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Optimal Growth, Renewable Resources and Sustainability

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Abstract

We study a growth model for a single resource-based economy, as an infinite-horizon optimal control problem. The resource is assumed to be governed by the standard model of logistic growth, and is related to the output of the economy through a Cobb-Douglass type production function with an exogenously driven knowledge stock. The problem involves unbounded controls and the non-concave Hamiltonian. These preclude direct application of the standard existence results and Arrow's sufficient conditions for optimality. We transform the original optimal control problem to an equivalent one with simplified dynamics and prove the existence of an optimal admissible control. Then we characterize the optimal paths for all possible parameter values and initial states by applying the appropriate version of the Pontryagin maximum principle. Our main finding is that only two qualitatively different types of behavior of sustainable optimal paths are possible depending on whether the resource growth rate is higher than the social discount rate or not.

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1 Introduction

Following the first analysis conducted by Ramsey [25], the mathematical problem of intertemporal resource allocation has attracted a significant amount of attention over the past decades, and has driven the evolution of first exogenous, and then endogenous growth theory (see [1, 13]). Endogenous growth models are typically identified by the production of economic output, the dynamics of the inputs of production, and the comparative mechanism of alternate consumption paths. Our framework considers a renewable resource, whose reproduction is logistic in nature, as the only input to production. The relationship of the resource with the output of the economy is explained through a Cobb-Douglas type production function with an exogenously driven knowledge stock. Alternate consumption paths are compared via a discounted utilitarian approach, with a logarithmic welfare function used to maximize economic growth. The question that we concern ourselves with for our chosen framework, is the following: what are the conditions of sustainability for optimal development?

Discounted utilitarianism has faced much criticism over the years. Perhaps one of the most famous critics of discounting is Ramsey himself, who described it as "ethically indefensible" [25]. Various alternatives to the discounted utilitarian approach have also been introduced, with some deviating marginally (e.g., hyperbolic discounting [19]), and other deviating more drastically (see for instance [17, 30]) from the original model. However, discounted utilitarianism has continued to be widely adopted by many researchers for both mathematical [19] and philosophical reasons [27]. In particular, discounted utilitarianism remains relevant because of the time-consistency, and also as a benchmark to evaluate alternative frameworks.

In the context of sustainability, the discounted utilitarian approach may propose undesirable solutions in certain scenarios. For instance, discounted utilitarianism has been reported to force consumption asymptotically to zero even when sustainable paths with non-decreasing consumption are feasible [11]. The Brundtland Commission defines sustainable development as development that meets the needs of the present, without compromising the ability of future generations to meet their own needs [14]. In this spirit, we employ the notion of sustainable development, as a consumption path ensuring a non-decreasing welfare for all future generations. This notion of sustainability is natural, and has also been used by various authors in their work. For instance, Valente [29] evaluates this notion of sustainability for an exponentially growing natural resource, and derives a condition necessary for sustainable consumption, namely that the rate of social discount must not exceed the sum of the resource growth rate and rate of resource-augmentation. We extend this model by allowing the resource to grow at a declining rate (the logistic growth model). We build on the work presented previously in [22] which proves the existence of an optimal path only in the case when the resource growth rate is higher than the social discount rate and admissible controls are uniformly bounded. We also introduce an exogenously driven knowledge stock in the model that makes the results more interpretable.

Our model is formulated as an infinite-horizon optimal control problem with logarithmic instantaneous utility. The model involves unbounded controls and the non-concave Hamiltonian. These preclude direct application of the standard existence results and Arrow's sufficient conditions for optimality. After establishing a precise notion of optimality, we transform the original optimal control problem to an equivalent one with simplified dynamics and prove the general existence result. Then we apply a recently developed version of the maximum principle [8–10] to our problem and describe the optimal paths for all possible parameter values and initial states in the problem. This allows us to establish a criterion of sustainability for the optimal paths which expands Valente's necessary condition for sustainable consumption [29].

Our analysis of the Hamiltonian phase space reveals that there are only two qualitatively different types of behavior of the sustainable optimal paths in the model. In the first case the instantaneous utility is a non-decreasing function in the long run along the optimal path (we call such processes *sustainable*). The second case corresponds to the situation when the optimal path is sustainable and in addition the resource stock is asymptotically nonvanishing (we call such processes *strongly sustainable*). We show that a strongly sustainable equilibrium is attainable only when the resource growth rate is higher than the social discount rate independent of the growth rate of the knowledge stock. This prescribes policy measures to increase resource growth and decrease social discount i.e., plan long term. When the condition for strong sustainable equilibrium is met, we obtain an optimal feedback law which steers the system asymptotically to the sustainable equilibrium i.e., a positive optimal consumption that can to be maintained indefinitely. When this condition is violated, we see that the optimal resource exploitation rate asymptotically follows the Hotelling rule of optimal depletion of an exhaustible resource [21]. In this case the sustainable consumption is possible only if the the depletion of the resource is compensated by an appropriate growth of the knowledge stock.

The paper is organized as follows. Section 2 sets up the problem and defines a notion of optimality. Section 3 establishes the equivalence of the problem with a simpler version, and applies the maximum principle after proving the existence of an optimal control. Section 4 presents an analysis of the Hamiltonian system of the maximum principle and formulates the optimal feedback law. Here we also present numerical simulations of the solution both when the condition of sustainable equilibrium is fulfilled and violated. We conclude in Section 5 where we develop conditions for sustainability and strong sustainability of the optimal paths in our model.

2 Problem formulation

Consider a society consuming a single renewable resource. The resource, whose quantity is given by $S(t) > 0$ at each instant of time $t \geq 0$, is governed by the standard model of logistic growth. In the absence of consumption, it regenerates at rate $r > 0$ and saturates at carrying capacity $K > 0$. The society consumes the resource by exerting effort (exploitation rate) $u(t) > 0$ resulting in a total consumption velocity of $u(t)S(t) > 0$ at time $t \geq 0$ respectively. The dynamics of the resource stock is then given by the following equation:

$$\dot{S}(t) = r\,S(t)\left(1 - \frac{S(t)}{K}\right) - u(t)S(t), \qquad u(t) \in (0, \infty).$$

The initial stock of the resource is $S(0) = S_0 > 0$.

In that follows we will treat the function $t \mapsto u(t)$, $t \geq 0$, as a control in our model. Notice that we do not assume any a priori upper bound for the values of control $u(\cdot)$ here. Consideration of such models with unbounded sets of control constraints is motivated by the fact that in this case the values $u_*(t)$, $t \geq 0$, of an optimal control $u_*(\cdot)$ (if such exists) could be used by a fictitious social planner for establishing of intertemporal quotas of the resource consumption.

We assume a single resource economy whose output $Y(t) > 0$ at instant $t \geq 0$ is related to the resource by the Cobb-Douglas type production function[1]

$$Y(t) = A(t)\big(u(t)S(t)\big)^{\alpha}, \qquad \alpha \in (0, 1]. \tag{1}$$

Here $A(t) > 0$ represents an exogenously driven knowledge stock at time $t \geq 0$. We assume that the knowledge stock $A(\cdot)$ grows not faster than exponentially, i.e. $\dot{A}(t) \leq \mu A(t)$, where $\mu \geq 0$ is a constant, and $A(0) = A_0 > 0$.

The whole output $Y(t)$ produced at each instant $t \geq 0$ is consumed and the welfare is measured by the aggregate discounted logarithmic utility function, maximizing which amounts to maximizing aggregate discounted future growth rates of consumption [5]. This leads to the following objective functional for the economy

$$\begin{aligned} \tilde{J}(S(\cdot), u(\cdot)) &= \int_0^{\infty} e^{-\rho t} \ln Y(t)\, dt \\ &= \int_0^{\infty} e^{-\rho t} \ln A(t)\, dt + \alpha \int_0^{\infty} e^{-\rho t} \left[\ln u(t) + \ln S(t)\right] dt, \end{aligned} \tag{2}$$

where $\rho > 0$ is a subjective discount rate.

