A SURVEY OF APPLICATIONS OF INTEGER AND COMBINATORIAL PROGRAMMING IN LOGISTICS

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Introduction

There are a number of definitional ground rules to be established before we enter into our survey. First, an application is taken to be a study in which concern over a real world problem caused the formulation of an integer or combinatorial programming model, the collection of data for this model, and the calculation of numerical solutions using a computer. This is in contrast to studies in other social science fields where mathematical models are used to obtain qualitative insights without necessarily requiring data and numerical calculations.

A second ground rule is to agree that we will not try to define logistics, but rather to consider specific illustrative applications which most of us would agree address logistics problems. These applications are chosen from the functional areas of distribution, location, scheduling, production/ inventory control, communications and reliability.

Another reason for considering illustrative applications is that the number of applications is enormous and a comprehensive survey is not possible. Our purpose instead is to discuss by example the underlying principles used in these applications. The principles are derived from the synergism that exists between mathematical programming theory as it relates to algorithms, the construction and use of computer systems, and the institutional aspects of the applications themselves.

Discrete Programming

In mathematical terms, the most general statement of the class of mathematical programming models we will discuss is the following. The object is to maximize the quantity f(x) where the vector x is chosen from a finite or denumerable set X contained in a finite dimensional space, say R^n. The set X may be given implicitly or defined explicitly by a set of constraint functions including integrality restrictions on the variable values. Discrete programming differs from nonlinear programming in that differential methods cannot be used directly to analyze the objective and constraint functions. Moreover, convex combinations of solutions from X may not themselves be points in X and therefore linear programming approximations may be inexact.

Within the class of discrete programming problems there are two overlapping subclasses: integer programming and combinatorial programming problems. We can think of integer programming problems as being of the form

$$\begin{aligned} \min \quad & f_1(x) + f_2(y) \\ \text{s.t.} \quad & A_1(x) + A_2(y) \geq b \\ & x \geq 0 \quad , \quad y \geq 0 \text{ and integer} \quad , \end{aligned}$$

where usually $A_2(y) = A_2y$, i.e., the function $A_2(y)$ is a linear function, and slightly less often $f_2(y) = f_2y$. For a system problem such as this one, one uses integer programming system theory including number theory and branch and bound (e.g., Geoffrion and Marsten (1972), Gorry, Northup and Shapiro (1973)).

By contrast, combinatorial programming problems have a less explicit mathematical statement. They contain network optimization problems as substructures including shortest route, maximal flow, minimum spanning tree and minimum cost flow problems. All of these network optimization problems can be solved by "good" algorithms which means algorithms with a number of steps upper bounded by a polynomial in the parameters of the problem (Edmonds (1971), Karp (1972)). An algorithm is not "good" if it is possible for the algorithm to require on some problems a number of steps that grows exponentially with the parameters of the problem. "Good" algorithms are good in a practical as well as theoretical sense and network optimization problems of significant size can often be solved in a matter of a few seconds on large scale computers (Glover et al (1974)).

There are other relatively simple combinatorial optimization problems which appear as subproblems in applications. These include simple covering and matching problems (Garfinkel and Nemhauser (1972)), discrete deterministic dynamic programming problems (Wagner (1969)), and others. Although "good" algorithms may not exist for these problems, they are often easy to solve relative to the complex combinatorial programming problems found in practice.

Specifically, the combinatorial programming models arising in logistics applications are often a synthesis of several similar or different problems of the above types, plus complicating constraints or relations. Practically all of these problems can be formulated as integer programming problems, but often the special structure of the problem is lost. A good example of this is the symmetric traveling salesman problem for which there is an integer programming formulation with approximately 2^n constraints, where n is the number of cities to be visited (Held and Karp (1970)). The majority of these constraints, however, describes a minimal spanning tree problem, and Held and Karp (1970, 1971) exploit this structure in a special purpose algorithm for the traveling salesman that involves the solution of an effective n constraint approximation of the problem.

The choice of an integer programming or combinatorial programming formulation of a discrete optimization problem is closely related to the choice one must make between a general purpose or special purpose algorithm for the given problem. Unfortunately, this choice cannot always be made as definitively as it can be for the traveling salesman problem. The conflict can be resolved in large part, however, by the modular design of integer programming and network optimization computer codes so that the synthesis required for a specific application can be made without a complete set-up. As we shall see, the synthesis of a model from its component parts can be effected by the application of dual or price directive decomposition methods of mathematical programming. Decomposition can also be effected by resource directive methods, but this approach has found little if any application. See Lasdon (1970) for a discussion of these approaches.

