corresponds to the fixed function $D(x)$.

By using the curves shown in Figures 3 and 4, it is possible to find the shape of the dikes $D(x)$, which corresponds

- to minimal cost of the building of the dikes (minimal investment),
- to minimal damage from the overflow,
- to minimal total cost of the building of the dike plus damage from overflow.

It is reasonable to note that simplification of the problem considered and the possibility of its transformation, for example, into the problem of linear programming, depends on the simplification of the equations (14) and (20) and finding the relation of the form

$$\bar{D} = M\,\bar{q} \tag{34}$$

where $\bar{D}$ and $\bar{q}$ are vectors with components D_j and q_j, which are the height of the dikes and overflow along the interval j of the distance [0,X], correspondingly, (see Figure 5), and M is the matrix of transformation.

It is necessary also to simplify (30), replacing it by linear

$$I = \sum_j k_j\,(D_j - D_j^o)\quad . \tag{35}$$

Integral (28) will take the form

$$W = -\sum_j q_j \tag{36}$$

The function (27) could be approximated by a linear relationship

$$S = \rho W = -\,\rho\sum_j q_j \tag{37}$$

After this simplification we could formulate three linear programming problems:

a)

$$\min I \equiv \sum_j k_j \; (D_j - D_j^o)$$

subject to

$$\bar{D} = M \, \bar{q}$$

$$S \equiv - \rho \sum_j q_j \leq S_{max} \tag{38}$$

$$q_j \leq 0$$

$$D_j \geq D_j^o \quad .$$

b)

$$\min S \equiv - \rho \sum_j q_j$$

subject to

$$\bar{D} = M \, \bar{q}$$

$$I \equiv \sum_j k_j \; (D_j - D_j^o) \leq I_{max} \tag{39}$$

$$q_j \leq 0$$

$$D_j \geq D_j^o \quad .$$

c)

$$\min (I + S) \equiv \sum_j \left[k_j \; (D_j - D_j^o) - \rho \sum_j q_j \right]$$

subject to

$$\bar{D} = M \, \bar{q}$$

$$q_j \leq 0 \tag{40}$$

$$D_j \geq D_j^o$$

Here k_j, ρ, D_j^o , are given constants, q_j and D_j are variables.

4. The flood protection problem in the river basin could be solved in different ways, depending upon the geography of the basin, technological possibilities, economic development plans, historical and various other factors. And practically speaking, that way will be more favourable which promises less investment to protect from the same flood damage.

Under the way of protection we mean not only the construction of storage reservoirs, dikes, diversion channels, etc., but also their combinations.

Above we considered the case when only dikes were used for protection. This problem seems to be important for such river basins as the Tisza, especially in Hungarian territory, where the geography of the river basin (very flat), and experiences from a long period of past time made this way of protection very reasonable.

But there are, of course, many practical, theoretical methodological questions estimating the investments and benefits from combined ways of protection. One very promising approach to this problem was presented by Walter Spofford*, when he considered the flood protection investment problem in cases where storage reservoirs and dikes are used for protection.

As seen in Figure 6, the main hydrodynamic parameters of the dike protection sub-problem u_o and h_o are generated by the storage protection sub-problem. On the other hand, the dike protection sub-problem generates the amount of the

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* W.O. Spofford "An Approach to the Flood Protection Investment Problem", IIASA, 20 August 1974

overflow, W, which defines the damage in the adjacent territory. If this damage is greater than the permissible value, then there are two alternatives: to increase the height of the dikes or to increase the flood capacity of the storage reservoir, (i.e., make a greater investment in dike or storage reservoir construction). The problem could be formulated as optimization or simulation depending upon the simplification which is acceptable.