Neglecting constant term $\int_0^{\infty} e^{-\rho t} \ln A(t)\, dt$ and positive scalar multiplier α our problem of optimal growth is thus set up as the following optimal control problem $(P1)$:

$$J(S(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt \to \max, \tag{3}$$

$$\dot{S}(t) = rS(t)\left(1 - \frac{S(t)}{K}\right) - u(t)S(t), \qquad S(0) = S_0, \tag{4}$$

$$u(t) \in (0, \infty). \tag{5}$$

Notice that the problem $(P1)$ does not depend at all on the knowledge stock $A(\cdot)$ and the elasticity parameter α.

[1] The case $\alpha < 1$ of diminishing returns to scale in production seems to be the most realistic. However, we retain the knife-edge case $\alpha = 1$ for completeness of the presentation.

By an *admissible control* in problem $(P1)$ we mean a Lebesgue measurable locally bounded function $u\colon [0,\infty) \mapsto \mathbb{R}^1$ which satisfies the control constraint (5) for all $t \geq 0$. As usual, the local boundedness of function $u(\cdot)$ means that $u(\cdot)$ is bounded on any finite time interval $[0,T]$, i.e. for arbitrary $T > 0$ there exists a constant $M_T \geq 0$ such that $|u(t)| \leq M_T$ for all $t \in [0,T]$. By definition, the corresponding to $u(\cdot)$ *admissible trajectory* is a (locally) absolutely continuous function $S(\cdot) : [0,\infty) \mapsto \mathbb{R}^1$ which is a Caratheodory solution (see [18]) to the Cauchy problem (4) on the whole infinite time interval $[0,\infty)$. Due to the local boundedness of the admissible control $u(\cdot)$ such admissible trajectory $S(\cdot)$ always exists and is unique (see [18, § 7]). A pair $(S(\cdot), u(\cdot))$ where $S(\cdot)$ is an admissible control and $S(\cdot)$ is the corresponding admissible trajectory is called an *admissible pair* or a *process* in problem $(P1)$. Due to (4) for any admissible trajectory $S(\cdot)$ the following estimate holds:

$$S(t) \leq S_{\max} = \max\{S_0, K\}, \qquad t \geq 0. \tag{6}$$

The integral in (3) is understood in improper sense, i.e.

$$J(S(\cdot), u(\cdot)) = \lim_{T\to\infty} \int_0^T \mathrm{e}^{-\rho t} \left[\ln S(t) + \ln u(t)\right] \, dt \tag{7}$$

if the limit exists.

To demonstrate that for any admissible pair $(S(\cdot), u(\cdot))$ the limit in (7) always exists we need the following auxiliary statement.

Lemma 1. *There is a decreasing function $\omega : [0,\infty) \mapsto (0,\infty)$ such that $\omega(t) \to +0$ as $t \to \infty$ and for any admissible pair $(S(\cdot), u(\cdot))$ the following inequality holds:*

$$\int_T^{T'} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] \, \mathrm{d}t < \omega(T), \qquad 0 \leq T < T'.$$

Proof. Indeed, due to inequality $\ln x < x$, $x > 0$, for arbitrary $0 \leq T < T'$ we have

$$\int_T^{T'} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] \, dt < \int_T^{T'} e^{-\rho t} u(t) S(t) \, dt.$$

Hence, substituting expression of $u(t)S(t)$ from (4) in the inequality above we get

$$\int_T^{T'} e^{-\rho t} \left[\ln S(t)) + \ln u(t)\right] \, dt < \int_T^{T'} e^{-\rho t} \left[r S(t) \left(1 - \frac{S(t)}{K}\right) - \dot{S}(t)\right] \, dt.$$

This implies

$$\begin{aligned}
\int_T^{T'} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] \, dt &< \int_T^{T'} e^{-\rho t} \left[r S(t) - \dot{S}(t)\right] \, dt \\
&\leq r S_{\max} \int_T^{T'} e^{-\rho t} \, dt - \int_T^{T'} e^{-\rho t} \dot{S}(t) \, dt \\
&= \frac{r S_{\max}}{\rho} \left(e^{-\rho T} - e^{-\rho T'}\right) - e^{-\rho t} S(t) \Big|_T^{T'} - \rho \int_T^{T'} e^{-\rho t} S(t) \, dt \\
&< \frac{r S_{\max}}{\rho} e^{-\rho T} + e^{-\rho T} S_{\max} = \frac{(r+\rho) S_{\max}}{\rho} e^{-\rho T} = \omega(T).
\end{aligned}$$

□

Now, let us show that for any admissible pair $(S(\cdot), u(\cdot))$ the limit in (7) exists.

Lemma 2. *For any admissible pair* $(S(\cdot), u(\cdot))$ *the limit in* (7) *exists and is either finite or equals* $-\infty$.

Proof. Let $(S(\cdot), u(\cdot))$ be an arbitrary admissible pair. For any $T > 0$ define $J_T(S(\cdot), u(\cdot))$ as follows:

$$J_T(S(\cdot), u(\cdot)) = \int_0^T \mathrm{e}^{-\rho t} \left[\ln S(t) + \ln u(t))\right] dt.$$

Let $\{\zeta_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that $\zeta_i \to \infty$ as $i \to \infty$ and

$$\lim_{i\to\infty} J_{\zeta_i}(S(\cdot), u(\cdot)) = \limsup_{T\to\infty} \int_0^T e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt.$$

Due to Lemma 1 we have the following estimate

$$\lim_{i\to\infty} J_{\zeta_i}(S(\cdot), u(\cdot)) \le \omega(0). \tag{8}$$

Analogously, let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that $\tau_i \to \infty$ as $i \to \infty$ and

$$\lim_{i\to\infty} J_{\tau_i}(S(\cdot), u(\cdot)) = \liminf_{T\to\infty} \int_0^T e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt.$$

Without loss of generality one can assume that $\tau_i < \zeta_i$, $i = 1, 2, \ldots$. Then we have

$$J_{\zeta_i}(S(\cdot), u(\cdot)) = J_{\tau_i}(S(\cdot), u(\cdot)) + \int_{\tau_i}^{\zeta_i} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt, \quad i = 1, 2, \ldots.$$

Due to Lemma 1 this implies

$$J_{\zeta_i}(S(\cdot), u(\cdot)) < J_{\tau_i}(S(\cdot), u(\cdot)) + \omega(\tau_i), \quad i = 1, 2, \ldots.$$

Since $\omega(\tau_i) \to 0$ as $i \to \infty$ taking the limit in the last inequality as $i \to \infty$ we get

$$\limsup_{T\to\infty} \int_0^T \mathrm{e}^{-\rho t} \left[\ln S(t) + \ln u(t))\right] dt \le \liminf_{T\to\infty} \int_0^T \mathrm{e}^{-\rho t} \left[\ln S(t) + \ln u(t))\right] dt.$$

As far as the opposite inequality

$$\liminf_{T\to\infty} \int_0^T \mathrm{e}^{-\rho t} \left[\ln S(t) + \ln u(t))\right] dt \le \limsup_{T\to\infty} \int_0^T \mathrm{e}^{-\rho t} \left[\ln S(t) + \ln u(t))\right] dt$$

is always true, the limit (7) exists, and due to (8) this limit is either finite or $-\infty$. $\square$

Since for any admissible pair $(S(\cdot), u(\cdot))$ in $(P1)$ we have $J(S(\cdot), u(\cdot)) \le \omega(0)$ (see (8)) and there is an admissible pair $(\hat{S}(\cdot), \hat{u}(\cdot))$ such that $J(\hat{S}(\cdot), \hat{u}(\cdot)) > -\infty$ the value $\sup_{(S(\cdot),u(\cdot))} J(S(\cdot), u(\cdot))$ is finite. This fact allows us to understand the optimality of an admissible pair $(S_*(\cdot), u_*(\cdot))$ in problem $(P1)$ in the strong sense [15]. By definition, an admissible pair $(S_*(\cdot), u_*(\cdot))$ is *strongly optimal* (or, for brevity, simply *optimal*) in the problem $(P1)$ if the functional (3) takes the maximal possible value on this pair , i.e.

$$J(S_*(\cdot), u_*(\cdot)) = \sup_{(S(\cdot),u(\cdot))} J(S(\cdot), u(\cdot)) < \infty.$$

Here the supremum is taken over all admissible pairs $(S(\cdot), u(\cdot))$ in problem $(P1)$.