Illustrative Application One:

Multi-item Production Scheduling and Inventory Control (Lasdon and Terjung (1971)).

Consider a manufacturing system consisting of I items for which production is to be scheduled over T time periods. The demand for item i in period t is the non-negative integer r_{it}; this demand must be met by stock from inventory or by production during the period. Let the variable x_{it} denote the production of item i in period t. The inventory of item i at the end of period t is.

$$y_{it} = y_{i,t-1} + x_{it} - r_{it} \qquad t=1,\ldots,T$$

where we assume $y_{i,0} = 0$, or equivalently, initial inventory has been netted out of the r_{it}. Associated with x_{it} is a direct unit cost of production c_{it}. Similarly, associated with y_{it} is a direct unit cost of holding inventory h_{it}. The problem is complicated by the fact that positive production of item i in period t uses up a quantity $a_i + b_i x_{it}$ of a scarce resource q_t to be shared among the I items. The parameters a_i and b_i are assumed to be non-negative.

Lasdon and Terjung (1971) applied this model to the scheduling of automobile tires production. The scarce resource in each period was machine capacity. The number of different items (tires) was approximately 400, and the planning horizon was approximately 6 periods.

This problem can be written as the mixed integer programming problem

$$v = \min \sum_{i=1}^{I} \sum_{t=1}^{T} (c_{it}x_{it} + h_{it}y_{it}) \quad (1.1a)$$

$$\text{s.t.} \sum_{i=1}^{I} (a_i\delta_{it} + b_i x_{it}) \leq q_t, \quad t=1,\ldots,T \quad (1.1b)$$

$$\sum_{t=1}^{s} x_{it} - y_{is} = \sum_{t=1}^{s} r_{it}, \quad s=1,\ldots,T \quad (1.1c)$$

$$x_{it} \leq M_{it}\delta_{it}, \quad t=1,\ldots,T \quad (1.1d)$$

$$x_{it} \geq 0, \quad y_{it} \geq 0, \quad \delta_{it} = 0 \text{ or } 1, \quad t=1,\ldots,T \quad (1.1e)$$

where $M_{it} = \sum_{s=t}^{T} r_{is}$ is an upper bound on the amount we would want to produce in period t. The constraints (1.1b) state that shared resource usage cannot exceed q_t. For simplicity, we have assumed a single resource to be shared in each production period. The model can clearly be used when there are K shared resources in each period. The constraints (1.1c) relate accumulated production and demand through period t to ending inventory in period t, and the non-negativity of the y_{it} implies demand must be met and not delayed (backlogged). The constraints (1.1d) ensure that $\delta_{it} = 1$ and therefore the fixed charge resource usage a_i is incurred if production x_{it} it positive in period t. Problem (1.1) is a mixed integer programming problem with IT zero-one variables, 2IT continuous variables and T + 2IT constraints. For the application of Lasdon and Terjung, these figures are 240 zero-one variables, 480 continuous variables and 486 constraints which is a mixed integer programming problem of significant size.

For future reference, define the set

$$N_i = \{(\delta_{it}, x_{it}, y_{it}), t=1,\ldots,T \mid \delta_{it}, x_{it}, y_{it} \text{ satisfy } (1.1c),(1.1d),(1.1e)\}. \tag{1.2}$$

This set describes a feasible production schedule for item i ignoring the joint constraints (1.1b).

The integer programming formulation (1) is not effective because it fails to exploit the special structure of the sets N_i. This can be accomplished by dual (price directive) decomposition which proceeds as follows. Assign prices $u_t \geq 0$ to the scarce resources q_t and place the constraints (1.1b) in the objective function to form the lagrangean.

$$L(u) = -\sum_{t=1}^{T} u_t q_t + \underset{(\delta_{it}, x_{it}, y_{it}) \in N_i}{\text{minimum}} \sum_{i=1}^{I} \sum_{t=1}^{T} \{(c_{it} + u_t b_i) x_{it} + u_t a_i \delta_{it} + h_{it} y_{it}\}.$$

Letting

$$L_i(u) = \underset{(\delta_{it}, x_{it}, y_{it}) \in N_i}{\text{minimum}} \sum_{t=1}^{T} \{(c_{it} + u_t b_i) x_{it} + u_t a_i \delta_{it} + h_{it} y_{it}\}, \tag{1.3}$$

the Lagrangean function clearly separates to become

$$L(u) = - \sum_{t=1}^{T} u_t q_t + \sum_{i=1}^{I} L_i(u).$$

Each of the problems (1.3) is a simple dynamic programming shortest route calculation for scheduling item i where the dual prices on shared resources adjust the costs as shown.