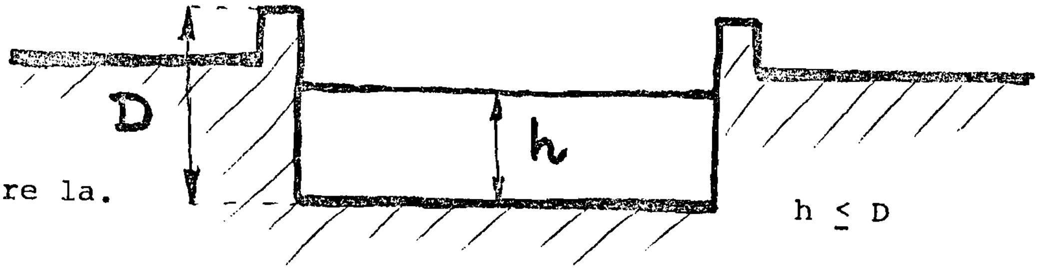


Figure 1a.

$h \leq D$

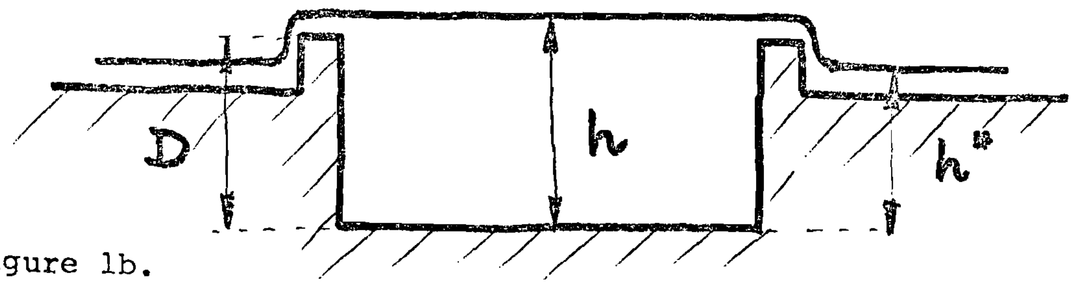


Figure 1b.

$h \geq D \geq h^*$

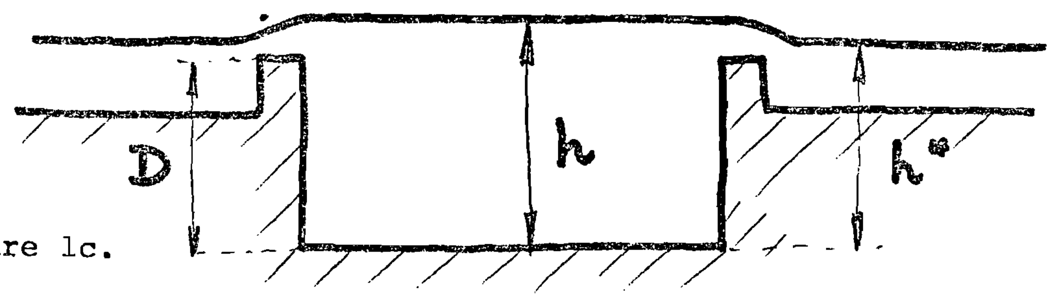


Figure 1c.

$h \geq h^* \geq D$

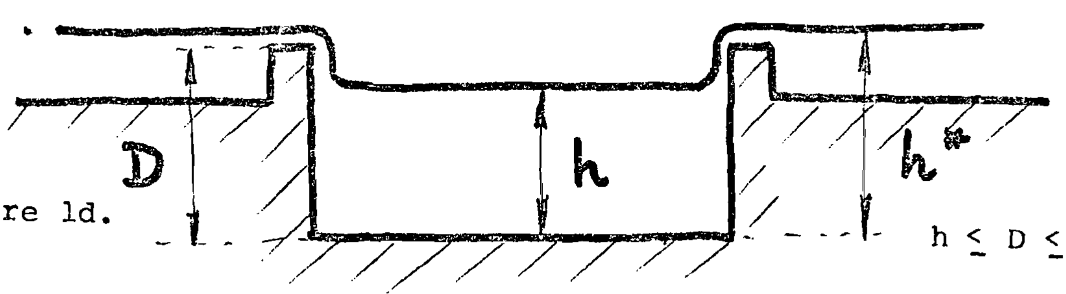


Figure 1d.