Note that the formulated infinite-horizon problem $(P1)$ possesses some important features that hamper application of standard results of the optimal control theory. In particular, the set of control constraints in problem $(P1)$ (see (5)) is nonclosed and unbounded. Due to this circumstance the standard existence theorems (see e.g. [12,16]) are not applicable to problem $(P1)$ directly. Moreover, the situation is complicated here by the fact that the Hamiltonian of problem $(P1)$ is non-concave in the state variable S. These preclude the usage of Arrow's sufficient conditions for optimality (see [15]).

Our analysis below is based on application of the recently developed existence result [3, 4] and the normal form version of the Pontryagin maximum principle [24] for infinite-horizon optimal control problems with adjoint variable specified explicitly via the Cauchy type formula (see [2, 5–10]). This formula completes relations of the maximum principle that gives us a possibility to characterize the optimal processes in problem $(P1)$ uniquely for all possible parameter values and all initial states. However, the correct application of the maximum principle assumes that the optimal control exists. Without such an existence result there is not any guarantee that one of the admissible pairs satisfying the necessary conditions will be a solution (see the corresponding discussion in [26]).

So, the proof of the existence of an optimal admissible pair $(S_*(\cdot), u_*(\cdot))$ in problem $(P1)$ will be our primary goal in the next section. We will show also that the optimal admissible pair $(S_*(\cdot), u_*(\cdot))$ (which exists) satisfies the conditions of the appropriate version of the maximum principle [8–10].

3 Existence of an optimal control and the maximum principle

To prove the existence of an optimal admissible control in problem $(P1)$ let us transform it into a more appropriate equivalent form. Recall, that the class of admissible controls in problem $(P1)$ consists of all locally bounded measurable functions $u\colon [0,\infty) \mapsto (0,\infty)$.

Due to (4) along any admissible pair $(S(\cdot), u(\cdot))$ we have

$$\frac{d}{dt}\left[e^{-\rho t}\ln S(t)\right] \stackrel{\text{a.e.}}{=} -\rho e^{-\rho t}\ln S(t) + re^{-\rho t} - e^{-\rho t}\left(\frac{r}{K}S(t) + u(t)\right), \quad t > 0.$$

Integrating this equality on arbitrary time interval $[0,T]$, $T > 0$, we obtain

$$e^{-\rho T}\ln S(T) - \ln S_0 = -\rho\int_0^T e^{-\rho t}\ln S(t)\,dt + r\int_0^T e^{-\rho t}\,dt - \int_0^T e^{-\rho t}\left(\frac{r}{K}S(t) + u(t)\right)dt$$

or, equivalently,

$$\int_0^T e^{-\rho t}\ln S(t)\,dt = \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \frac{r}{\rho^2}\left(1 - e^{-\rho T}\right) - \int_0^T e^{-\rho t}\left(\frac{r}{\rho K}S(t) + \frac{u(t)}{\rho}\right)dt.$$

Hence, for any admissible pair $(S(\cdot), u(\cdot))$ and arbitrary $T > 0$ we have

$$\begin{aligned} \int_0^T e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt = \frac{\ln S_0 - e^{-\rho T} \ln S(T)}{\rho} + \frac{r}{\rho^2}\left(1 - e^{-\rho T}\right) \\ - \frac{r}{\rho K} \int_0^T e^{-\rho t} S(t)\, dt + \int_0^T e^{-\rho t} \left(\ln u(t) - \frac{u(t)}{\rho}\right) dt. \end{aligned} \tag{9}$$

Due to Lemma 2 the limits of the left-hand and the right-hand sides in (9) as $T \to \infty$ exist and equal either a finite number or $-\infty$ simultaneously.

Further, due to (6) we have $\limsup_{T\to\infty} e^{-\rho T} \ln S(T) \le 0$. Hence, two cases are possible: either (i) $\liminf_{T\to\infty} e^{-\rho T} \ln S(T) = 0$ or (ii) $\liminf_{T\to\infty} e^{-\rho T} \ln S(T) < 0$.

Consider case (i). In this case $\lim_{T\to\infty} e^{-\rho T} \ln S(T) = 0$. Hence, passing to the limit in (9) as $T \to \infty$ we get

$$\begin{aligned} \int_0^\infty e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt = \frac{\ln S_0}{\rho} + \frac{r}{\rho^2} \\ - \frac{r}{\rho K} \int_0^\infty e^{-\rho t} S(t)\, dt + \int_0^\infty e^{-\rho t} \left(\ln u(t) - \frac{u(t)}{\rho}\right) dt, \end{aligned} \tag{10}$$

where both the left-hand and the right-hand sides in (10) are equal to a finite number or $-\infty$ simultaneously.

Consider case (ii). In this case there are a sequence of positive numbers $\{T_i\}_{i=1}^\infty$, $\lim_{i\to\infty} T_i = \infty$, and an $\varepsilon > 0$ such that

$$\lim_{i\to\infty} e^{-\rho T_i} \ln S(T_i) \le -\varepsilon < 0. \tag{11}$$

In this situation let us consider initial state $S_0 > 0$ and carrying capacity $K > 0$ as parameters in problem $(P1)$, and define the optimal value function $V(\cdot,\cdot)$ of two variables $S_0 > 0$ and $K > 0$ as follows:

$$V(S_0, K) = \sup_{u(\cdot)} \int_0^\infty e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt, \qquad S_0 > 0, \quad K > 0.$$

Here, the supremum is taken over all admissible controls $u(\cdot)$ in $(P1)$.

The following (uniform in $K > 0$) estimate holds true:

$$V(S_0, K) < S_0 + \frac{r}{\rho^2} \qquad S_0 > 0, \quad K > 0. \tag{12}$$

Indeed, introducing the auxiliary variable $y(t) = e^{-rt} S(t)$, $t \ge 0$, due to (4) we get

$$\dot{y}(t) \overset{\text{a.e.}}{\le} -u(t) y(t), \qquad y(0) = S_0.$$

Hence,

$$\begin{aligned} V(S_0, K) &= \sup_{u(\cdot)} \int_0^\infty e^{-\rho t} \left[rt + \ln(u(t) y(t))\right] dt \\ &< \sup_{u(\cdot)} \int_0^\infty e^{-\rho t} u(t) y(t)\, dt + \frac{r}{\rho^2} \le \sup_{u(\cdot)} \left[-\int_0^\infty e^{-\rho t} \dot{y}(t)\, dt\right] + \frac{r}{\rho^2} \le S_0 + \frac{r}{\rho^2}. \end{aligned}$$

Thus, estimate (12) is proved.