It is easily shown that L(u) is a lower bound on the minimal objective function cost v in problem (1.1). The best choice of prices is a vector u* which provides the greatest lower bound; namely, a vector u* which is optimal in the dual problem

$$\begin{aligned} w = \max\ & L(u) \\ \text{s.t. } & u \geq 0, \end{aligned} \tag{1.4}$$

where clearly $w \geq v$. The reason for this selection of prices is that if the maximal dual objective function value w equals the minimal primal objective function value v, then it is possible to solve (1.1) by calculation of $L_i(u^*)$ for each item i. Approximate equality between v and w obtains when the number of items I is significantly greater than the number of joint constraints (1.1b) in the planning problem.

The dual problem (1.4) can be solved in a number of ways. One algorithm is generalized linear programming, otherwise known as Dantzig-Wolfe decomposition (Lasdon (1970)). This is the approach taken by Lasdon and Terjung who, in addition, used the generalized upper bounding technique (Lasdon (1970)) to solve the linear programming subproblems which arise in the use of this algorithm. Further discussion about generalized linear programming and duality is contained in Magnanti, Shapiro and Wagner (1973).

If there is a substantial duality gap between the primal problem (1.1) and the dual problem (1.4) (i.e., if v - w is a large positive number), then problem (1.1) becomes more difficult to solve. In this case, the dual decomposition approach needs to be combined with branch and bound (see Fisher, Northup and Shapiro (1974)). To the best of my knowledge, the model (1.1) has never been used to analyze a real-life logistics problem where the number of joint constraints (1.1b) is of the same order of magnitude as the number of items for which production is being scheduled and a large duality gap is likely.

Another application of combinatorial methods to production is contained in Mueller-Merbach (1973). He considers a production system consisting of a hierarchy of assemblies to be merged into final products. The assembly process is described as a network for the purposes of analyzing explosion of material requirements and costs.

Illustrative Application Two:

Warehouse Location and Multi-Commodity distribution (Geoffrion and Graves (1973)).

In the previous application, we considered a discrete optimization problem for which the mixed integer programming formulation was inefficient because it failed to exploit special structure. We consider now an application in which mixed integer programming was successfully applied. The model used in the application is an example of a large class called location-allocation problems (see Lea (1973) for an extensive bibliography).

The application of Geoffrion and Graves involved a two-level distribution system with plants each producing a number of different commodities to be shipped to warehouses from which wholesale customers are supplied. These decisions to be made were: (1) what warehouse sites should be used; (2) what should be the size of each warehouse; (3) which customers should be served by each warehouse; and (4) what is the optimal pattern of multi-commodity transportation flows?

Let i be the index for commodities, j the index for plants, k the index for possible warehouse sites and l the index for customers. Define the variables x_{ijkl} as the non-negative amount of commodity i produced in plant j for delivery to customer l via a warehouse at site k. Let the zero-one variable j_k determine whether (z_k = 1) or not

(z_k = 0) a warehouse is constructed at location k. Let the zero-one variable Y_{kl} determine whether (Y_{kl} = 1) or not (Y_{kl} = 0) customer l is supplied from warehouse k.

The warehouse location and multi-commodity distribtuion problem can be written as the mixed integer programming problem

$$\min \quad \sum_{ijkl} c_{ijkl}\, x_{ijkl} + \sum_k \{f_k z_k + v_k \sum_{il} d_{il}\, y_{kl}\} \qquad (2.1a)$$

$$\text{s.t.} \quad \sum_{kl} x_{ijkl} \le s_{ij} \qquad \text{all } ij \qquad (2.1b)$$

$$\sum_j x_{ijkl} = d_{il}\, y_{kl} \qquad \text{all } ikl \qquad (2.1c)$$

$$\sum_k y_{kl} = 1 \qquad \text{all } l \qquad (2.1d)$$

$$\underline{v}_k\, z_k \le \sum_{il} d_{il}\, y_{kl} \le \bar{v}_k\, z_k \qquad \text{all } k \qquad (2.1e)$$

