Working Paper

MATHEMATICAL SYSTEM THEORY
AND SYSTEM MODELING

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International Institute for Applied Systems Analysis
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FOREWORD

Choosing models related effectively to the questions to be addressed is a central issue in the craft of systems analysis. Since the mathematical description the analyst chooses constrains the types of issues he can deal with, it is important for these models to be selected so as to yield limitations that are acceptable in view of the questions the systems analysis seeks to answer.

In this paper, John L. Casti of Princeton University gives an overview of the central issues affecting the question of model choice. To this end, he discusses model components and a wide variety of possible mathematical system descriptions. After discussing both local and global aspects of these model types, he addresses basic questions and perspectives of system theory. The paper concludes with a sketch of a systematic response to the question: What model to choose?

To provide a thorough overview of systems analysis, the International Institute for Applied Systems Analysis is preparing a Handbook of Systems Analysis in three volumes: 1. Overview; 2. Methods; 3. Cases. This essay is a contribution to the second volume of this Handbook. It is circulated in this informal way for review and comment. Please direct all response to the author, c/o IIASA Survey Project.

Hugh J. Miser
Survey Project

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MATHEMATICAL SYSTEM THEORY AND SYSTEM MODELING

by

John L. Casti

I. Model Components

The implications of existing knowledge in fields such as biology, psychology, business, economics and political science, not to mention "hard" scientific disciplines like physics and chemistry, are so complex that it is no longer possible for the human mind to digest them without extensive abstraction, i.e., without mathematics. Here by "mathematics," we do not mean data analysis, numerical formulas, graphical methods and other pedestrian (although often useful) tools frequently termed mathematics, but rather the use of conceptual ideas from set theory, algebra, topology and analysis to construct and analyze abstract versions of real-world situations with the goal of understanding the essential relations among their constituent parts. Such constructions and analyses are the province of the mathematical modeler (read: system theorist)—the keeper of the abstract processes.

When confronted by a particular process whose behavior is of interest, the modeler's first task is to separate the various aspects of the process into major subsystems, which can then be modeled at a more detailed level. A convenient high-level decomposition of a general situation is depicted in Figure 1. Part of the art of system modeling lies in the establishment of useful boundaries between the major components indicated in Figure 1.

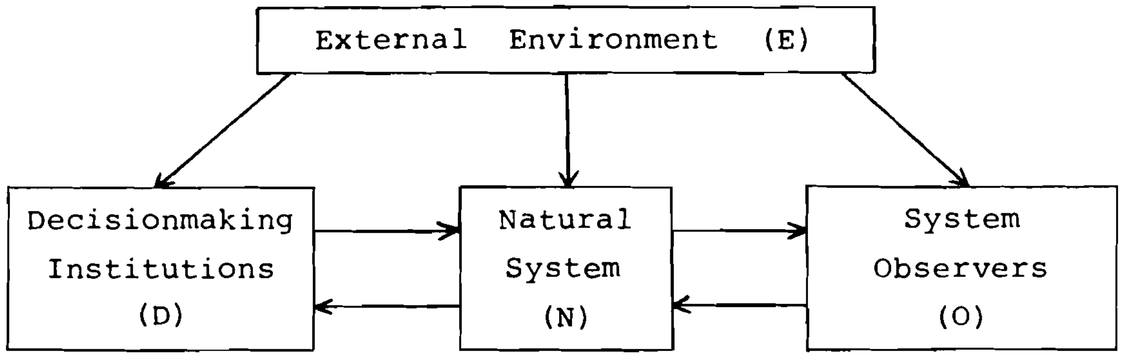

Figure 1. High-level System Decomposition

In actuality, these boundaries are totally artificial and what constitutes a useful separation is, in general, highly context-dependent as we shall see below. Nevertheless, the divisions indicated in Figure 1 do provide a helpful guideline upon which to focus the remainder of the modeling effort.

According to current system-theoretic thinking, the Natural System (N) is usually considered to be that part of a given process which is not directly accessible to external influence or observation. From a certain point of view, one might say that only the physically observable causes and effects reside in the Decisionmaking (D) and Observing (O) components, with the Natural System playing the role of a mediator. An alternate interpretation is to regard the Natural System as a "black box," whose inner workings we attempt to explore by applying inputs from the Decisionmaking and/or External (E) components and measuring outputs in the Observing component.

We can loosely delineate the model components N, D, O and E as follows:

- Natural System — a collection of variables and relationships perceived by D and O as having "internal" dynamics and couplings to E through measuring apparatus, control mechanisms and "forcing functions." This is an open system definition and there is no pretense that the boundaries separating N from D, O and E have been chosen in a knowledgeable or even intelligent fashion.

- Decisionmaking Institution — a system which processes information, develops models and exerts controlling actions on N, chosen with respect to the models and to objectives which may be at least partially established by E.

- System Observer — a component which monitors both N and E and which provides information to D about the behavior of the system N.

- External Environment — a collection of relationships that affect and are affected by both N and D, yet are not generally perceived as "part of the problem" by D. In a very real sense, E can be viewed as "everything that goes on in the world."

The foregoing definitions are far from being entirely satisfactory, but the only crucial point is that natural systems are arbitrary objects of analysis, whose formalization in N-D-O-E terms hinges critically upon the questions which the model is required to answer.

Probably the best way in which to gain a feel for the decomposition of a given problem into the compartments outlined above is to consider a few representative examples.

A. <u>Fishery Management</u> — a simple model of interspecific competition between two species is provided by the Gauss-type logistic equations

$$\frac{dx}{dt} = rx(1 - \frac{x}{K}) + \alpha xy - h_1(t),$$

$$\frac{dy}{dt} = sy(1 - \frac{y}{L}) + \beta xy - h_2(t),$$

where x and y are the two fish populations, r and s are growth rates, K and L are maximum population levels which the environment can support, α and β are measures of the extent to which each species interferes with the other's use of the external resource (food supply) and $h_1(\cdot)$ and $h_2(\cdot)$ are harvesting functions.

A plausible separation of the above model into the macro-components outlined above is to consider the variables x and y, together with their growth rates r and s and interference parameters α and β, as the natural system N. The decisionmaking institution D clearly is composed of the harvesting functions h_1 and h_2, while the observer O may be thought of as the variables x and y, since it is reasonable to suppose that the fish populations may be measured directly. Finally, the environmental carrying capacities K and L comprise the external environment E.

The above decomposition of the model variables illustrates the important point that the system components N, D, O and E are not necessarily disjoint; here we see that the variables x and y belong naturally to both components N and O.

B. National Income Dynamics — consider a vastly simplified picture of the dynamics of national income in which the total national income in year k is denoted by y_k and is the sum of the consumption expenditures w_k and the investment expenditures u_k, i.e.,

$$y_k = w_k + u_k \quad .$$

We assume that consumption expenditures depend upon the national income of the previous year as

$$w_k = by_{k-1} \quad ,$$

where b is a constant measuring the marginal propensity to consume. Clearly,

$$\begin{aligned} y_k &= by_{k-1} + u_k \\ &= u_k + bu_{k-1} + b^2 y_{k-2} \\ &= u_k + bu_{k-1} + \ldots + b^k y_0 \quad . \end{aligned}$$

The above relation defines an elementary input/output model of national income dynamics, in the sense that the output (total income) is determined as a (linear) function of past inputs (investment expenditures).

In terms of our earlier system components, it would be natural to regard y as the system observer O, u as the decision-making body D, and b as comprising the external environment E. What is of interest here is that the natural system N is only implicitly determined as some "mechanism" which generates the measured output y from the inputs u. Thus, while it is impossible to escape the feeling that N must be part of the problem, it is not explicitly represented by the variables defining the input/output relation, but rather must be mathematically <u>inferred</u> from them. We return to this "realization" problem later.

C. <u>Euler Arch</u> — here we consider the classical physics problem of describing the static behavior of two rigid arms of unit length supported at the ends and pivoted together at the center, with a spring of modulus μ tending to keep the arms apart at 180° (see Figure 2).

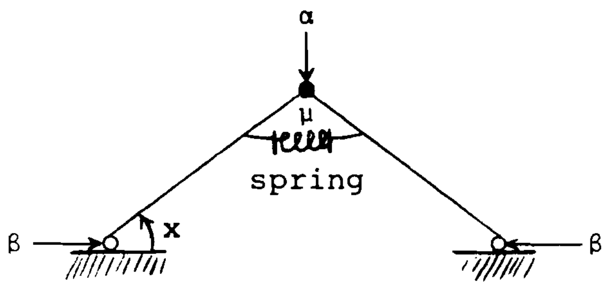

Figure 2. The Simple Euler Arch

If the ends are compressed with a gradually increasing horizontal force β then the arms will remain horizontal until β reaches a critical value, at which point they will begin to buckle upwards (or downwards). If β is now fixed and a gradually increasing vertical load α is applied to the pivot, then the arch will support the load until α reaches a critical value, when the arch will suddenly snap catastrophically into the downwards position. We wish to model the equilibrium positions of the arch as a function of α and β.