Further, representing $S(t) = S_0\tilde{S}(t)$, $t \geq 0$, we get

$$\tilde{S}(t) \overset{\text{a.e.}}{=} r\tilde{S}(t)\left(1 - \frac{S_0\tilde{S}(t)}{K}\right) - u(t)\tilde{S}(t), \quad t \geq 0.$$

As far as $\tilde{S}(0) = 1$ estimate (12) implies the following relations:

$$V(S_0, K) = \frac{\ln S_0}{\rho} + V(1, \frac{K}{S_0}) < \frac{\ln S_0}{\rho} + 1 + \frac{r}{\rho^2}.$$

Hence, for any admissible pair $(S(\cdot), u(\cdot))$ and arbitrary $T > 0$ we have

$$\begin{aligned}\int_T^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt &= e^{-\rho T}\int_0^\infty e^{-\rho t}\left[\ln S(t+T) + \ln u(t+T)\right] dt \\ &\leq e^{-\rho T}V(S(T), K) < e^{-\rho T}\left[\frac{\ln S(T)}{\rho} + 1 + \frac{r}{\rho^2}\right]. \qquad (13)\end{aligned}$$

Since the limit in the left hand-side of (9) as $T \to \infty$ always exists and equals either a finite number or $-\infty$ inequalities (11) and (13) imply

$$\lim_{i\to\infty}\int_{T_i}^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt \leq -\varepsilon < 0. \qquad (14)$$

If the integral $\int_0^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt$ converges (to a finite number) then for any $i = 1, 2, \ldots$ we have

$$\int_0^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt = \int_0^{T_i} e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt + \int_{T_i}^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt.$$

In this case due to (14) passing to the limit in the last inequality as $i \to \infty$ we get

$$\begin{aligned}\int_0^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt &\leq \int_0^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt - \varepsilon \\ &< \int_0^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt.\end{aligned}$$

But this inequality is contradictive. Hence, in this case

$$\int_0^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt = -\infty,$$

and due to (9)

$$\lim_{T\to\infty}\int_0^T e^{-\rho t}\left(\ln u(t) - \frac{u(t)}{\rho}\right) dt = -\infty.$$

Hence, if $\liminf_{T\to\infty} e^{-\rho T}\ln S(T) < 0$ then equality (10) holds as $-\infty = -\infty$.

If the integral $\int_0^\infty e^{-\rho t}\left[\ln S(t) + \ln u(t)\right]\,dt$ diverges to $-\infty$ then due to (9)

$$\lim_{T\to\infty}\int_0^T e^{-\rho t}\left(\ln u(t) - \frac{u(t)}{\rho}\right)\,dt = -\infty.$$

and equality (10) also holds as $-\infty = -\infty$.

Thus, equality (10) holds in the general case.

Neglecting now the constant terms in the right-hand side of (10) we obtain the following optimal control problem $(\tilde{P}1)$ which is equivalent to $(P1)$:

$$\tilde{J}(S(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t}\left[\ln u(t) - \frac{u(t)}{\rho} - \frac{r}{\rho K}S(t)\right]\,dt \to \max,$$

$$\dot{S}(t) = rS(t)\left(1 - \frac{S(t)}{K}\right) - u(t)S(t), \qquad S(0) = S_0, \tag{15}$$

$$u(t) \in (0, \infty). \tag{16}$$

The class of admissible controls in problem $(\tilde{P}1)$ is the same as in $(P1)$. It consists of all locally bounded measurable functions $u\colon [0,\infty) \mapsto (0,\infty)$.

It is easy to see that if $u_1(\cdot)$ and $u_2(\cdot)$ are admissible controls in $(\tilde{P}1)$ such that $u_1(t) \overset{\text{a.e.}}{\geq} u_2(t)$, $t \geq 0$, then by virtue of (15) the opposite inequality $S_1(t) \leq S_2(t)$, $t \geq 0$, holds for the corresponding trajectories $S_1(\cdot)$ and $S_2(\cdot)$. Further, $u \mapsto ln\, u - u/\rho$ is an increasing function on $(0, \rho]$ and it reaches the global maximum on $(0,\infty)$ at point $u_* = \rho$. Hence, any optimal control $u_*(\cdot)$ in $(\tilde{P}1)$ (if such exists) must satisfy to inequality $u_*(t) \geq \rho$ for almost all $t \geq 0$.

Thus, without loss of generality the control constraint (16) in $(\tilde{P}1)$ (and hence the control constraint (5) in $(P1)$) can be replaced by the control constraint $u(t) \in [\rho, \infty)$. Thus we arrive to the following optimal control problem $(P2)$:

$$J(S(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t}\left[\ln u(t) + \ln S(t)\right]\,dt \to \max,$$

$$\dot{S}(t) = rS(t)\left(1 - \frac{S(t)}{K}\right) - u(t)S(t), \qquad S(0) = S_0,$$

$$u(t) \in [\rho, \infty). \tag{17}$$

Here the class of admissible controls in problem $(P2)$ consists of all locally bounded functions $u(\cdot)$ satisfying the control constraint (17) for all $t \geq 0$.

As it is shown above the problem $(P2)$ is equivalent to problem $(P1)$ in the following sense.

Lemma 3. *The sets of optimal admissible pairs $(S_*(\cdot), u_*(\cdot))$ in problems $(P1)$ and $(P2)$ coincide.*

Note that Lemma 3 gives the lower bound for an optimal admissible control $u_*(\cdot)$ in $(P1)$ which corresponds to the Hotelling rule [21], i.e. $u_*(t) \geq \rho$ for a.e. $t \geq 0$, and this bound does not depend on the regeneration rate $r > 0$.

Now, to simplify dynamics in problem $(P2)$ let us introduce the new state variable $x(\cdot)$ as follows:

$$x(t) = \frac{1}{S(t)}, \qquad t \geq 0.$$

As it can be verified directly, in terms of the state variable $x(\cdot)$ problem $(P2)$ can be rewritten as the following (equivalent) problem $(P3)$:

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t} \left[\ln u(t) - \ln x(t)\right] \, dt \to \max, \tag{18}$$

$$\dot{x}(t) = [u(t) - r] \, x(t) + a, \qquad x(0) = x_0 = \frac{1}{S_0}, \tag{19}$$

$$u(t) \in [\rho, \infty). \tag{20}$$

Here $a = r/K$. The class of admissible controls $u(\cdot)$ in problem $(P3)$ is the same as in $(P2)$. It consists of all measurable locally bounded functions $u\colon [0, \infty) \mapsto [\rho, \infty)$.

Due to Lemma 3 all three problems $(P1)$, $(P2)$ and $(P3)$ are equivalent. Thus, in that follows we will focus our analysis on problem $(P3)$ with simplified (linear in x) dynamics (see (19)) and the closed set of control constraints (see (20)). Notice, that because of the non-concavity of the instantaneous utility in (18) in the state variable x the Hamiltonian of problem $(P3)$ is also non-concave in x.

Due to the linearity of system (19) for arbitrary admissible control $u(\cdot)$ the corresponding admissible trajectory $x(\cdot)$ can be expressed via the Cauchy formula [20] as follows:

$$x(t) = x_0 e^{\int_0^t u(\xi)\, d\xi - rt} + a e^{\int_0^t u(\xi)\, d\xi - rt} \int_0^t e^{-\int_0^s u(\xi)\, d\xi + rs} \, ds, \qquad t \geq 0. \tag{21}$$

The constructed problem $(P3)$ is a particular case of the following autonomous infinite-horizon optimal control problem $(P4)$ with exponential discounting:

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t} g(x(t), u(t)) \, dt \to \max,$$

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0, \tag{22}$$

$$u(t) \in U.$$

Here U is a nonempty closed subset of $\mathbb{R}^m$, $x_0 \in G$ is an initial state which belongs to a given open convex subset G of $\mathbb{R}^n$, and $f : G \times U \mapsto \mathbb{R}^n$ and $g : G \times U \mapsto \mathbb{R}^m$ are also given functions. The class of admissible controls in $(P4)$ consists of all measurable locally bounded functions $u\colon [0, \infty) \mapsto U$. The optimality of admissible pair $(x_*(\cdot), u_*(\cdot))$ is understood in the strong sense [15].

Problems of type $(P4)$ arise in many fields of economics. Such problems were intensively studied in last decades (see [2–10]). In this paper we will use the existence result and the variant of the Pontryagin maximum principle for problem $(P4)$ developed recently in [3, 4] and [8–10] respectively. For application of these results we need to verify the following three conditions.