$h \leq D \leq h^*$

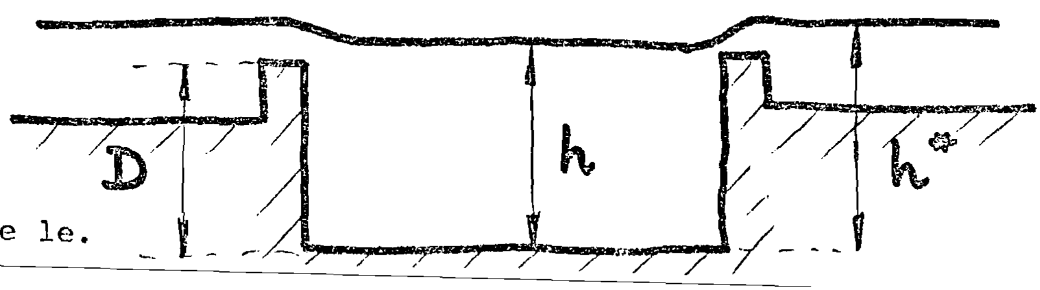
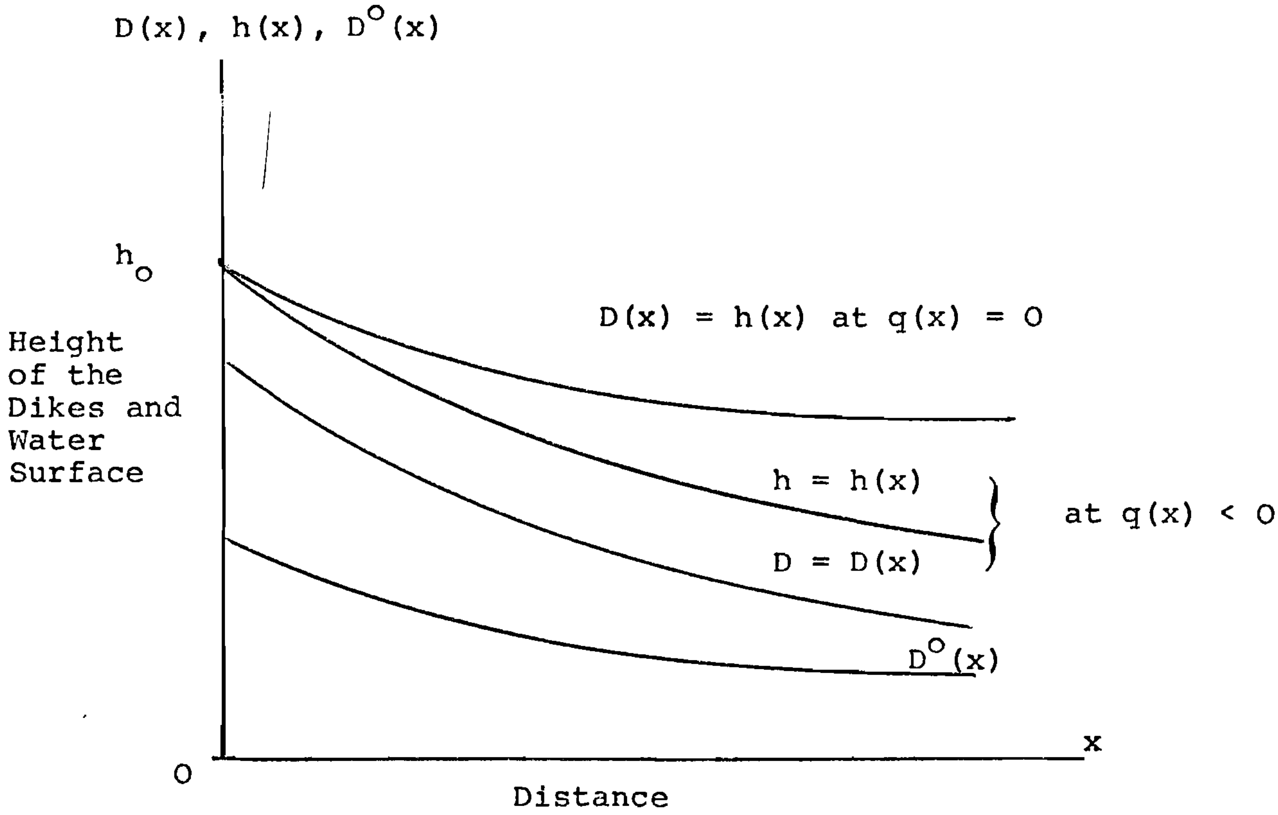


Figure 1e.