Linear configuration constraints on y and z (2.1f)

$$x_{ijkl} \ge 0 \qquad \text{for all } ijkl$$

$$y_{kl} = 0 \text{ or } 1 \qquad \text{for all } k, l \qquad (2.1g)$$

$$z_k = 0 \text{ or } 1 \qquad \text{for all } k$$

The constraints (2.1b) limit the supply of commodity that can be shipped from plant j. The constraints (2.1c) and (2.1d) together state that the demand for commodity i by customer l must be met and by shipment from exactly one warehouse. The constraints (2.1e) state that if warehouse site k is selected ($z_k = 1$), then total storage of all commodities for all customers supplied from k must be between the lower and upper limits $\underline{v}_k$ and $\overline{v}_k$. The constraints (2.1f) are a variety of logical constraints on the zero-one decision variables such as $\sum_{k \varepsilon K^1} z_k \leq 1$ implying no more than one warehouse site can be selected from a subset K^1 of the possible sites. Finally, the objective function (2.1a) consists of linear terms and fixed change terms involving the variables y_{kl} and z_k.

For the application of Geoffrion and Graves, there were 17 different commodities, 14 plants, 45 possible warehouse sites and 121 customers. The mixed integer programming problem (2.1) consisted of 11,854 rows, 727 binding variables and 25,513 continuous variables. These large figures are somewhat misleading because the continuous part of the problem consists of a number of transportation problems with simple structure. Fortunately, it was possible to exploit these structures, and at the same time solve the mixed integer programming problem, by the use of Benders' method for mixed integer programming as shown schematically in figure 1.

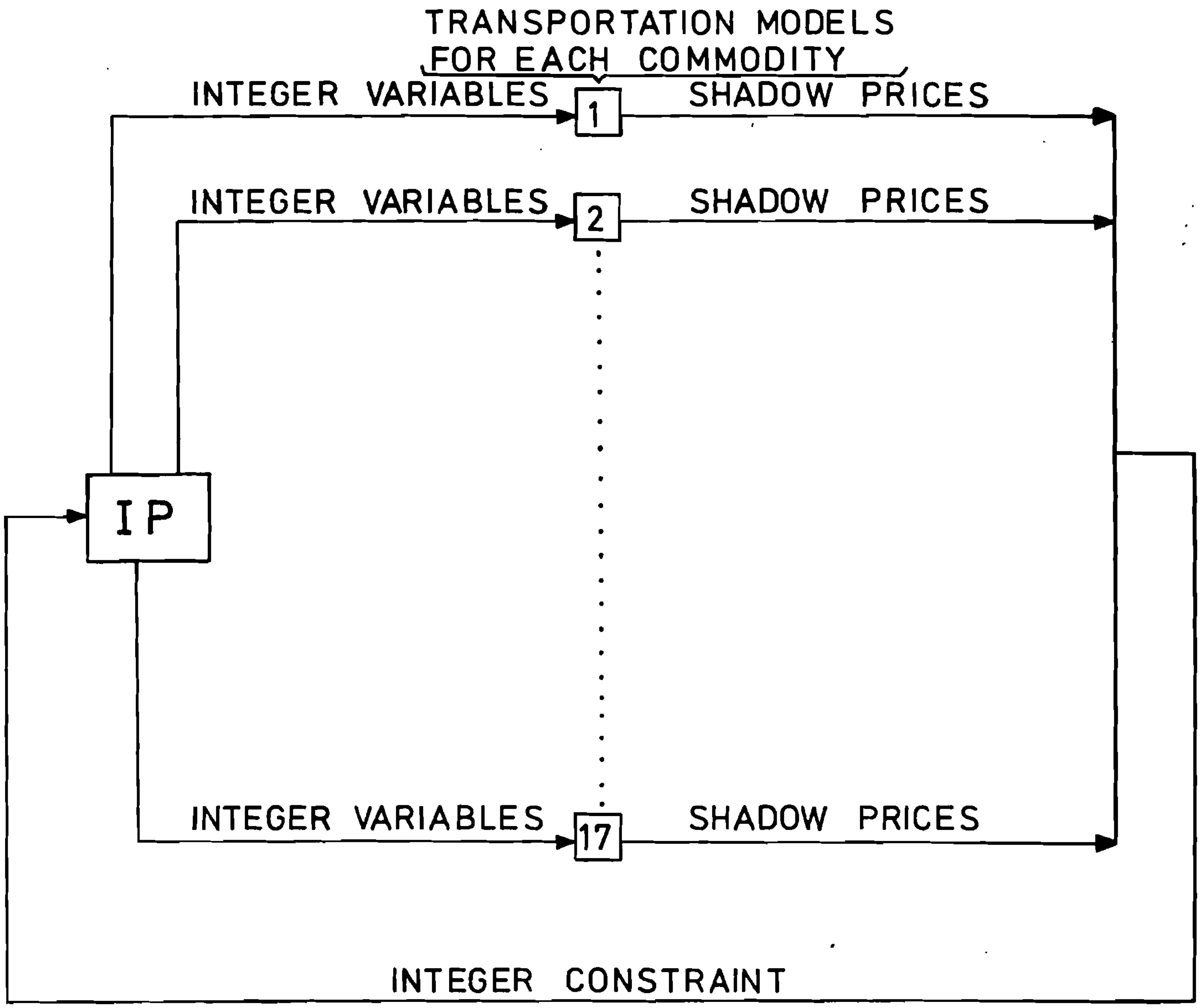
TRANSPORTATION MODELS
FOR EACH COMMODITY
INTEGER VARIABLES
1
SHADOW PRICES
INTEGER VARIABLES
2
SHADOW PRICES
IP
INTEGER VARIABLES
17
SHADOW PRICES
INTEGER CONSTRAINT