The first condition characterize regularity of functions $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ (see [8–10]).

(A1) *The functions* $f(\cdot,\cdot)$ *and* $g(\cdot,\cdot)$ *together with their partial derivatives* $f_x(\cdot,\cdot)$ *and* $g_x(\cdot,\cdot)$ *are continuous and locally bounded on* $G \times U$.

The second condition characterizes the growth of the instantaneous utility $g(\cdot,\cdot)$ in some "neighborhood" of an admissible (not necessary optimal) pair $(x_*(\cdot), u_*(\cdot))$ (see [8–10]).

(A2) *There exist a number* $\beta > 0$ *and a nonnegative integrable function* $\lambda : [0,\infty) \mapsto \mathbb{R}^1$ *such that for every* $\zeta \in G$ *with* $\|\zeta - x_0\| < \beta$ *equation* (22) *with* $u(\cdot) = u_*(\cdot)$ *and initial condition* $x(0) = \zeta$ *(instead of* $x(0) = x_0$*) has a solution* $x(\zeta;\cdot)$ *on* $[0,\infty)$ *in* G*, and*

$$\max_{\theta \in [x(\zeta;t), x_*(t)]} \left| e^{-\rho t} \langle g_x(\theta, u_*(t)), x(\zeta;t) - x_*(t) \rangle \right| \overset{a.e.}{\leq} \|\zeta - x_0\| \lambda(t).$$

Here $[x(\zeta;t), x_*(t)]$ *denotes the line segment with vertices* $x(\zeta;t)$ *and* $x_*(t)$.

The third condition provides a uniform estimate on the "tail" of the integral utility functional in $(P4)$ (see [3–5]).

(A3) *There is a positive function* $\omega(\cdot)$ *decreasing on* $[0,\infty)$ *such that* $\omega(t) \to +0$ *as* $t \to \infty$ *and for any admissible pair* $(x(\cdot), u(\cdot))$ *the following estimate holds:*

$$\int_T^{T'} e^{-\rho t} g(x(t), u(t))\, dt \leq \omega(T), \qquad 0 \leq T \leq T'.$$

Obviously, condition $(A1)$ is satisfied in problem $(P3)$. Indeed, in this case $f(x,u)) = [u - r]\, x + a$, $g(x,u) = \ln u - \ln x$, $f_x(x,u) = u - r$ and $g_x(x,u) = -1/x$ are continuous locally bounded functions on $(0,\infty) \times [\rho,\infty)$.

Let us show that condition $(A2)$ holds for any admissible pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$. Set $\beta = x_0/2$ and define the nonnegative integrable function $\lambda : [0,\infty) \mapsto \mathbb{R}^1$ as follows: $\lambda(t) = 2e^{-\rho t}/x_0$, $t \geq 0$. Then, as it can be seen directly, for any real ζ: $|\zeta - x_0| < \beta$, the Cauchy problem (19) with $u(\cdot) = u_*(\cdot)$ and the initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta;\cdot)$ on $[0,\infty)$ and

$$\max_{\theta \in [x(\zeta;t), x_*(t)]} \left| e^{-\rho t} g_x(\theta, u_*(t))\, (x(\zeta;t) - x_*(t)) \right| \overset{\text{a.e.}}{\leq} |\zeta - x_0| \lambda(t).$$

Hence, for any admissible pair $(x_*(\cdot), x_*(\cdot))$ condition $(A2)$ is also satisfied.

Validity of condition $(A3)$ follows from Lemma 1 in the case of $(P3)$ directly.

Along arbitrary admissible pair $(x(\cdot), u(\cdot))$ consider the following linear differential equation:

$$\dot{z}(t) = -\left[f_x(x(t), u(t))\right]^* z(t) = \left[-u(t) + r\right] z(t). \tag{23}$$

Since $u(\cdot)$ is a locally bounded function on $[0,\infty)$, the normalized at instant $t = 0$ fundamental solution $Z(\cdot)$ to equation (23) is defined on $[0,\infty)$ as follows:

$$Z(t) = e^{-\int_0^t u(\xi)\, d\xi + rt}, \qquad t \geq 0. \tag{24}$$

Due to (21) and (24) for any admissible pair $(x(\cdot),u(\cdot))$ we have

$$\begin{aligned}&\left|e^{-\rho t}Z^{-1}(t)g_x(x(t),u(t))\right|\\&\qquad=\left|\frac{e^{-\rho t}e^{\int_0^t u(\xi)\,d\xi-rt}}{x_0e^{\int_0^t u(\xi)\,d\xi-rt}+ae^{\int_0^t u(\xi)\,d\xi-rt}\int_0^t e^{-\int_0^s u(\xi)\,d\xi+rs}\,ds}\right|\le\frac{e^{-\rho t}}{x_0},\quad t\ge 0.\quad(25)\end{aligned}$$

Hence, for any $T>0$ the function $\psi_T:\ [0,T]\mapsto\mathbb{R}^1$ defined as

$$\begin{aligned}\psi_T(t)&=Z(t)\int_t^T e^{-\rho s}Z^{-1}(s)g_x(x(s),u(s))\,ds\\&=-e^{-\int_0^t u(\xi)\,d\xi+rt}\int_t^T\frac{e^{\int_0^s u(\xi)\,d\xi-rs}e^{-\rho s}}{x(s)}\,ds,\qquad t\in[0,T],\quad(26)\end{aligned}$$

is absolutely continuous.

Analogously, due to (25) the function $\psi\colon\ [0,\infty)\mapsto\mathbb{R}^1$ defined as

$$\begin{aligned}\psi(t)&=Z(t)\int_t^\infty e^{-\rho s}Z^{-1}(s)g_x(x(s),u(s))\,ds\\&=-e^{-\int_0^t u(\xi)\,d\xi+rt}\int_t^\infty\frac{e^{\int_0^s u(\xi)\,d\xi-rs}e^{-\rho s}}{x(s)}\,ds,\qquad t\ge 0,\quad(27)\end{aligned}$$

is locally absolutely continuous.

Define the normal form Hamilton-Pontryagin function $\mathcal{H}:[0,\infty)\times(0,\infty)\times[\rho,\infty)\times\mathbb{R}^1\mapsto\mathbb{R}^1$ and the normal-form Hamiltonian $H:[0,\infty)\times(0,\infty)\times\mathbb{R}^1\mapsto\mathbb{R}^1$ for problem $(P3)$ in the standard way:

$$\mathcal{H}(t,x,u,\psi)=\psi f(x,u)+e^{-\rho t}g(x,u)=\psi[(u-r)x+a]+e^{-\rho t}[\ln u-\ln x],$$

$$H(t,x,\psi)=\sup_{u\ge\rho}\mathcal{H}(t,x,u,\psi),$$

$$t\in[0,\infty),\quad x\in(0,\infty),\quad u\in[\rho,\infty),\quad \psi\in\mathbb{R}^1.$$

Now we are ready to prove a general theorem on existence of an optimal admissible control in problem $(P3)$ (and hence in $(P2)$ and in $(P1)$).