D(x), h(x), D^{o}(x)
h$_o$
Height of the Dikes and Water Surface
D(x) = h(x) at q(x) = 0
h = h(x)
D = D(x)
at q(x) < 0
D^{o}(x)
x
0
Distance

Figure 2.

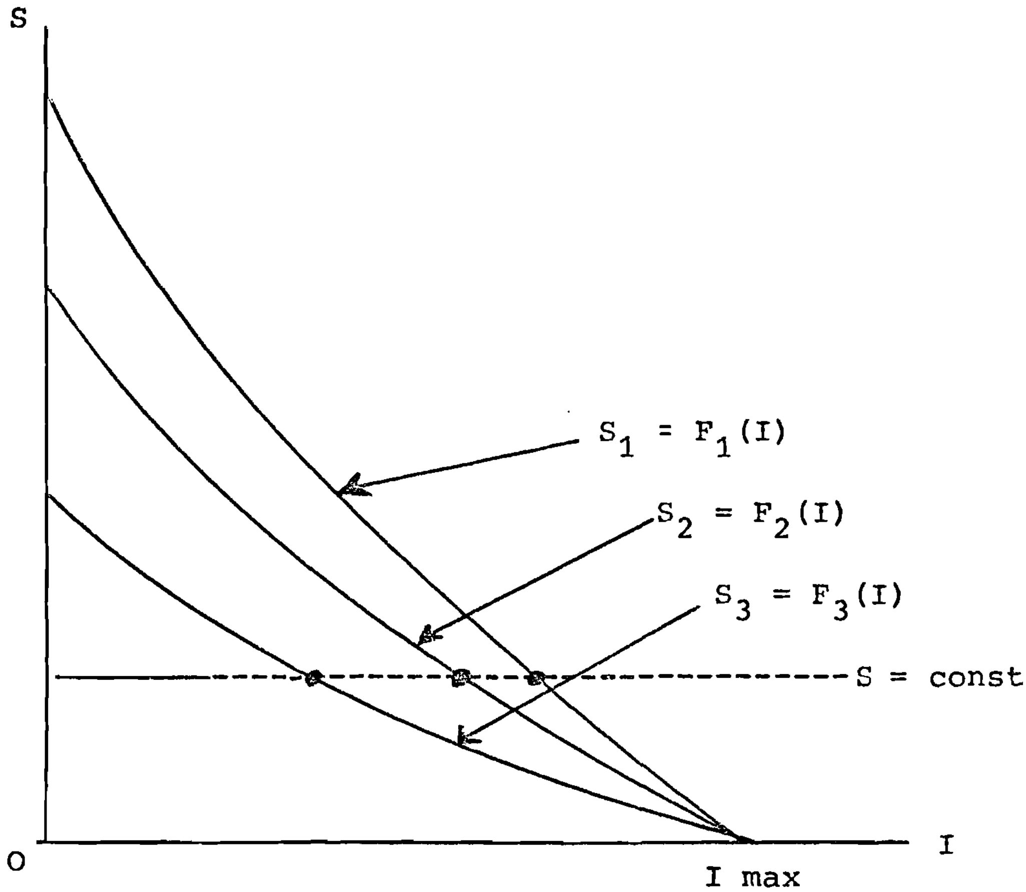


Figure 3.