FIGURE 1

The integer programming subproblem (IP) involved the variable y_{kl} and z_k and the constraints (2.1d), (2.1e), (2.1f) and the zero-one constraints in (2.1g) plus constraints approximating the objective function (2.1a) from below. The transportation models consisted for each commodity i of (2.1b) and (2.1c) where the variables y_{kl} were fixed at zero-one values. The objective functions consisted of the linear terms $\sum_{jkl} c_{ijkl} x_{ijkl}$ for each commodity i. Benders' method proceeds by alternatively solving the integer programming subproblem and the continuous transportation problem. It stops when the integer constraint derived from the transportation subproblems does not cut off the previously optimal solution to the integer programming subproblem.

As we indicated, each solution of IP produced a better lower bound to the optimal objective function value in (2.1). Moreover, each solution of the 17 transportation problems produced a feasible mixed integer programming solution to (2.1). Thus, it is possible to terminate computation before optimality is reached (or proven), and have a bound on the objective function cost loss due to non-optimality.

Illustrative Application Three:

Optimal Design of Offshore Natural-Gas Pipeline System (Rothfarb et al (1970)).

The previous two examples have involved continuous as well as discrete decision variables and therefore they required mixed methods of solution. Specifically, dual pricing of scarce resources was required in order to adjust the costs on discrete decision variables. By contrast, the application to be discussed here is purely discrete and requires combinatorial algorithms adapted from algorithms for simpler problems of similar type. Moreover, the complexity of the problem necessitates the use of heuristic methods because optimality is too costly to obtain.

Figure 2 depicts a typical design of a pipeline system connecting offshore gas fields (nodes) to an onshore separation and compressor plant. The location of the fields is assumed given and the graph of the system is always a tree (i.e., one and only one path from a gas field to the plant). The pipeline system is required to carry known flow per day from each gas field according as

$$\text{flow} = K \, \frac{(\text{pressure change})^2}{\text{pipe length}} \, \text{pipe diameter} \qquad (3.1)$$

where K is a proportionality constant.