Theorem 1. *There is an optimal admissible control $u_*(\cdot)$ in problem $(P3)$. Moreover, for any optimal admissible pair $(x_*(\cdot),u_*(\cdot))$ the following inequality holds:*

$$u_*(t)\overset{a.e.}{\le}\left(1+\frac{1}{Kx_*(t)}\right)(r+\rho),\qquad t\ge 0.\quad(28)$$

Proof. Let us show that there are a continuous function $M\colon[0,\infty)\mapsto\mathbb{R}^1$, $M(t)\ge 0$, $t\ge 0$, and a function $\delta\colon\ [0,\infty)\mapsto\mathbb{R}^1$, $\delta(t)>0$, $t\ge 0$, $\lim_{t\to\infty}(\delta(t)/t)=0$, such that for any admissible pair $(x(\cdot),u(\cdot))$, satisfying on a set $\mathfrak{M}\subset[0,\infty)$, $\operatorname{meas}\mathfrak{M}>0$, to inequality $u(t)>M(t)$, for all $t\in\mathfrak{M}$ we have

$$\inf_{T>0:\,t\le T-\delta(T)}\left\{\sup_{u\in[\rho,M(t)]}\mathcal{H}(t,x(t),u,\psi_T(t))-\mathcal{H}(t,x(t),u(t),\psi_T(t))\right\}>0,\quad(29)$$

where the function $\psi_T(\cdot)$ is defined on $[0,T]$, $T>0$, by equality (26).

Let $(x(\cdot),u(\cdot))$ be an arbitrary admissible pair in $(P3)$. Then due to (21) and (24), for any $T>0$ and arbitrary $t\in[0,T]$ we get (see (26))

$$\begin{aligned}-x(t)\psi_T(t) &= \left[x_0 + a\int_0^t e^{-\int_0^s u(\xi)\,d\xi + rs}\,ds\right]\int_t^T \frac{e^{-\rho s}}{x_0 + a\int_0^s e^{-\int_0^\tau u(\xi)\,d\xi + r\tau}\,d\tau}\,ds\\ &\geq x_0\int_t^T \frac{e^{-\rho s}}{x_0 + a\int_0^s e^{r\tau}\,d\tau}\,ds \geq \frac{rx_0 e^{-(r+\rho)t}}{(rx_0+a)(r+\rho)}\left[1-e^{-(r+\rho)(T-t)}\right]. \end{aligned}\tag{30}$$

For arbitrary $\delta>0$ define the function $M_\delta\colon[0,\infty)\mapsto\mathbb{R}^1$ as follows:

$$M_\delta(t) = \frac{(rx_0+a)(r+\rho)}{rx_0\left[1-e^{-(r+\rho)\delta}\right]}e^{rt} + \frac{1}{\delta}, \qquad t\geq 0. \tag{31}$$

Then for any $T>\delta$, $t\in[0,T-\delta]$ and arbitrary admissible pair $(x(\cdot),u(\cdot))$ the function $u\mapsto\mathcal{H}(t,x(t),u,\psi_T(t))$ reaches its maximal value on $[\rho,\infty)$ at the point (see (30))

$$u_T(t) = -\frac{e^{-\rho t}}{x(t)\psi_T(t)} \leq \frac{(rx_0+a)(r+\rho)}{rx_0\left[1-e^{-(r+\rho)(T-t)}\right]}e^{rt} \leq M_\delta(t) - \frac{1}{\delta}. \tag{32}$$

Now, for a fixed $\delta>0$ set $\delta(t)\equiv\delta$ and $M(t)\equiv M_\delta(t)$, $t\geq 0$. Let $(x(\cdot),u(\cdot))$ be an admissible pair such that inequality $u(t)>M_\delta(t)$ holds on a set $\mathfrak{M}\subset[0,\infty)$, $\operatorname{meas}\mathfrak{M}>0$.

Let us show that for any $t\in\mathfrak{M}$ inequality (29) holds. Indeed, for arbitrary fixed $t\in\mathfrak{M}$ define the function $\Phi\colon[t+\delta,\infty)\mapsto\mathbb{R}^1$ as follows

$$\begin{aligned}\Phi(T) &= \sup_{u\in[\rho,M(t)]}\mathcal{H}(t,x(t),u,\psi_T(t)) - \mathcal{H}(t,x(t),u(t),\psi_T(t))\\ &= \psi_T(t)u_T(t)x(t) + e^{-\rho t}\ln u_T(t) - \left[\psi_T(t)u(t)x(t) + e^{-\rho t}\ln u(t)\right], \qquad T\geq t+\delta.\end{aligned}$$

Due to (32) we have

$$\begin{aligned}\Phi(T) = -e^{-\rho t} &+ e^{-\rho t}\left[-\rho t - \ln(-\psi_T(t)) - \ln x(t)\right]\\ &- \left[\psi_T(t)u(t)x(t) + e^{-\rho t}\ln u(t)\right], \qquad T\geq t+\delta.\end{aligned}$$

Hence, due to (26) and (32) for a.e. $T\geq t+\delta$ we get

$$\begin{aligned}\frac{d}{dT}\Phi(T) &= -\frac{e^{-\rho t}}{\psi_T(t)}\frac{d}{dT}\left[\psi_T(t)\right] - u(t)x(t)\frac{d}{dT}\left[\psi_T(t)\right]\\ &= x(t)\frac{d}{dT}\left[\psi_T(t)\right]\left[\frac{e^{-\rho t}}{-\psi_T(t)x(t)} - u(t)\right] = x(t)\frac{d}{dT}\left[\psi_T(t)\right](u_T(t)-u(t)) > 0.\end{aligned}$$

Hence,

$$\begin{aligned}\inf_{T>0:\,t\leq T-\delta}&\left\{\sup_{u\in[\rho,M(t)]}\mathcal{H}(t,x(t),u,\psi_T(t)) - \mathcal{H}(t,x(t),u(t),\psi_T(t))\right\}\\ &= \inf_{T>0:\,t\leq T-\delta}\Phi(T) = \Phi(t+\delta) > 0.\end{aligned}$$

Thus, for any $t \in \mathfrak{M}$ inequality (29) is proved.

As far as the instantaneous utility in (18) is concave in u, the control system (19) is affine in u, the set U is closed (see (20)), conditions $(A1)$ and $(A3)$ are satisfied, and since $(A2)$ also holds for any admissible pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$, all conditions of [3, Theorem 1] (see also [4]) are fullfiled[2]. Hence, there is an optimal admissible control $u_*(\cdot)$ in $(P3)$ and, moreover, $u_*(t) \overset{a.e.}{\le} M_\delta(t)$, $t \ge 0$. Passing to a limit in this inequality as $\delta \to \infty$ we get (see (31))

$$u_*(t) \overset{\text{a.e.}}{\le} \left(1 + \frac{1}{Kx_0}\right)(r+\rho)e^{rt}, \qquad t \ge 0. \tag{33}$$

Further, it is easy to see that for any $\tau > 0$ the pair $(\tilde{x}_*(\cdot), \tilde{u}_*(\cdot))$ defined as $\tilde{x}_*(t) = x_*(t+\tau)$, $\tilde{u}_*(t) = u_*(t+\tau)$, $t \ge 0$, is an optimal admissible pair in the problem $(P3)$ taken with initial condition $x(0) = x_*(\tau)$. Hence, repeating the same arguments as above we get the following inequality for $(\tilde{x}_*(\cdot), \tilde{u}_*(\cdot))$ (see (33)):

$$\tilde{u}_*(t) \overset{\text{a.e.}}{\le} \left(1 + \frac{1}{K\tilde{x}_*(0)}\right)(r+\rho)e^{rt}, \qquad t \ge 0.$$

Hence, for arbitrary fixed $\tau > 0$ we have

$$u_*(t) = \tilde{u}_*(t-\tau) \overset{\text{a.e.}}{\le} \left(1 + \frac{1}{Kx_*(\tau)}\right)(r+\rho)e^{r(t-\tau)}, \qquad t \ge \tau.$$

Due to arbitrariness of $\tau > 0$ this implies (28). $\square$

The following result follows from the normal form version of the maximum principle developed in [8–10].