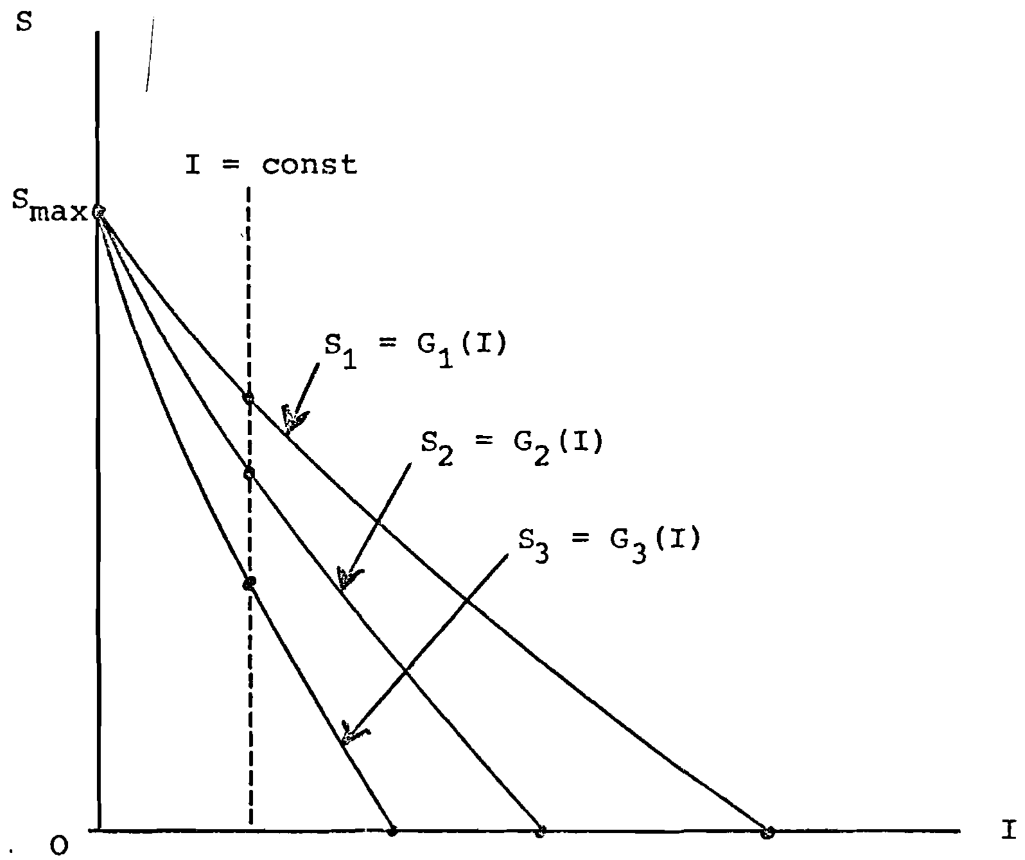


Figure 4.