It is convenient to model the global behavior of the Euler arch by considering the total energy of the system. We have the

$$\text{total spring energy} = \frac{1}{2}\mu(2x)^2 \quad ,$$

$$\text{energy gained by loading} = \alpha \sin x \quad ,$$

$$\text{energy lost by compression} = -2\beta(1 - \cos x) \quad .$$

Thus,

$$\text{total energy } V = 2\mu x^2 + \alpha \sin x - 2\beta(1 - \cos x) \quad .$$

The surface M of equilibrium positions is determined as

$$M = \left\{ x : \frac{\partial V}{\partial x} = 0 \right\} \quad .$$

In fact, it can be shown that M is a so-called cusp catastrophe surface. The main point to note now is that the local dynamics of the Euler arch are deduced from the global energy function V, together with the variational principle that the system assumes an equilibrium position such that V is at a local minimum. Following through the implications of this fact, it turns out that the arch buckles when $\beta = 2\mu$.

For the Euler arch, the natural system N is the position x of the arms of the arch. The decisionmaking body or mechanism D consists of variation of the input parameters α and β, while the observer O is x, the same as N. Finally, we think of the spring constant μ as representing E, the external environment, although we might also wish to include other fixed elements such as the length of the arms (here taken to be unity), the gravitational constant and other possibly influential factors to be part of E.

D. <u>Transportation Network</u> — in Figure 3 we display a section of a typical urban street network. The arrows indicate the allowable directions of traffic flow within the street network. For

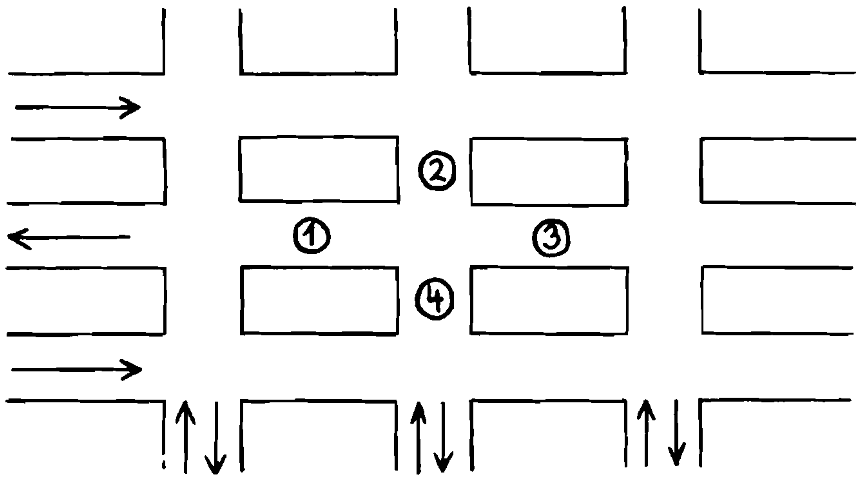

Figure 3. Urban Traffic Network

purposes of analyzing traffic flow through such a network, it is convenient to represent it abstractly by the directed graph (digraph) of Figure 4. Here the nodes 5 and 6 have been added to account for trip initiation or termination. An arc connects nodes i and j if it is possible to pass from node i to node j in a trip of at most one block.

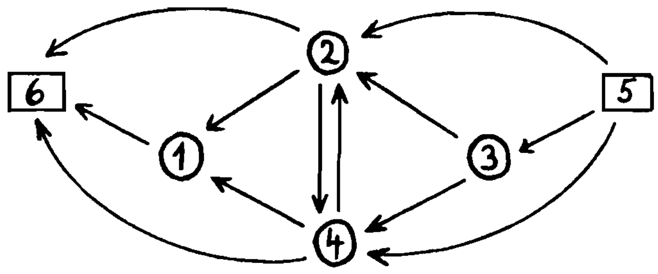

Figure 4. A Graphical Representation of the Traffic Network

If costs c_{ij} are assigned to each arc (i,j), representing the travel time, say, between node i and j, then a number of questions related to assignment of network traffic, regulation of traffic signals, bottleneck intersections and so forth can be approached through the digraph model depicted above.

In N, D, O, E terms, the foregoing transportation network might be decomposed as:

N = the set of streets, together with the allowable directions of traffic flow and the time of traverse along each link.

D = the assignment of automobile trips to the streets of the network by, say, regulation of the traffic signals.

O = the measured traffic flow along each arc of the network or, equivalently, the measurement of traffic passing through each intersection.

E = the traffic flowing into and out of the system through nodes 5 and 6.

Examination of the preceding examples leads to the observation that each system model exhibits the characteristic features of inputs (decisions), outputs (measurements) and states. The explicit separation of variables into these categories forms the cornerstone of modern mathematical modeling and distinguishes the current view of modeling from an earlier, semi-archaic approach pioneered in operations research and mathematical programming, in which all variables are treated equally with no explicit acknowledgement of their individual role in the problem. We shall continually emphasize the importance of this point throughout the chapter.

The preceding discussion, together with the examples, shows that there are generally many different types of mathematical representations which may lay claim to the title of a "model" for a given process. The particular type of model employed will usually depend upon the specific questions we wish to have answered about the process and the accuracy demanded in these answers. Before turning to a detailed examination of the basic system-theoretic questions one might wish to consider, let us examine in more detail the various types of mathematical descriptions which might be employed.

II. Types of Mathematical Descriptions of a System

A. Sets and Relations — at the most primitive level of mathematical description, we may choose to model a system as a relation (or collection of relations) defined on two (or more) finite sets.

The general set-up for such a model is to choose two sets $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, whose elements have some relevance to the variables of the problem, and define a binary relation λ on the cartesian product $X \times Y$ of X and Y. The relation λ, which is a subset of $X \times Y$, is defined in some meaningful way within the context of the particular problem and it is often convenient to represent it by an incidence matrix Λ, as described in Section V, p. 50. The transportation example cited above is a model of this type, wherein we might choose $X = Y =$ the street intersections in the transportation network (the nodes of the graph). The relation λ for this example is such that $(x_i, y_j) \in \lambda$ if and only if node x_i may be reached from node y_j by a single block passage through the network according to the allowable traffic flow. Thus, we see that almost all graph-theoretic descriptions are inherently of the sets/relations type, with $X = Y$.

Generally, $X \neq Y$ and the relation λ links quite different aspects of the system under study. For example, in an analysis of the relative strength of various board positions in the game of chess, it is useful to let

$$X = \{\text{playing pieces}\} = \{P,N,B,R,Q,K\} \ ,$$

$$Y = \{\text{board squares}\} = \{1,2,\ldots,64\} \ .$$

In this case, several different relations may be defined on $X \times Y$, representing various views of the players Black and White. Details of this example may be found in the chapter notes for the interested reader.

A particularly valuable aspect of the sets/relations view of a system is the ease with which such a description enables the analyst to deal with hierarchical system structure. The key idea in such a hierarchical system decomposition is the concept of a <u>set cover</u> and the natural stratification it induces on a set. We say that a collection of subsets $\{A_i\}$ forms a cover of the set X if and only if

i) $A_i \in P(X)$, the power set of X,

ii) $X = \bigcup_i A_i$.

Thus, we may now define a hierarchy H by relations of the type $(A_i, x_j) \in \mu \Leftrightarrow x_j \in A_i$. The general idea can also be extended in an obvious way to additional hierarchical levels and diagonally across levels as indicated in Figure 5.

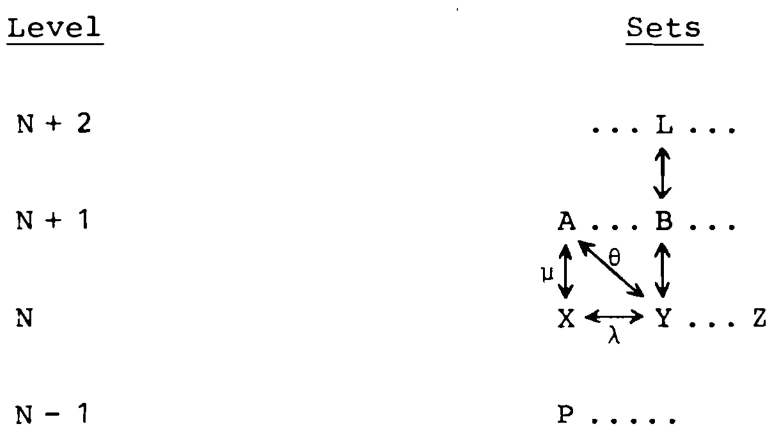

Figure 5. Hierarchical Levels of Sets and Relations

B. Input/Output — closely related to the sets/relations type of description is that in which we describe the system inputs and outputs by elements in certain spaces Ω and Γ, say, and define a map $f : \Omega \to \Gamma$, which associates inputs with the corresponding outputs. Such a description differs from the sets/relations type only in that additional algebraic and/or topological structure is usually imposed on the sets Ω and Γ, depending upon the application. Most commonly, Ω and Γ are assumed to be finite-dimensional vector spaces of some sort, with f a linear map. Such is the case, for example, in the so-called input/output models in the Leontief theory of global economic processes.

Unfortunately, both the sets/relations and input/output types of mathematical descriptions, while of considerable value in analyzing certain structural and connective features of large systems, are somewhat deficient in dealing with dynamical considerations. Furthermore, as these system descriptions are basically phenomenological, as expressed through the binary relation λ (or the map f), such models are inherently limited in their predictive powers, i.e., they offer no real explanation of the means by which inputs are transformed into observable outputs. Thus, the need arises for a more detailed description accounting for the "inner workings" of the system under study.