Theorem 2. *Let $(x_*(\cdot), u_*(\cdot))$ be an optimal admissible pair in problem $(P3)$. Then the function $\psi : [0,\infty) \mapsto \mathbb{R}^1$ defined for pair $(x_*(\cdot), u_*(\cdot))$ by formula (27) is (locally) absolutely continuous and satisfies the conditions of the normal form maximum principle, i.e. $\psi(\cdot)$ is a solution of the adjoint system*

$$\dot{\psi}(t) = -\mathcal{H}_x\left(x_*(t), u_*(t), \psi(t)\right), \tag{34}$$

and the maximum condition holds:

$$\mathcal{H}(x_*(t), u_*(t), \psi(t)) \overset{a.e.}{=} H(x_*(t), \psi(t)). \tag{35}$$

Proof. Indeed, as it is already have been shown above condition $(A1)$ is satisfied and $(A2)$ holds for any admissible pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$. Hence, due to the variant of the maximum principle developed in [8–10] the function $\psi : [0,\infty) \mapsto \mathbb{R}^1$ defined for pair $(x_*(\cdot), u_*(\cdot))$ by formula (27) satisfies the conditions (34) and (35). $\square$

[2]The proof of [3, Theorem 1] is given in the case $M(t) \equiv M = \text{const}$, $t \ge 0$. However, the result is true also in the case when $M(\cdot)$ is a continuous function of t. In this case the proof word to word coincides with the proof presented in [3] with replacement of M by $M(t)$, $t \ge 0$.

Note that the Cauchy type formula (27) provides more precise information on the adjoint variable $\psi(\cdot)$ than the standard transversality conditions at infinity. In particular, due to (27) we have (see (21) and (24))

$$
\begin{aligned}
0 > \psi(t) &= -e^{-\int_0^t u_*(\xi)\,d\xi + rt} \int_t^\infty \frac{e^{-\rho\tau} e^{\int_0^\tau u_*(\xi)\,d\xi - r\tau}}{e^{\int_0^\tau u_*(\xi)\,d\xi - r\tau} \left[x_0 + a \int_0^\tau e^{-\int_0^\theta u_*(\xi)\,d\xi + r\theta}\,d\theta \right]}\,d\tau \\
&= -e^{-\int_0^t u_*(\xi)\,d\xi + rt} \int_t^\infty \frac{e^{-\rho\tau}}{x_0 + a \int_0^\tau e^{-\int_0^\theta u_*(\xi)\,d\xi + r\theta}\,d\theta}\,d\tau \\
&> -\frac{e^{-\int_0^t u_*(\xi)\,d\xi + rt}}{x_0 + a \int_0^t e^{-\int_0^\theta u_*(\xi)\,d\xi + r\theta}\,d\theta} \int_t^\infty e^{-\rho\tau}\,d\tau = -\frac{e^{-\rho t}}{\rho x_*(t)}, \qquad t \ge 0. \qquad (36)
\end{aligned}
$$

Thus, formula (27) implies the following condition on $\psi(\cdot)$:

$$
0 < -\psi(t) x_*(t) < \frac{e^{-\rho t}}{\rho}, \qquad t \ge 0. \tag{37}
$$

Obviously, estimate (37) is a stronger fact than both the standard transversality conditions at infinity

$$
\lim_{t\to\infty} \psi(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \psi(t) x_*(t) = 0.
$$

Note also that due to [2, Corollary to Theorem 3] formula (27) implies the following stationarity condition for the Hamiltonian (Michel's version of the transversality condition at infinity (see [7, 23])):

$$
H(t, x_*(t), \psi(t)) = \rho \int_t^\infty e^{-\rho s} g(x_*(s), u_*(s))\,ds, \qquad t \ge 0. \tag{38}
$$

It can be shown directly that if an admissible pair (not necessary optimal) $(x(\cdot), u(\cdot))$ in problem $(P3)$ together with an adjoint variable $\psi(\cdot)$ satisfies the core conditions (34) and (35) of the maximum principle and $\lim_{t\to\infty} H(t, x(t), \psi(t)) = 0$ then condition (38) holds for the triple $(x(\cdot), u(\cdot), \psi(\cdot))$ as well (see details in [7, Section 3]).

Further, due to the maximum condition (35) for a.e. $t \ge 0$ we have

$$
u_*(t) = \arg\max\nolimits_{u \in [\rho, \infty)} \left[\psi(t) x_*(t) u + e^{-\rho t} \ln u \right].
$$

This implies (see (37))

$$
u_*(t) \overset{a.e.}{=} -\frac{e^{-\rho t}}{\psi(t) x_*(t)} > \rho, \qquad t \in [0, \infty). \tag{39}
$$

Substituting this formula for $u_*(\cdot)$ in (19) and in (34) due to Theorem 2 we get that any optimal in $(P3)$ trajectory $x_*(\cdot)$ together with the corresponding adjoint variable $\psi(\cdot)$ must satisfy to the following Hamiltonian system of the maximum principle:

$$
\begin{aligned}
\dot{x}(t) &= -r x(t) - \frac{e^{-\rho t}}{\psi(t)} + a, \\
\dot{\psi}(t) &= r \psi(t) + \frac{2e^{-\rho t}}{x(t)}.
\end{aligned} \tag{40}
$$

Moreover, the estimate (37) and the stationarity condition (38) must hold as well.

In the terms of the current value adjoint variable $\lambda(\cdot)$, $\lambda(t) = e^{\rho t}\psi(t)$, $t \geq 0$, one can rewrite system (40) and estimate (37) as follows:

$$\begin{aligned} \dot{x}(t) &= -rx(t) - \frac{1}{\lambda(t)} + a, \\ \dot{\lambda}(t) &= (\rho + r)\lambda(t) + \frac{2}{x(t)}, \end{aligned} \tag{41}$$

and

$$0 < -\lambda(t)x_*(t) < \frac{1}{\rho}, \qquad t \geq 0. \tag{42}$$

Accordingly, the optimal control $u_*(\cdot)$ can be expressed via the current value adjoint variable $\lambda(\cdot)$ as follows (see (39)):

$$u_*(t) \overset{a.e.}{=} -\frac{1}{\lambda(t)x_*(t)}, \qquad t \geq 0. \tag{43}$$

Define the normal form current value Hamiltonian $M : (0, \infty) \times \mathbb{R}^1 \mapsto \mathbb{R}^1$ for problem $(P3)$ in the standard way (see [7, Section 3]):

$$M(x, \lambda) = e^{\rho t} H(t, x, \psi), \quad x \in (0, \infty), \quad \lambda \in \mathbb{R}^1. \tag{44}$$

Then in the current value terms the stationarity condition (38) takes the form

$$M(x_*(t), \lambda(t)) = \rho e^{\rho t} \int_t^\infty e^{-\rho s} g(x_*(s), u_*(s))\, ds, \qquad t \geq 0. \tag{45}$$

In the next section we will analyze the system (41) coupled with the estimate (42) and the stationarity condition (45). We will show that for any values of parameters in the model and for arbitrary initial state $x_0 > 0$ the corresponding optimal process $(x_*(\cdot), u_*(\cdot))$ in $(P3)$ (which exists) is unique, and there is a unique solution $(x_*(\cdot), \lambda_*(\cdot))$ of the system (41) that corresponds to the pair $(x_*(\cdot), u_*(\cdot))$ due to Theorem 2. We will characterize all optimal processes in $(P3)$ (and hence in $(P1)$) and show that there are only two qualitatively different types of behavior of the optimal paths that are possible. If $r > \rho$ then the optimal path asymptotically approaches an optimal nonvanishing steady state while the corresponding optimal control tends to $(r + \rho)/2$ as $t \to \infty$. If $r \leq \rho$ then the optimal path $x_*(\cdot)$ goes to infinity, while the corresponding optimal control $u_*(\cdot)$ tends to ρ as $t \to \infty$, i.e. asymptotically it follows the Hotelling rule of optimal depletion of an exhaustible resource [21] in this case.

4 Analysis of the Hamiltonian system

Since, the state variable $x(\cdot)$ takes positive values and the values of the current value adjoint variable $\lambda_*(\cdot)$, that corresponds to an optimal pair $(x_*(\cdot), u_*(\cdot))$ due to the maximum principle (Theorem 2), are negative (see (36)), we will restrict analysis of system (41) to the open set $\Gamma = \{\, (x, \lambda) \colon x > 0, \lambda < 0\}$ in the phase plane $\mathbb{R}^2$.