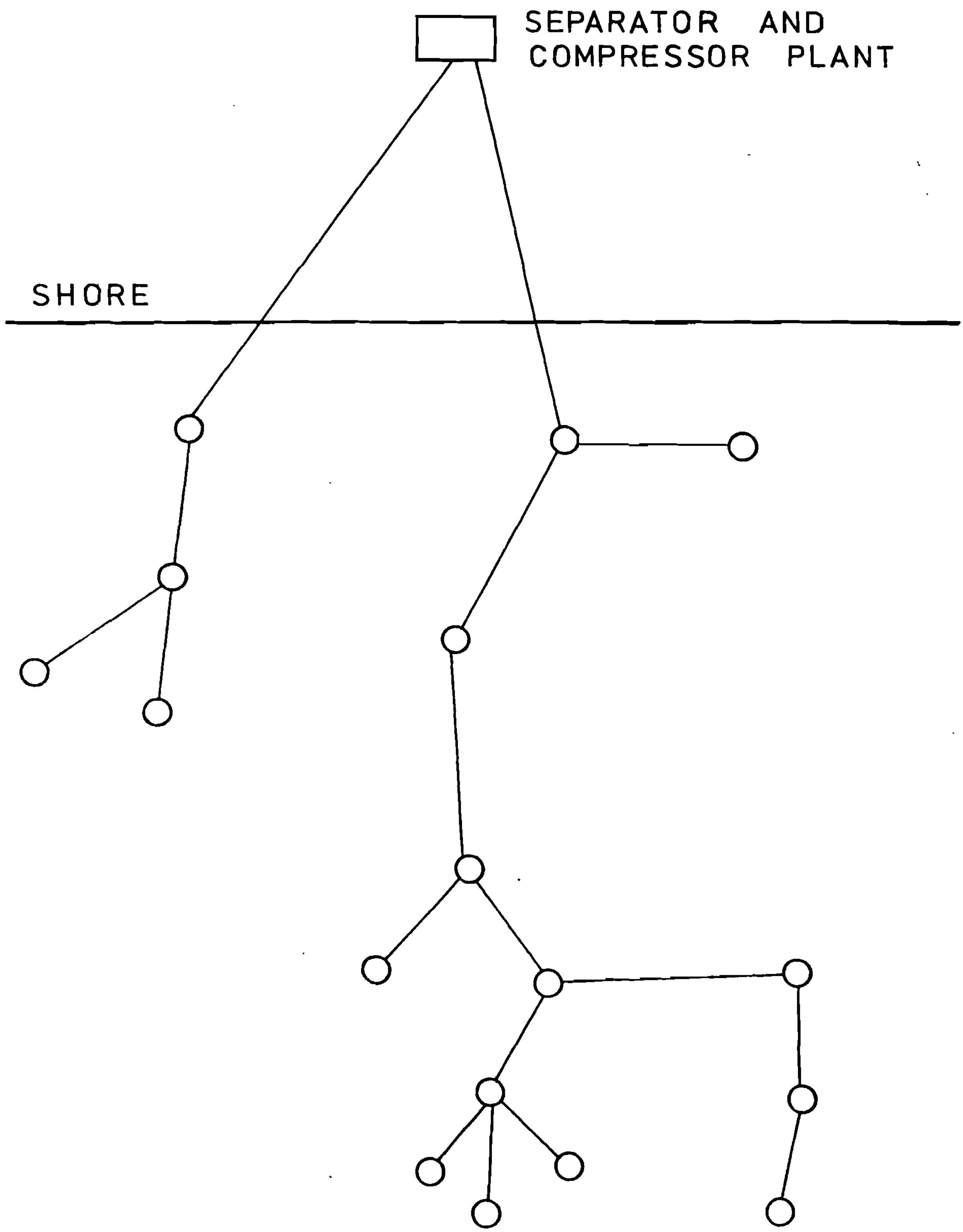

FIGURE 2

The variables on the right side of (3.1) are the design parameters. In addition, there are upper bound constraints on pressure due to safety and design considerations, and lower bounds due to delivery requirements at the plant. The cost of a pipeline link depends on its diameter and the depth of the water. The plant costs depend upon flow and delivered pressure.

The two main problems addressed by Rothfarb et al were:

(1) the selection of minimal cost pipeline diameter given a pipeline network and delivery requirements;

(2) the design of a minimal cost pipeline network given gas field locations and delivery requirements.

Problem (1) is a subproblem of problem (2).

Problem (1) was surprisingly difficult to optimize because the relation (3.1) and the pipe costs are nonlinear, the number of different pipe diameters was 7 and the number of gas fields was 20 or more. As a result, the number of design combinations was quite large and the nonlinearities made it difficult to identify dominating subsets of the combinations. Heuristic rules were developed to eliminate apparent uneconomical diameter combinations without exhaustive enumeration. The heuristics were based on looking at critical paths which are those to the ends of the trees where the flow and therefore the pipe diameters are smallest. The heuristics entailed local optimization at these ends followed by a merging of the nodes at the end into a single node with a aggregate design and flow requirements. The analysis was then repeated on the reduced network.