C. Potential Functions — occupying an intermediate position between purely phenomenological descriptions and detailed internal descriptions of system behavior are potential (or energy) function descriptions, which have at their basis the teleological principle that a system's dynamic is such that the system "moves" to a minimum of a suitably defined energy function.

Such models, of course, have a long tradition in classical mechanics, arising from the well known variational principles of Fermat, d'Alembert, Hamilton, Lagrange and others. Considerable ingenuity, imagination and wishful thinking have been expended in recent years in an attempt to develop corresponding variational principles for more general processes occurring in biology, ecology and the social sciences. The basic problem, of course, is to find some invariants of motion for such processes. Various thermodynamic arguments, interspersed with concepts from information theory have also been employed in this regard. Perhaps surprisingly, there have been some limited successes in such modeling efforts, with interesting results reported in population dynamics, cell differentiation and chemical reactions.

Mathematically, a potential function description of a process assumes that there exists a function $V(x_1, x_2, \ldots, x_n)$, where the x_i are microscopic system variables, such that the equilibrium states x^* of the process are given by the set $M = \left\{x : \frac{\partial V}{\partial x_i} = 0\right\}$. Dynamically, this means that the transient motion of the system variables is described by the set of differential equations

$$\frac{dx_i}{dt} = -\frac{\partial V}{\partial x_i}, \qquad i = 1, 2, \ldots, n,$$

or more compactly,

$$\frac{dx}{dt} = -\operatorname{grad} V .$$

Thus, we see that the existence of a potential function V induces a dynamic upon the system. The converse question, namely, given a dynamic

$$\dot{x} = f(x) \quad ,$$

does a potential function V exist such that

$$f(x) = -\operatorname{grad} V(x) \quad ,$$

is of some importance also, especially in view of the dependence of Thom's theory of catastrophes upon such "gradient" systems. The answer to the above question is provided by the following simple test: the system $\dot{x} = f(x)$ is a gradient system if and only if the Jacobian matrix $J(x) = \frac{\partial f}{\partial x}$ is symmetric, i.e., $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all x in the region of interest.

D. Internal Descriptions — passing from a local dynamic induced from a global system potential function, we next consider a system description based entirely upon local interactions, a so-called "state variable" model. In continuous-time, such a model takes the form of a system of differential equations

$$\dot{x} = g(x,u,t) \quad , \qquad x(0) = x_0 \quad ,$$

where $x(t) \in R^n$ is the system state, while $u(t) \in R^m$, is the vector of system inputs. Usually, there is also an output $y(t) \in R^p$, generated by the states (and possibly the inputs), given as

$$y(t) = h(x,u,t) \quad .$$

The critical point to note here is the way in which a mathematical artifice, the state vector x, has been introduced as a vehicle to mediate between the inputs and outputs. The importance of this observation cannot be overemphasized since it is precisely the introduction of the notion of state which enables an internal description to provide a predictive model of system behavior. Schematically, we have the "black-box" situation in Figure 6, with the

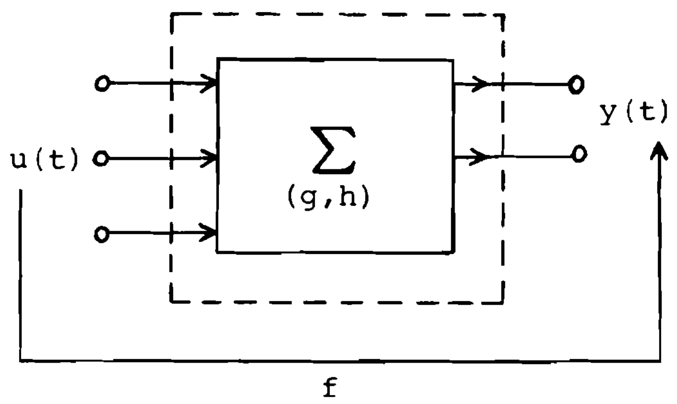

Figure 6. Black-Box Model of the System

system input/output model being described by the map f. The internal system description is provided by the two maps (g,h), which together give an "explanation" of f in the sense that the input/output behavior of Σ generated by (g,h) agrees with that of f. The question of how to determine (g,h), given f, will be discussed in the next section.

The attraction of internal system models is quite clear: if the functions (g,h) can be obtained on the basis of physical laws and observed data, then the internal description allows us to predict the future behavior of the process as a function of the inputs (decisions) applied. In short, the functions (g,h) describe the internal "wiring diagram" of the system Σ and uniquely determine its future outputs, given the current state and future inputs.

E. Finite-State Descriptions — for a number of technical reasons, the problem of determining an internal model, given an input/output map f, is complicated when the problem state-space is infinite (even if finite-dimensional), except in certain cases where special structure, e.g., linearity, is present. In addition, there are a number of practical situations in which it is natural to consider a finite state-space model as, for example, when modeling the workings of a digital computer. Such considerations lead to finite-state descriptions of dynamical processes. The usual ingredients of a finite-state description are

U, a finite set of admissible inputs,

Y, a finite set of admissible outputs,

Q, the finite set of states,

$\lambda : Q \times U \to Q$, the next-state function,

$\gamma : Q \times U \to Y$, the output function.

Since the finite sets U, Y and Q have no interesting topological structure, the analysis of a system described in the above terms is a purely algebraic matter, relying heavily upon the theory of finite semigroups. As computational considerations ultimately force us to reduce all descriptions of systems to the above terms, it is of considerable importance to understand as much as possible about the underlying structure and behavior of such finite-state descriptions. The famous Krohn-Rhodes Decomposition Theorem provides a starting point for analyzing the inherent structure of the finite-state models, as it insures the existence of certain coordinatizations of the state set Q, which are advantageous for computation of the action of system inputs upon the states.

F. Operations Research Descriptions — as already noted, the typical type of mathematical description arising in areas such as game theory, mathematical programming, decision analysis and other fields, which we collectively term operations research, generally consists of an exhaustive listing of all variables relevant to the system at hand, an account of various identities, inequalities and constraints existing between the variables and the presentation of a cost function involving some subset of the variables. A fundamental conceptual deficiency in such a description is the lack of discrimination between inputs, outputs and states. While a posteriori analysis can usually separate the original variables into the appropriate classes, the fact that such a separation has not been carried out prior to the analysis of the process prejudices, in our view, the entire methodological approach to the problem.

As illustration of such a modeling approach, consider an elementary linear programming problem.

A certain factory manufactures two products, "gadgets" and "gizmaccis." In each case the product is first processed on a cutting machine, then a hole is drilled into it on a drill press. The times required for these operations, the total time available per week, and the profit per gadget or gizmacci are as below.

machine	gadget	gizmacci	time available
cutter	3	5	15
drill press	5	2	10
profit per unit	5	3	

How can the manufacturer maximize his profit?

Introducing the variables x_1 and x_2 as x_1 = number of gadgets to produce, x_2 = number of gizmaccis to produce, we have the relations

$$3x_1 + 5x_2 \leq 15 \quad ,$$

$$5x_1 + 2x_2 \leq 10 \quad ,$$

where $x_1 \geq 0$, $x_2 \geq 0$, with the profit being

$$5x_1 + 3x_2 \quad .$$

Elementary graphical means or a routine application of standard algorithms yield the optimal solution $x_1^* = \frac{20}{19}$, $x_2^* = \frac{45}{19}$. (Here, for simplicity, we assume that gadgets and gizmaccis are continuously divisible.)

While the preceding model certainly deals with the problem as stated, the distinction between what constitutes the decisions, the states and the outputs is clearly blurred, at best. One might ask, 'so what?' The reply is that by neglecting the basic distinction between variables, it is very difficult to naturally incorporate dynamical considerations into the model and, what is worse, without the concept of a system state, it is next to impossible to consider feedback decisionmaking or stochastic effects.

As an illustration of how the preceding problem could have been formulated in more system-theoretic terms, let us introduce the variables

u = the amount of time available on the cutting machine,

v = the amount of time available on the drill press,

and the function

$f_n(u,v)$ = the profit obtained when u units of cutter time, v units of drill press time are available, n types of items are to be produced and an optimum decision rule is employed.

Then it is easy to see that

$$f_2(u,v) = \max_{\substack{0 \le 5x_2 \le u \\ 0 \le 2x_2 \le v}} [3x_2 + f_1(u - 5x_2, v - 2x_2)] \quad ,$$

$$f_1(u,v) = \max_{\substack{0 \le 3x_1 \le u \\ 0 \le 5x_1 \le v}} [5x_1] \quad .$$

Computation of the functions f_1 and f_2 for all values of (u,v) in the range $0 \leq u \leq 15$, $0 \leq v \leq 10$ enables us to solve a family of problems for all cutting and drilling times in the indicated ranges. Furthermore, the concept of a system dynamic is introduced through the idea of manufacturing "gadgets," followed by "gizmaccis." Thus, the solution proceeds one item at a time, rather than attempting to compute all production levels in one fell swoop. This dynamic approach is a direct consequence of introducing the state variables u and v, along with the decision variables x_1 and x_2.