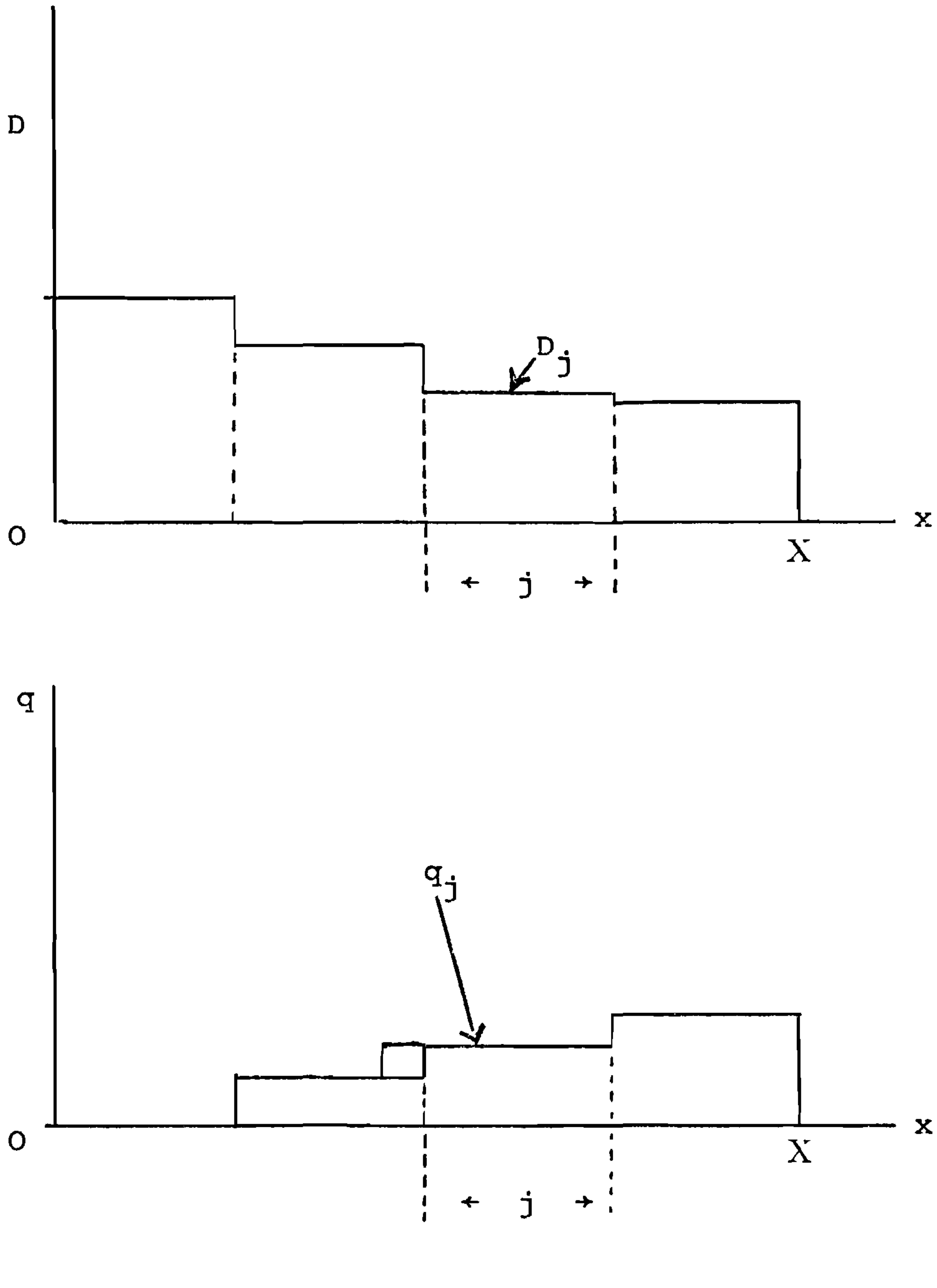
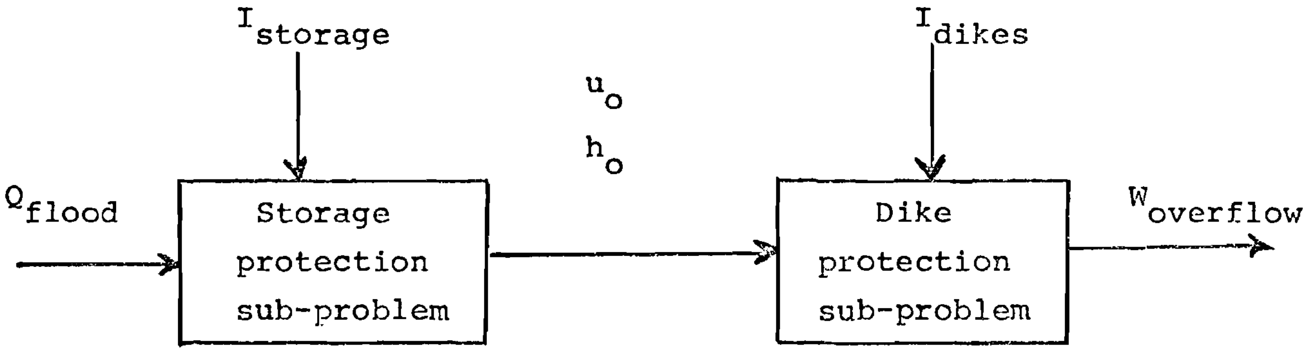


Figure 5



I storage
I dikes
u o
h o
Q flood
Storage protection sub-problem
Dike protection sub-problem
W overflow

Figure 6.