Problem (2) subsumes problem (1) and required additional heuristics. First, it is known that the pipes connected directly to the plants, called arms, play an important role in determining overall cost. It is assumed either that these are given by the user, or problem (2) must be solved for all possible combinations of arms. An automatic tree generator is used to generate a distribution of candidates for solution. Two guidelines were used:

(1) efficient trees have low total pipe length; and

(2) efficient trees have nearly equal flow in their arms.

If the first guideline were the only criterion, then the problem of pipeline network design could be solved as a minimum spanning tree problem by a "good" algorithm.

This illustrative application is only one of many examples of network design and analysis for problems where exact optimization is difficult. An attractive possibility is to use man-machine interactive computer programs to find satisfactory designs. Such a program has been constructed by Schneider et al (1972) to design urban transportation networks.

A class of network design problems from an entirely different application area giving rise to optimization problems with similar mathematical structure are computer communications network design problems. A number of remote terminals are to be attached to a central computer by a communications network. The costs to be minimized are line costs plus concentrator costs for those nodes where many lines are accumulated. See Frank et al (1971) for a discussion of models of this type.

Application Four:

Routing Problems.

We have not found in the literature a single application of the routing problem illustrating many of its aspects. A simple version of this problem is the following. A trucking company must deliver a quantity q_i of a single commodity to customer i for $i = 1,\ldots,m$. The company has an unlimited number of trucks of capacity Q which can transport the commodity from the warehouse to the customers. We assume $q_i \leq Q$ for all i and orders cannot be split between two or more delivery trucks. The objective is to minimize the total distance traveled by the delivery trucks. Let $d_{ij} = d_{ji}$ denote the distance from customer i to customer j where d_{0j} is the distance from the warehouse to customer j. Figure 3 depicts a typical problem of this type with a solution involving four trucks.

An integer programming formulation of the problem has been given by Balinski and Quandt (1964). A generic activity a_j, called a tour, is an m-vector with components

$$a_{ij} = \begin{cases} 1 & \text{if delivery route } j \text{ visits customer } i \\ 0 & \text{otherwise} \end{cases}$$

where the a_{ij} satisfy $\sum_{i=1}^{m} a_{ij}\, q_i \leq Q$. The objective function