The disadvantage of the dynamic programming (DP) formulation just given is that the computational algorithms are not nearly as efficient as for the previous linear programming (LP) set-up, which can employ the simplex method. However, if the costs and/or constraints are nonlinear or if stochastic effects enter, then the DP formulation comes into its own. The point to observe here is that it is a fundamental mistake to swear religious adherence to any one particular orthodoxy: flexibility in modeling must be maintained if best results are to hoped for.

III. Local Considerations

Once a particular mathematical description has been chosen for a given process, a number of important system-theoretic questions involving both local and global phenomena present themselves. The manner in which these questions appear in the model depends, of course, upon the type of description employed; however, the abstract phenomena are relatively invariant under a change of description, so we shall attempt to discuss the main issues in as context-free a manner as possible in the next two sections.

One set of system phenomena that any model must cope with are issues which are best termed "local" system properties. Here by local we mean that the phenomena either manifest themselves in some restricted region of the system model or, in some meaningful sense, can be analyzed by considering only interactions between system components in the immediate neighborhood of a restriced piece of the entire system. On the other hand, global aspects require consideration of the entire system for their analysis; no smaller subsystem will suffice. We shall consider global properties later. For now, let us examine some of the local considerations more closely.

i) stochastic effects — typically, the influence of the external part of a system, which we do not fully understand or cannot account for in the system descriptions sketched above, is often assumed to be a random perturbation whose effect is locally felt upon the system.

For illustration, assume we have modeled a process by an internal description as

$$x(t+1) = g(x(t), u(t), t) \quad ,$$

$$y(t) = h(x(t), t) \quad .$$

To account for the fact that the functions g and h may not be known exactly, the above model may be replaced by

$$z(t+1) = g(z(t), u(t), t) + r(t) \quad ,$$

where $r(\cdot)$ is a stochastic process with appropriate statistical properties. Here the local effect of the noise r is felt by the system in state z(t), i.e., r acts as a perturbation in a local neighborhood of the state z(t). Such a disturbance is in the E part of the system Σ.

Another manner in which stochastic effects locally influence Σ is through the D component. Here we assume exact knowledge of the dynamics g and the observation function h is known exactly, but the theoretically desirable control law u(t) cannot be applied because of computational inaccuracies or otherwise. Similarly, it may not be possible to measure the system output with complete precision owing to noise corruption in the measuring apparatus. These are again local effects in the sense that they affect only a neighborhood of a point in the control or output space.

With a more elementary level of system description the stochastic features assume a somewhat different form. For instance, if the description is sets/relations then there may be uncertainty as to whether or not a particular pair $(x_i, y_j) \in \lambda$ or, if an input/output model is used, then uncertainties in the map f may arise. What is important here is not the fact that stochastic influences appear, but that their influence is exerted at a point of the system, not throughout the entire system simultaneously. This is the essence of what we mean by a local effect.

ii) constraints — restrictions on the system inputs and/or states come in two varieties: local, in which the immediate decision or state is constrained to lie in some admissible region or global, in which some overall function of the control or state must remain bounded within given limits. We shall illustrate both types.

Consider an internal system model

$$\dot{x} = x + u \quad , \qquad x(0) = x_0 \quad ,$$

and assume that it is desired to transfer this system to the origin. Further, assume that the magnitude of controlling action is limited by

$$|u(t)| \leq M \quad .$$

Such a limitation may arise as a result of considerations such as maximum stress factors, finite resource availability, maximum tolerable unemployment rates and so on. In any case, the constraint locally restricts the amount of control action that can be exerted to modify the system's dynamical behavior. Here local is interpreted in the temporal sense, as the magnitude limit M must be obeyed at each time instant t.

Now consider the same problem with the local constraint replaced by the condition

$$\int_0^\infty u^2(t)\, dt \leq K \quad .$$

Here we have an example of a global constraint. There is now no restriction on the instantaneous value of the control u, only the condition that the total control energy expended in transferring the system to the origin remain bounded by K. Thus, we have now traded a constraint on the local value of a control function for a condition on the entire function u itself.

If the system description is of the sets/relations type then the constraints are almost automatically built into the schema through the relation λ. This would be regarded as a global constraint, as it restricts those elements of the basic sets X and Y which can be λ- related. In other descriptions, as in operations research, the constraints enter in both a local and global form, as was indicated in the LP example of the preceding section, where we had the local nonnegativity constraints $x_1 \geq 0$, $x_2 \geq 0$, with the global resource constraints on the available cutting machine and drill press time.

iii) time-lags — a fundamental principle of large systems is: control takes time. Thus, the theoretical assumption implicitly built into most internal models that the system output measurement and determination and application of the control action take place coterminously must be regarded as only a convenient mathematical idealization in real problems. Happily, such an approximation works well in many cases, especially in classical physics and engineering. However, in decentralized processes with many components and decisionmakers the "simultaneity" hypothesis can no longer be accepted and explicit account must be taken of the time-lag effect. This is particularly true in models arising from social-science situations.

To illustrate the manner in which time lags can affect a control law, consider the internal model

$$\dot{x} = -2x(t) + u(t) \quad ,$$

$$x(0) = 1 \quad , \qquad 0 \le t \le 2 \quad ,$$

where it is desired to choose the input u(t) so that the terminal state x(2) is as small as possible, subject to the constraint

$$|u(t)| \le 1 \quad , \qquad 0 \le t \le 2 \quad .$$

It is an elementary exercise to see that the optimal choice is

$$u^*(t) = -1 \quad , \qquad 0 \le t \le 2 \quad .$$

Now consider exactly the same problem with the sole change that the system dynamics have a unit time lag in the state, i.e., the dynamics are

$$\dot{x} = -2x(t-1) + u(t) \quad ,$$

with the initial condition now being

$$x(t) = 1 \quad , \qquad -1 \leq t \leq 0 \quad .$$

It is a somewhat less elementary, although straightforward, exercise to determine the optimal control for this problem as

$$u(t) = \begin{cases} 1, & 0 \leq t \leq \frac{1}{2} \quad , \\ -1, & \frac{1}{2} \leq t \leq 2 \quad . \end{cases}$$

Thus, we find that introduction of a time lag has resulted in a <u>qualitative</u> change in the structure of the optimal decision by introducing a switching point from max control to min control at $t = \frac{1}{2}$. Furthermore, we note that this change in control strategy is a local effect in that it is applied to the system when it is in its state $x(\frac{1}{2})$ (which happens to be $x = \frac{1}{2}$ in this case).

It is a common feature of processes involving time lags that the presence of a delay may cause the appearance of self-exciting oscillations, then exertion of too much control, followed by complete instability of the system. Such undesirable phenomena can only be avoided by application of inputs timed in such a manner so as to counteract the influence of the after effects on the state due to the delay. This is an important aspect of controlling large, complex systems and one to which considerable theoretical work is currently being directed.

IV. Global Aspects of System Structure

In contrast to the local issues involving system structure and behavior in a restricted region of its definition, we must also consider aspects of the system which cannot be confined to any particular part of the structure, but which are properties of the entire system. If the preceding section dealt with topics associated with a "reductionist" view of the system, then the current section will look at the system from a "holist's" vantage point, taking into account properties possessed by no single component or subsystem of the total system. We have in mind system properties such as conservation/dissipation laws, hierarchical structure, singularities and process time scales. Let us examine each of these global features in a bit more detail.

i) conservation/dissipation laws — a good part of mathematical physics is anchored by the laws of conservation of mass, energy, charge, baryon number and so on. These are all restrictions imposed upon the global behavior of physical processes. On the other hand, equally basic principles involving dissipation effects also pervade classical physics. Here we refer to increase of entropy in closed systems as dictated by the Second Law of Thermodynamics and the transformation of mechanical energy to heat via various types of functional effects. Again, such dissipative principles are constraints which the global dynamical behavior of a process must adhere to. The conservation/dissipation laws impose no restriction on the local behavior of a process; they simply say that the total motion must be such that certain functions of that motion are invariant or non-decreasing.

The search for extensions of the conservation laws of classical physics for more general systems has been the topic of much study in the systems literature. Generally speaking, the further the given system is from a classical physical process, the more fanciful the proposed conservation principles seem. Nonetheless, interesting results have been obtained in some areas. For instance, for the well known Lotka-Volterra predator-prey dynamics

$$\dot{x} = (a - by)\,x \quad ,$$

$$\dot{y} = (cx - d)\,y \quad ,$$

it can be shown that the following function is constant along solution curves

$$H(x,y) = cx + by - d\log x - a\log y \quad .$$

The constancy of H imposes certain global structural features upon the dynamics of the system, e.g., no limit cycles, each trajectory is a closed orbit, etc.

In a quite different direction, Ashby developed his Law of Requisite Variety in an attempt to introduce thermodynamic considerations into system theory. The basic idea is to define the variety of a finite set A to be $\log_2$ (card A), where card A = number of elements in A. Then if Ω and E are, respectively, the set of inputs (decisions) and external disturbances for a given system, the Law of Requisite Variety states that only variety in Ω can force down variety due to E. In other words, if the variety in the control is $\log r$ and that of E is $\log c$, then the variety in the output is <u>at least</u> log r - log c. An account of the derivation of this basic rule, together with its connections to entropy and information theory can be found in the works of Ashby cited at the end of the chapter.

The preceding examples show that conservation principles can yield important information on the structure and behavior of a given system if we are either clever or lucky enough to find them. Regrettably, as yet there appears to be no uniform procedure to employ for generation of such laws for general classes of processes.

ii) <u>hierarchical structure</u> — almost all large systems, biological, business, economic, political or sociological, share the property of hierarchical organization. Decisionmakers exist on all levels communicating instructions and receiving information from subordinate levels. From a modeling standpoint, we are interested in questions such as how the hierarchical structure influences the flow of information throughout the system, what effect the hierarchical organization has upon the system's ability to react to external disturbances, the sensitivity of the system output to changes in the connective structure of the hierarchy and so forth.

The hierarchical organization of a given system is clearly a global feature which cannot be analyzed by local tools. In mathematical system studies, it appears that ideas taken from algebra and geometry will prove most effective in studying questions related to system hierarchy. As was indicated earlier, techniques from algebraic topology can be employed in a sets/

relations description to quantitatively study hierarchical organization. It is tempting to conjecture that once the global tools have mapped out the overall hierarchical structure and connective pattern for a system then local tools from analysis may be brought to bear upon considerations such as system stability. However, since we don't wish to begin dreaming in print, we leave this as only a speculative possibility and move on to other global system properties.

iii) <u>singularities</u> — for systems modeled by differential (or difference) equations, perhaps the most noticeable feature is the set of points in state, parameter or control space at which qualitative changes in system behavior occur.

For instance, consider the internal model

$$\dot{x} = g(x,a) \quad , \qquad x(0) = c \quad ,$$

where a is a vector of parameters. The steady-state equilibrium of the system will be the state $x(\infty) = x^*(c,a)$, where we explicitly indicate the dependence of x^* upon the parameters a and the initial state c. Furthermore, $x^*(c,a)$ will be a solution of the equation

$$g(x,a) = 0 \quad .$$

For <u>fixed</u> a and c, any equilibrium x^* can be regarded as a singularity of the process. This is the view taken in classical stability theory. On the other hand, we may also consider the map

$$x^* : \begin{array}{l} (c,a) \longmapsto R^n \\ (c,a) \longmapsto x^*(c,a) \end{array} \quad ,$$

whereby a and c are regarded as variables. In this setting, those values of c and a at which the map x^* is discontinuous or multivalued are also considered to be singularities of the system, although of a very different type.

In either of the above cases, it can be shown that the singularities of the system determine to a large degree its entire dynamical behavior and it is only a small exaggeration to think of the transient motion as being forced upon the system by the particular structure of its singularities. In addition, one should note that no local coordinate changes can remove singularities of the above type: they are global invariants of the process. Thus, it is of the utmost importance to understand the number and nature of all system singularities if we wish to control any system in an effective manner.

iv) time constants — an almost universal feature of large systems is that the variables seem to separate into a fast-time/ slow-time dichotomy. This qualitative distinction between variables is so pervasive that, for most engineering problems, the "fast" variables are usually considered state variables, while the "slow" variables are generally treated as parameters. Here again we see a system property which cannot be localized to a particular region of variable definition.

In mathematical modeling, it is of some importance to isolate the slow variables since, depending upon the application, it may be possible to "factor" them out of the problem, at least in a computational sense. For example, if there are $n + m$ variables which describe the evolution of the system and m of them are slow variables which can be regarded as parameters, then we have the option of considering a single problem with a state space of dimension $n + m$, or m problems of dimension n. In many cases, the first version may be computationally intractable, or at least impractical. Those analysts familiar with dynamic programming procedures for control processes will recognize the fast/slow separation of variables as one way in which we may hope to lift the "curse of dimensionality." The catastrophe theory applications of Thom, Zeeman and others are also a good illustration of time-constant exploitation, wherein the slow variables are regarded as inputs (or decisions), while the fast variables are the observed outputs. All intermediate speed variables (the states) are suppressed in what is essentially an input/output theory.

Having now had a look at some of the main local and global aspects of large-scale systems, we turn to a more extensive discussion of the basic system-theoretic questions which the mathematical model must address.

V. Basic Questions and Perspectives of System Theory

Models are constructed because there are aspects of the system which we don't understand and wish to explore. But what type of questions can models of the foregoing type address and to what extent do modern system-theoretic tools enable us to speak with confidence about the connection between the system model and the system? These issues lie at the very heart of the "systems approach", and it is possible to provide only a partial glimpse of the overall situation in such a brief chapter. So, in this section we shall sketch a broad array of questions which can be approached using mathematical models of various sorts. This overview should provide the needed perspective for the reader to pursue the technical literature with some confidence.

The basic questions of theoretical and applied system science, when broadly interpreted, are surprisingly few in number and, in one way or another, all center about the interaction of the system with its environment. For purposes of exposition, it is convenient to group the questions into the following main categories:

- reachability/controllability — the identification of those system behaviors which are achievable by application of admissible inputs;

- observability/detectability — the determination of those system behaviors which are identifiable from measured physical outputs;

- realization/identification — the generation of the class of models which could "explain" a given set of input/output data;

- optimality — determining how efficiently a system can perform a specified function, subject to physical and theoretical operating constraints;

- stability/sensitivity — calculating the way in which errors and disturbances affect the equilibrium behavior of a system.

We now examine each of the above categories within the context of specific types of mathematical descriptions. As we proceed, it will become evident that the type of mathematical description employed will strongly flavor the precise technical form of the question, but the invariant essence of the problem will be sufficiently clear as to leave no doubt as to which category the question belongs.

A. Reachability/Controllability

Abstractly, the question of reachability may be formulated in the following terms: given a set Ω of admissible inputs and a set X of system states, the transition map ϕ of the system associates a particular state with each element $\omega \in \Omega$, assuming the system starts in some agreed upon initial state x_0 (usually taken to be the origin if X is a vector space). Thus, the map $\phi : \Omega \to X$ determines the effect of the input ω on the system, transferring x_0 to the state $\phi(x_0;\omega)$. The problem of reachability is to characterize the range of ϕ. In the event the map ϕ is onto, i.e., the range of ϕ is all of X, then we say that the system is completely reachable, i.e., any state in X may be "reached" by application of some admissible input from Ω. The corresponding problem of controllability is similar: given that the system is in a state $x \in X$, does there exist an input from Ω which transfers the system to x_0? If so, x is called a controllable state. If all $x \in X$ are controllable, then the system is said to be completely controllable. (Remark: the preceding definitions are incomplete in the sense that the initial and terminal time should also be taken into account. We omit this aspect for two reasons. First, it is relevant only for non-autonomous systems described in internal form and secondly, it requires an extended notation which is needlessly elaborate for our current needs. The technical treatments of mathematical system theory cited in the bibliography will supply the reader with all relevant definitions and details.)

Certainly, the best-structured concretization of the above abstract set-up is for a linear system given in internal form. Here the system dynamics are

$$\dot{x} = Fx + Gu \quad , \qquad x(0) = x_0 = 0 \quad ,$$

and it is a well known result that the set of reachable states $\mathcal{R}$ is precisely the set of elements in R^n spanned by the vectors $G, FG, F^2G, \ldots, F^{n-1}G$, i.e.,

$$\mathcal{R} = \text{span}\,\{G, FG, F^2G, \ldots, F^{n-1}G\} \quad .$$

Furthermore, a consequence of linearity and continuous-time is that, if a state $x \in \mathcal{R}$, then x may be reached in an arbitrarily short length of time. As an illustration of the above result, consider the system in R^4 given by

$$F = \begin{bmatrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 6 & -1 & 2 & 1 \\ -14 & -5 & -1 & 0 \end{bmatrix} , \qquad G = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Here the set

$$\mathscr{R} = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}3\\-4\\6\\-14\end{pmatrix}, \begin{pmatrix}5\\-8\\20\\-28\end{pmatrix}, \begin{pmatrix}7\\-12\\50\\-50\end{pmatrix}\right\}$$

$$= R^4 \quad ,$$

since the vectors G, FG, F^2G, F^3G are linearly independent. Hence, the above system is completely reachable.

For more general processes given in internal form, the situation cannot usually be resolved by linear algebraic techniques alone. For instance, the reachable set of the nonlinear system

$$\dot{x} = f(x,u) \quad , \qquad x(0) = x_0 \quad ,$$

with f analytic in x and u, can be characterized using techniques from differential geometry and Lie algebras of vector fields.

Systems described by finite-state machines have a reachability theory that parallels that for internal descriptions, and which is usually termed "strongly connected" in the automata literature.

In the event we have a system description of a more general type as, for instance, sets and relations, then we may no longer have as much structure in the sets Ω and X and, consequently, it may be somewhat more complicated to characterize reachability. Say, for example, that in a sets/relations description we take the state space X to consist of a vector Q whose components (positive integers) characterize the number of connected components which exist at dimension level q in the simplicial complex associated with the relation λ. (The notion of q-connectivity is elaborated in detail in the works cited in the references.) The set Ω may consist of various modifications that one could make to the incidence matrix Λ, e.g., addition or deletion of vertices, modifications of entries from 0 to 1 or vice-versa. Then the reachability problem would be to ask if a prescribed structure vector Q could be obtained by admissible changes in the relation λ. This is a far different technical problem than that sketched earlier. Nonetheless, the abstract structure of the question is the same.