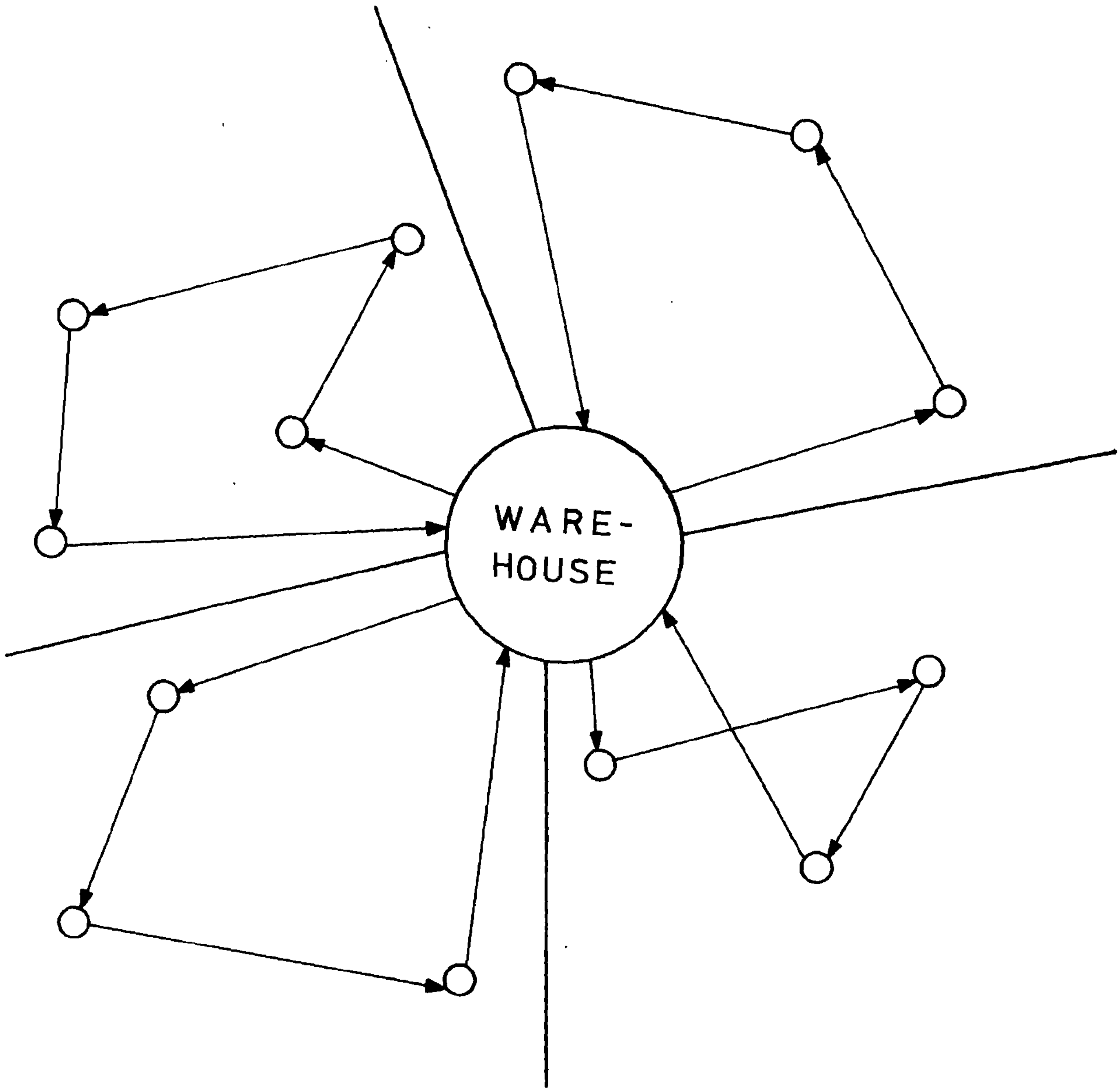

FIGURE 3

coefficient c_j associated with a_j is the shortest distance tour, starting and ending at the warehouse, of the customers visited by the activity. The calculation of c_j is a traveling salesman problem. The delivery problem is solved by solving the set partitioning problems.

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j = 1 \qquad 1 = 1,\ldots,m \qquad (4.1)$$

$$x_j = 0 \text{ or } 1, \qquad j = 1,\ldots,n$$

where n is the total number of tours satisfying $\sum_{i=1}^{n} a_{ij} q_i \leq Q$. This number can be quite large and Balinksi and Quandt suggest a column generation technique similar to the one discussed in the multi-item production scheduling example. Of course, there are a number of generalizations of the problem as stated including the use of trucks of different sizes, multi-commodity delivery, etc.

Hausman and Gilmour (1967) applied a model of this general type to the problem of scheduling fuel-oil delivery to home customers. The costs of delivery included a fixed cost for each delivery in addition to distance traveled, and the frequency of delivery was a factor in the problem. The optimal tour distance for each group of customers serviced by a single delivery truck was approximated by multiple regression, using a few simple statistics for the group. Practical problems involving 120 customers were solved with a substantial cost reduction over hand solutions.

An important class of routing problems with the form (4.1) are the airline crew scheduling problems (e.g., see Arabeyre et al (1969) and Simpson (1969)). For these problems, the "customers" are cities and the "warehouse" is a home base for crews and planes. A route map is given with the existing flights, and their times, which must be flown between cities during a given time period, usually a few days or a week. An activity a_j corresponds to a sequence of cities connected by flights that can be flown by a crew without violating safety and union constraints. The cost c_j of such an activity are the bonuses, per-diem and overtime payments. In practical applications of the airline crew scheduling problem, there can be more than one home base for crews, and additional constraints limiting the number of crews which can begin and end their tours at each home base.

Laderman (1966) and Lasdon (1973) have formulated and solved some large routing problems for ships on the Great Lakes. Mevert (1974) reports on a large trans-Atlantic shipping problem which has been formulated as a problem of the type (4.1) with a number of side constraints.

Conclusions

We have tried to present applications of integer and combinatorial programming in logistics which illustrate the current state-of-the-art of these methods and some principles to be applied to new applications. There are a number of application areas which were not mentioned including, for example, reliability (Kershenbaum and Van Slyke (1972)), decision CPM (Crowston (1970)), and the setting of traffic signals (Little (1966)). Finally, we have tried to indicate a representative rather than an exhaustive list of references. Extensive bibliographies can be found in Garfinkel and Nemhauser (1972) and Scott (1970).

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