B. Observability/Detectability

The question of reachability revolves about what can be accomplished using admissible inputs. Problems of observability focus upon what can be done with system outputs. More precisely, each state $x \in X$ of a system generates a certain output via the system output map

$$\eta : X \to \Gamma \quad ,$$

where Γ is the output set. Questions of observability deal with the issue of whether or not two (or more) distinct states x and x' give rise to the same output. In set-theoretic terms, we are concerned with whether the map η is 1 - 1. In most practical problems, it is impossible to physically monitor the entire system state. We must settle for measurements of accessible variables or aggregates such as sums of various state components. Observability properties of the system then determine whether it is theoretically possible to reconstruct the entire state from output measurements.

As one might suspect, for the constant linear system

$$\dot{x} = Fx \quad , \qquad x(0) = x_0 \quad ,$$

$$y = Hx \quad ,$$

the observability question can be settled by purely algebraic means. It is an easy exercise to verify that a given state x^* is <u>unobservable</u> if and only if $\eta(x^*) = 0$. Thus, the unobservable states are precisely those elements forming the kernel of the matrix

$$[H', F'H', (F^2)'H', \ldots, (F^{n-1})'H'] \quad .$$

In other words x_0 is unobservable if and only if it is mapped to zero by the above matrix. Otherwise, measurement of the output $y(t)$, over an arbitrarily short interval, will suffice to uniquely determine x_0.

The foregoing result strongly suggests a dual relationship between the concepts of reachability and observability upon making the transformations $F \to F'$, $G \to H'$, $p \to m$ (recall: H is $p \times n$, G is $n \times m$). A precise duality theory (in the vector space sense) can be developed by following up this observation and it can be seen that a system is completely reachable if and only if its dual is completely observable. Heuristically, this result is equivalent to interchanging the system inputs and outputs and reversing the flow of time.

As usual, the observability question for more general processes is not so well understood and its very discussion would require more mathematics than we have room for here. In more general contexts, such as potential functions, sets/relations, etc., even a precise statement of the problem remains to be formulated, although the general notion of the output map being 1 - 1 provides a starting point.

C. Realizations/Identification

The construction of an internal description from input/output data is the very essence of mathematical modeling. In technical terms, this is the "realization" (electrical engineering terminology) of the data. A special subcase of the general problem is when the model structure is given and only the values of parameters within the model need to be determined by the data. This is the parameter identification problem. In either case, the objective is to provide a model which, in some sense, "explains" the observed data.

The form of the realization depends, of course, upon the type of model one is attempting to obtain. Generally, we are given an external description, i.e., a map

$$f : \Omega \to \Gamma \quad ,$$

where Ω and Γ are the system input and output spaces, respectively, and the task is to construct an internal model whose input/output behavior reproduces that of the map f. If f is linear, it turns out that the problem is remarkably easy: there are an infinite number of non-equivalent internal models which will have external behavior identical to that of f. However, all ambiguity is removed (modulo a coordinate change in the state space X) if we further demand that the realization be both completely reachable and completely observable. Such a realization is called canonical and is equivalent to demanding that the dimension of the state space X be as small as possible.

Example. Suppose a single-input/single-output linear system is presented with the input

$$u(t) = \begin{cases} 1, & t = 0 \\ 0, & t > 0 \end{cases} ,$$

and the observed output is the sequence of natural numbers, i.e.,

$$y(t) = t \quad , \qquad t = 1,2,\ldots \quad .$$

The problem is to realize an internal model

$$x(t+1) = Fx(t) + Gu(t) \quad ,$$

$$y(t) = Hx(t) \quad ,$$

whose input/output behavior generates the natural numbers starting with a unit input. Application of standard algorithms soon yields the canonical model

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} , \qquad G = \begin{pmatrix} 1 \\ 2 \end{pmatrix} , \qquad H = \begin{pmatrix} 1 & 0 \end{pmatrix} .$$

which can easily be seen to be reachable and observable.

The realization problem takes on a much more complicated character once we pass out of the realm of linear theory. For some classes of nonlinear processes, some procedures exist which mimic the linear case as long as sufficient structure is present in the input/output map, e.g., multilinear or polynomial. However, almost everything remains to be done in the way of making these methods practically operational.

If the model we are trying to generate is not an internal "differential-equation type," then the realization problem enters the realm of system theory research. For instance, in a sets/relation context, the realization problem would be that of generating the system incidence matrix Λ, given the two finite sets X and Y together with an input/output relation between them. Here it is not even entirely clear what constitutes the input/output map f, but a plausible beginning would be to take the structure vector Q mentioned above. Difficult mathematical questions then arise as to whether or not Q contains sufficient information to determine Λ (up to a permutation matrix).

In another direction, we might wish to determine a potential function such that observed system equilibria (the measured data) agree with the stationary states of the potential function. Depending upon the setting, this is equivalent to solving the so-called "inverse problem" of the calculus of variations. Much work has been done in this area, but the problem is by no means completely settled.

D. Optimality

The imposition of some measure of system performance upon a process changes dramatically our view of the choice of system inputs. Now, instead of choosing an input to transfer the system to some specified state, we select the input to minimize a measure of system cost. (Of course, the reachability problem for a fixed terminal state could be viewed as a special case of the optimality problem by introducing a distance measure from the desired state as the cost function; however, it is generally more illuminating to regard the reachability issue separately as we have done above.)

In general terms, the optimality problem goes as follows: we are given a cost measure $J : \Omega \to R$ which associates a real number (the process cost) with each admissible input. The problem is to determine those inputs (controls) which yield the minimal cost. The existence and uniqueness of optimal controls for various classes of maps J and various types of internal and external dynamics has been studied for many years and a considerable body of knowledge, termed "optimal control theory," has arisen as a modern outgrowth of the classical calculus of variations, within which the results and techniques are codified. Again, the most extensive results are available for those processes described in internal form, as we now illustrate.

Consider the problem of minimizing

$$J = \int_0^T q(x,u,t)\, dt \quad ,$$

over all piecewise-continuous functions $u(t)$ on $[0,T]$. Assume that the system dynamics are

$$\dot{x} = f(x,u,t) \quad , \qquad x(0) = c \quad .$$

It has been shown that a necessary condition which any candidate optimal control must satisfy, is that it yield the pointwise minimum of the system Hamiltonian

$$H(x,u,t) = q(x,u,t) + \lambda(t)\, f(x,u,t) \quad ,$$

where $\lambda(t)$ is an arbitrary piecewise-continuous multiplier function to be determined. This is a scalar version of the famous Pontryagin Maximum Principle. Under convexity conditions on q, this principle can also be shown to be a sufficient condition for optimality, as well. With a little bit of analysis, it can be shown that the solution of the above problem reduces to solving the nonlinear two-point boundary-value problem

$$\dot{x} = \frac{\partial H}{\partial \lambda} \quad , \qquad x(0) = c \quad ,$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \quad , \qquad \lambda(T) = 0 \quad ,$$

with the minimizing control $u^*(t)$ belonging to the set

$$\mathcal{U} = \left\{ u : \frac{\partial H}{\partial u} = 0 \right\} \quad .$$

Thus, the Pontryagin Principle is an up-dated extension of Hamilton's equations from classical mechanics. We note, in passing, that the same problem can also be approached using dynamic programming or even via nonlinear programming methods.

In the event the system is described by a potential function, then the dynamics themselves are governed by a variational principle and we can express them as

$$\dot{x} = -\operatorname{grad}_x V(x,u) \quad ,$$

where V(x,u) is the appropriate potential. Here we generally regard the inputs u as parameters and the optimal control problem might be posed as the nonlinear programming problem of finding the best set of parameter values, with the above system dynamics as a constraint. However, if the inputs are functions then the Pontryagin approach sketched above could also be employed.

The more general setting of sets/relations or a graph description introduces the problem of suitable definition of a criterion, together with the serious technical difficulties of determining the type of inputs which will optimize the chosen performance measure. The difficulty is one of a lack of continuity, a typical obstacle in combinatorial problems. Since there is no notion of "nearness" upon which one can construct a variational theory, it is necessary to employ various algebraic means to attempt to isolate the best system input. Unfortunately, these methods are still in their infancy and nothing approaching a comprehensive set of results is yet available.

E. Stability/Sensitivity

One of the most fundamental of all system-theoretic questions is that of determining the effect of changes in the model upon the system structure and observed behavior. Such stability problems take on myriad forms, depending upon the type of system disturbance, the observed output, the structural feature under consideration, the type of mathematical description chosen and so forth. Here we shall indicate only a few of the more common types of stability problems.

Consider a system whose dynamics are described by the potential function $V(x,u)$, where the components of the vector u are a set of system parameters, i.e., the system evolves according to the gradient dynamics

$$\dot{x} = -\operatorname{grad}_x V(x,u) \quad .$$

The equilibrium states M of such a system correspond to the critical points of the potential V and the particular location $x^* \in M$ of the equilibria depends upon the vector u, i.e., $x^* = x^*(u)$, where $M = \{x : \operatorname{grad}_x V = 0\}$. An important stability problem is to determine those values of u such that the map $u \longrightarrow x^*(u)$ is discontinuous. Such values of u are called "catastrophe" points and are the focus of the recent catastrophe theory of Thom and Zeeman.

The catastrophe theory set-up is a special case of another type of stability concept, structural stability, in which one studies how changes in the system dynamics, themselves, influence the geometric character of the system trajectories. For example, consider the damped oscillator described by the equation

$$\ddot{u} + a\dot{u} + u = 0 \quad ,$$

$$u(0) = c \quad , \qquad \dot{u}(0) = 0 \quad , \qquad a \geq 0 \quad .$$

If $a > 0$, the phase-plane portrait of the trajectories is as in Figure 7a. For the undamped case $a = 0$, we have the situation depicted in Figure 7b.

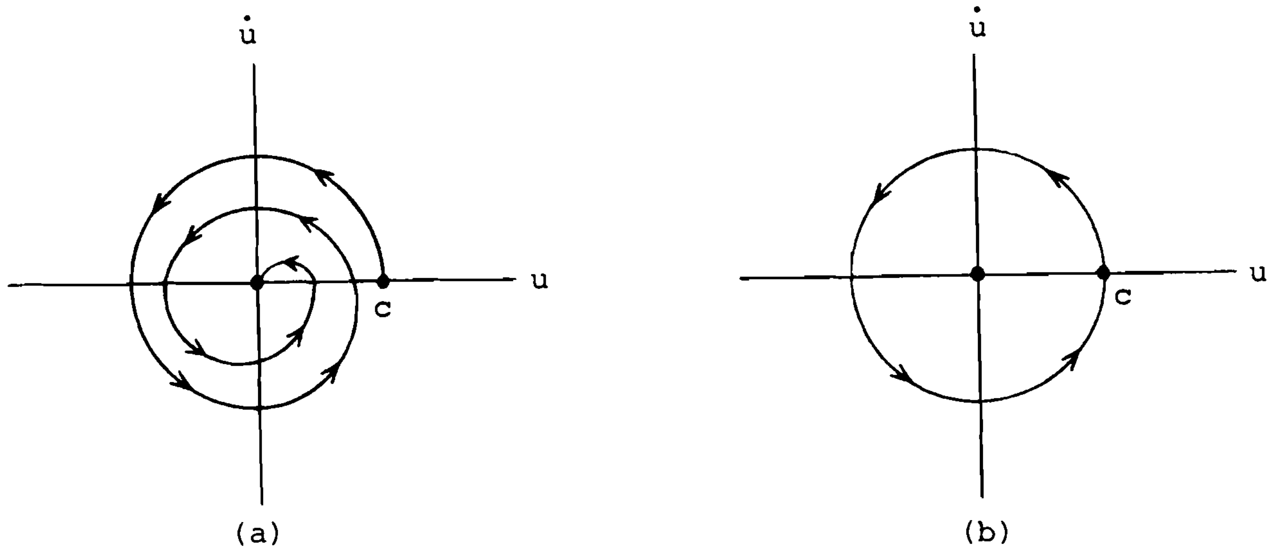

Figure 7. The Damped and Undamped Harmonic Oscillator

The equilibrium at the origin is of an entirely different topological character in the above two cases: in case (a), the origin is a focus, while in case (b) it is a center. Thus, the undamped harmonic oscillator is not structurally stable with respect to perturbations in the damping coefficient a, since any departure from a = 0 changes the character of the system trajectory. On the other hand, the damped oscillator is structurally stable with respect to changes in a, since for any a > 0, there is a nearby value of a such that the system trajectory is still a focus. Higher-dimensional generalizations of the above idea form the essence of multiparameter bifurcation theory, of which catastrophe theory is an important special case.

The most classical stability questions involve a system given in internal form

$$\dot{x} = f(x) \quad , \qquad x(0) = c \quad ,$$

where it is assumed that $f(0) = 0$, i.e., the origin is an equilibrium point. If the initial state $c \neq 0$, then it is of interest to know if the system state $x(t) \to 0$ as $t \to \infty$ and, if so, at what rate does the state approach the origin. These are stability problems in the sense of Lyapunov and many effective techniques exist for answering the above questions and many more.

From an applied systems analysis viewpoint, perhaps the most interesting aspect of classical stability is the determination of the domain of attraction of the origin, i.e., the determination of those initial states c which will eventually go to the origin. If the system dynamic f contains parameters, i.e., $f = f(x,u)$, then the variation of the boundary of the domain of attraction with changes in u brings us back to the catastrophe theory setting under appropriate hypotheses on the analytic structure of f.

If the basic system model is not of the internal type but, say, is a graph or simplicial complex, then the stability problems are of a somewhat different sort. For instance, consider the energy demand model characterized by the graph of Figure 8.

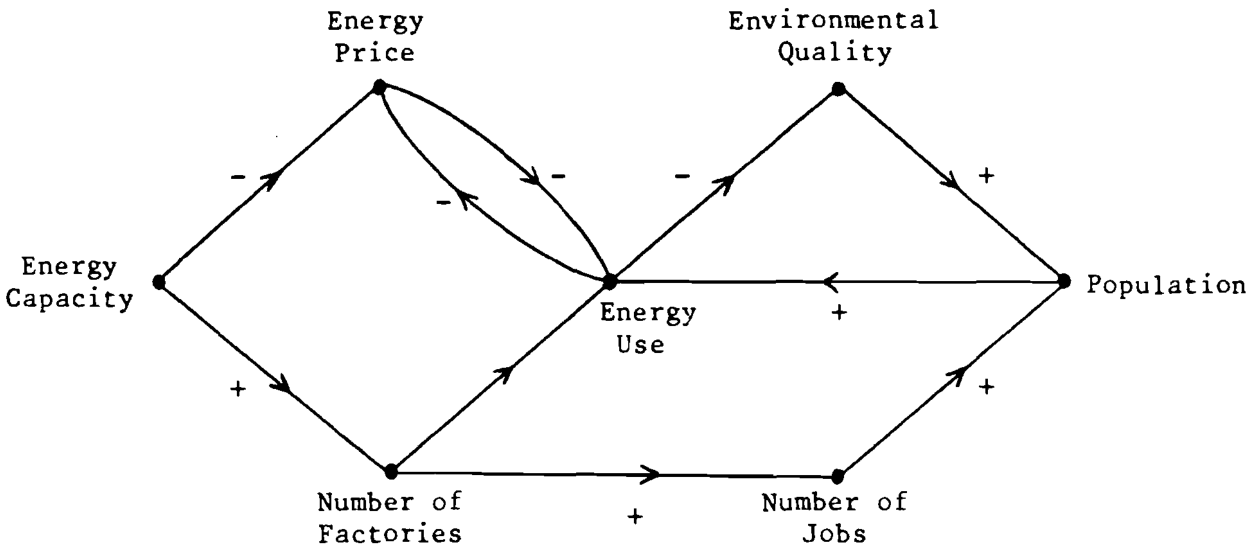

Figure 8. Graph Model of Energy Demand

Here a "+" on a directed arc from node i to j means that an increase in the value of variable i tends to increase the value of variable j, all other factors being held constant, while a "-" means an increase in i tends to reduce the level of j. This is an example of a <u>signed digraph</u>. A stability question of interest in connection with such a situation is whether a unit pulse introduced into the system at a given node (e.g., population) results in the value of any variable ultimately becoming unbounded. If not, then we say the system is <u>value stable</u>. A related concept, called <u>pulse stability</u> looks at basically the same question but with respect to the sequence of <u>changes</u> in values at a vertex from one time period to another. Both of these stability concepts can be attacked by algebraic means, utilizing the connection between the properties of a planar digraph and the properties of associated matrices.

In the more general case of a system described by a simplicial complex, the stability problems center upon changes in the connection pattern induced by perturbations of vertex values and/or changes in the defining relation λ. To illustrate, consider a pair of sets $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, with the defining relation $\lambda \subset Y \times X$ being characterized by the incidence matrix

$\Lambda =$

	x_1	x_2	x_3	x_4
y_1	1	1	0	0
y_2	0	0	0	1
y_3	0	0	0	0
y_4	0	0	0	1

Decomposing this complex into its dimensional components, we find that there are 3 distinct components at the 0-level, and one component at the 1-level. This is easily seen from the geometrical representation of the complex shown in Figure 9.

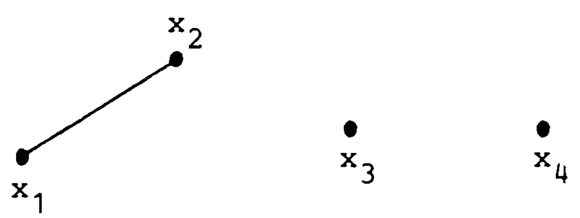

Figure 9. The Simplicial Complex of the Relation Λ

The first structure vector of this complex is then $Q = (1 \quad 3)$, indicating a low level of connection at the 0-dimensional level.

A stability problem that may arise in connection with a problem of this sort is whether or not the components of Q remain unchanged if we vary some elements in Λ. When stated in this form, it is also clear that the stability and reachability problems are related as we may wish to arrange the modifications in Λ to achieve the structure vector $Q = (1 \quad 1)$, which would indicate a more tightly connected system. Regrettably, a systematic methodology for answering this type of question remains to be developed.

VI. What Model to Choose?

In the preceding sections we have presented a number of alternative descriptions for modeling applied system processes and discussed a variety of basic questions which the models address. However, in the final analysis the modeler must choose one or another type of description, which then constrains the type of question with which he can effectively deal. As a guide to the selection of a particular system description, we present the Table below, in which the strengths and weaknesses of the model classes presented above are summarized. The reader should consider the Table only as a rough guide, since in any individual case, peculiarities of the problem may require a modeling approach departing from the general guidelines of the Table.

Model Type	Strengths	Weaknesses
External (Input/Output)	Deals only with observed data; does not require introduction of state variables, thereby reducing computational burden.	Provides no explanatory mechanism or prediction procedure; reachability/ observability questions hard to formulate.
Internal (State Variable)	Explicitly postulates a mechanism whereby inputs are transformed into outputs; highly developed mathematical theory for analyzing most basic system questions; not difficult to naturally incorporate global system constraints such as conservation laws, non-local effects, connectivity structures.	Requires detailed knowledge of dynamics and the way system inputs and outputs are processed; computational burden very high unless special structure (e.g., linearity) present; hard to model non-dynamical situations, e.g., art, music, game-playing, etc.
Potential Functions	Easy to synthesize local dynamics from global variational principle.	Difficult to justify in cases when no apparent variational principle exists; hard to formulate meaningful reachability/ observability problems.
Sets/Relations (Graphs)	Can be employed in very general settings; easy to analyze overall connection pattern between system components; readily accommodates hierarchical decomposition of system structure; computational aspects relatively straightforward.	Hard to incorporate dynamical effects in a natural way; provides little predictive power.
Operations Research (Mathematical Programming)	Can handle very large problems if sufficient structure (e.g., linearity) present; computational procedures well advanced; require relatively modest mathematical background to understand and employ.	Fails to distinguish between inputs, outputs, states; also, makes no distinction between open-loop and feedback control; no natural way to include stochastic/adaptive aspects; notions of reachability/ observability nonexistent.

Table 1. Relative Merits/Demerits of Model Types

NOTES AND REFERENCES

Section I

Extensive discussions of general systems principles, together with their implications for modeling are given in the works

Weinberg, G. (1975), An Introduction to General Systems Thinking, John Wiley, New York.

Vemuri, V. (1978), Modeling of Complex Systems: An Introduction, Academic Press, New York.

Roberts, P. C. (1978), Modelling Large Systems, Taylor and Francis, London.

The classic work

Ashby, W. R. (1956), An Introduction to Cybernetics, Chapman and Hall, London,

should also be consulted for many insightful examples carrying a strong information-theoretic flavor.

Section II

The sets/relation description of a system is due to R. H. Atkin and has been extensively pursued in his works

Atkin, R. H. (1973), Mathematical Structure in Human Affairs, Heinemann, London.

Atkin, R. H. (1977), Combinatorial Connectivities in the Social Sciences, Birkhäuser, Basel.

Atkin, R. H. (to appear 1980), Multidimensional Man, Penguin, London.

Input/output models are extensively utilized in economic modeling. For a survey of these efforts, see

Leontief, W. (1954), Mathematics in Economics, Bulletin Amer. Math. Soc., 60, 215-233.

Isard, W., and P. Kaniss (1973), The 1973 Nobel Prize for Economic Science, Science, 182, 568-569, 571.

Potential function descriptions have a long and venerable tradition in mathematical physics. For some interesting remarks and examples, see

Lanczos, C. (1966), The Variational Principles of Mechanics, 3rd ed., University of Toronto Press, Toronto.

Thompson, J. M. T., and G. Hunt (1973), A General Theory of Elasticity, John Wiley, London.

State variable descriptions are discussed in detail in

Casti, J. (1977), Dynamical Systems and Their Applications: Linear Theory, Academic Press, New York.

Kalman, R., P. Falb, and M. Arbib (1969), Topics in Mathematical System Theory, McGraw-Hill, New York.

Brockett, R. (1970), Finite-Dimensional Linear Systems, John Wiley, New York.

For a treatment of finite-state processes, see the Kalman, et al. book cited above, as well as

Ginsburg, S. (1962), An Introduction to Mathematical Machine Theory, Addison-Wesley, Reading, Mass.

Hartmanis, J., and R. Stearns (1966), The Algebraic Structure Theory of Sequential Machines, Prentice-Hall, Englewood-Cliffs, N.J.

Davis, M. (1953), Computability and Unsolvability, McGraw-Hill, New York.

Operations research models can be found in any number of outstanding texts and references on the subject. A random sampling includes

Dantzig, G. (1963), Linear Programming and Extensions, Princeton University Press, Princeton.

Vadja, S. (1961), Mathematical Programming, Addison-Wesley, Reading, Mass.

Bellman, R., and S. Dreyfus (1962), Applied Dynamic Programming, Princeton University Press, Princeton.

Section III

A general reference for the items covered in this section is

Casti, J. (1979), Connectivity, Complexity, and Catastrophe in Large-Scale Systems, John Wiley, London.

Interesting treatments of uncertainty and its inclusion into system models are given in

Kushner, H. (1971), Introduction to Stochastic Control, Holt, Rinehart and Winston, New York.

Kickert, W. J. M. (1978), Fuzzy Theories on Decision-Making, Martinus Nijhoff, Leiden.

Haken, H. (1978), Synergetics, Springer, Berlin.

The presence of time-lags in the system dynamics can have a pronounced effect upon all qualitative aspects of the temporal behavior. For a fairly complete account of the types of effects, see

El'sgol'ts, L., and S. Norkin (1973), Introduction to the Theory and Application of Differential Equations With Deviating Arguments, Academic Press, New York.

Kirillova, F., and R. Gabasov (1977), The Qualitative Theory of Optimal Processes, Dekker, New York.

Section IV

Ashby's Law of Requisite Variety is quite closely related to the information-theoretic ideas of Shannon. For a detailed study, with many examples, see the Ashby book cited under Section I, as well as the paper

Porter, B. (1976), Requisite Variety in the Systems and Control Sciences, Int. J. Gen. Syst., 2, 225-229.

Questions of hierarchical structure are surveyed in

Pattee, H., ed. (1973), Hierarchy Theory, Braziller, New York.

Some of the connections between hierarchical structure and the concept of system complexity are explored in the classic paper

Simon, H. (1962), The Architecture of Complexity, Proc. Amer. Philos. Soc., 106, 467-482.

The relationship between system singularities and the behavior they force upon a system are detailed in

Post, E. J. (1967), Geometry and Physics, in M. Bunge, ed., Foundations of Physics, Springer, Berlin and New York.

Some related ideas outside of mathematical physics are given in

Thom, R. (1975), Structural Stability and Morphogenesis, Addison-Wesley, Reading, Mass.

Section V

A recent survey of reachability/observability questions and methods is

Casti, J. (1979), Illusion or Reality? The Mathematics of Attainable Goals and Irreducible Uncertainties, Int. J. Gen. Syst., 5, 75-92.

Other recent results are presented in

Hermann, R., and A. Krener (1977), Nonlinear Controllability and Observability, IEEE Trans. Auto. Control, AC-22, 728-740.

Mayne, D., and R. Brockett, eds. (1973), Geometric Methods in System Theory, Reidel, Dordrecht.

Brockett, R. (1976), Nonlinear Systems and Differential Geometry, Proc. IEEE, 64, 61-72.

Realization theory, together with its close relative identification theory, is treated in

Mehra, R., and D. Lainiotis, eds. (1976), System Identification: Advances and Case Studies, Academic Press, New York.

Rajbman, N. (1976), The Application of Identification Methods in the USSR: A Survey, Automatica, 12, 73-95.

Åström, K., and P. Eyekhoff (1971), System Identification — A Survey, Automatica, 7, 123-162.

Optimal control theory has been the subject of many textbooks, research works and surveys. For a representative treatment of the main approaches, we recommend the classic works

Bellman, R. (1957), Dynamic Programming, Princeton University Press, Princeton.

Pontryagin, L., et al. (1961), The Mathematical Theory of Optimal Processes, Interscience, New York.

More up-to-date textbook treatments are

Bryson, A., and Y. C. Ho (1969), Applied Optimal Control, Blaisdell, Waltham, Mass.

Barnett, S. (1975), Introduction to Mathematical Control Theory, Clarendon Press, Oxford.

Stability and sensitivity theory are well covered in the following works.

Hahn, W. (1967), Stability of Motion, Springer, New York.

La Salle, J., and S. Lefschetz (1961), Stability by Lyapunov's Direct Method With Applications, Academic Press, New York.

Hirsch, M., and S. Smale (1974), Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, New York.

Siljak, D. (1978), Large-Scale Dynamic Systems: Stability and Structure, North-Holland, New York.

Poston, T., and I. Stewart (1978), Catastrophe Theory and Its Applications, Pitman, London.

Zeeman, E. C. (1977), Catastrophe Theory: Selected Papers 1972-77, Addison-Wesley, Reading, Mass.