Let us introduce functions $y_1\colon (1/K,\infty) \mapsto (-\infty,0)$ and $y_2\colon (0,\infty) \mapsto (-\infty,0)$ as follows (recall that $a = r/K$):

$$y_1(x) = \frac{1}{a - rx}, \quad x \in \left(\frac{1}{K}, \infty\right), \qquad y_2(x) = -\frac{2}{(\rho + r)x}, \quad x \in (0,\infty).$$

Due to (41) the curves $\gamma_1 = \{(x,\lambda)\colon\ \lambda = y_1(x), x \in (1/K,\infty)\}$ and $\gamma_2 = \{(x,\lambda)\colon\ \lambda = y_2(x), x \in (0,\infty)\}$ are the nullclines at which the derivatives of variables $x(\cdot)$ and $\lambda(\cdot)$ vanish respectively.

Two qualitatively different cases are possible: (i) $r > \rho$ and (ii) $r \leq \rho$.

Consider case (i). In this case the nullclines γ_1 and γ_2 have a unique intersection point $(\hat{x},\hat{\lambda})$ which is a unique equilibrium of system (41) in Γ:

$$\hat{x} = \frac{2r}{(r-\rho)K}, \qquad \hat{\lambda} = \frac{(\rho - r)K}{(\rho + r)r}. \tag{46}$$

The corresponding equilibrium control $\hat{u}(\cdot)$ is

$$\hat{u}(t) \equiv \hat{u} = \frac{\rho + r}{2}, \qquad t \geq 0. \tag{47}$$

The nature of the equilibrium can be deduced by analyzing the linearization of (41) around $(\hat{x},\hat{\lambda})$. It can be seen that the eigenvalues of the linearized system are given by

$$\sigma_{1,2} = \frac{\rho}{2} \pm \frac{1}{2}\sqrt{2r^2 - \rho^2},$$

which are real and distinct with opposite signs when $r > \rho$. Hence, by the Grobman-Hartman theorem in a neighborhood Ω of the equilibrium state $(\hat{x},\hat{\lambda})$ the system (41) is of saddle type (see [20, Chapter 9]).

The nullclines γ_1 and γ_2 divide the set Γ in four open regions:

$$\Gamma_{-,-} = \left\{(x,\lambda) \in \Gamma\colon\ \lambda < y_1(x), \frac{1}{K} < x \leq \hat{x}\right\} \bigcup \left\{(x,\lambda) \in \Gamma\colon\ \lambda < y_2(x), \hat{x} < x < \infty\right\},$$

$$\Gamma_{+,-} = \left\{(x,\lambda) \in \Gamma\colon\ \lambda < y_2(x), 0 < x \leq \frac{1}{K}\right\} \bigcup \left\{(x,\lambda) \in \Gamma\colon\ y_1(x) < \lambda < y_2(x), \frac{1}{K} < x < \hat{x}\right\},$$

$$\Gamma_{+,+} = \left\{(x,\lambda) \in \Gamma\colon\ y_2(x) < \lambda < 0, 0 < x \leq \hat{x}\right\} \bigcup \left\{(x,\lambda) \in \Gamma\colon\ y_1(x) < \lambda < 0, \hat{x} < x < \infty\right\},$$

$$\Gamma_{-,+} = \left\{(x,\lambda) \in \Gamma\colon\ y_2(x) < \lambda < y_1(x), x > \hat{x}\right\}.$$

Any solution $(x(\cdot),\lambda(\cdot))$ of (41) in Γ has definite signs of derivatives of its (x,λ)-coordinates in the sets $\Gamma_{-.-}$, $\Gamma_{-.+}$, $\Gamma_{+,+}$, and $\Gamma_{-,+}$. These signs are indicated by the corresponding subscript indexes. Thus, $\Gamma_{-,-}$ is the set of all points $(x,\lambda) \in \Gamma$ at which both signs of the derivatives $\dot{x}$ and $\dot{\lambda}$ are negative, $\Gamma_{-,+}$ is the set of all points $(x,\lambda) \in \Gamma$ at which the sign of the derivative $\dot{x}$ is negative and the sign of the derivative $\dot{\lambda}$ is positive, and so on.

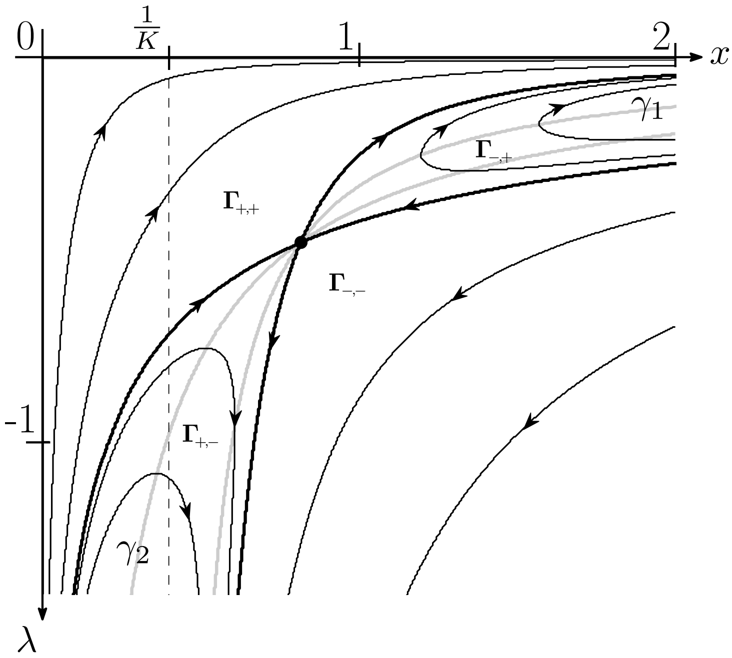


Figure 1: Phase portrait of (41) around $(\hat{x}, \hat{\lambda})$. Here $r = 5$, $\rho = 0.1$, and $K = 2.5$.

A graphical representation of the phase plane, along with the stable and unstable manifolds of the saddle point, is shown in Figure 1, when this condition is met.

Obviously,

$$\Gamma = \Gamma_{-,-} \bigcup \Gamma_{+,-} \bigcup \Gamma_{+,+} \bigcup \Gamma_{-,+} \bigcup \gamma_1 \bigcup \gamma_2.$$

For any initial state $(\xi, \beta) \in \Gamma$ there is a unique solution $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ of the system (41) satisfying initial conditions $x(0) = \xi$, $\lambda(0) = \beta$, and due to the standard extension result this solution is defined on some maximal time interval $[0, T_{\xi,\beta})$ where $0 < T_{\xi,\beta} \leq \infty$ (see [20, Chapter 2]).

Let us consider asymptotic behaviors of solutions $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ of system (41) for all possible initial states $(\xi, \beta) \in \Gamma$ as $t \to T_{\xi,\beta}$.

The standard analysis of system (41) in each of the sets $\Gamma_{-.-}$, $\Gamma_{-.+}$, $\Gamma_{+,+}$, and $\Gamma_{-,+}$ shows that only three types of asymptotic behavior of solutions $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ of (41) in Γ as $t \to T_{\xi,\beta}$ are possible:

1) $(x_{\xi,\beta}(t), \lambda_{\xi,\beta}(t)) \in \Gamma_{-,-}$ or $(x_{\xi,\beta}(t), \lambda_{\xi,\beta}(t)) \in \Gamma_{+,-}$ for all sufficiently large times $t < T_{\xi,\beta}$ depending on the initial state (ξ, β). In this case $T_{\xi,\beta} = \infty$ and $\lim_{t\to\infty} \lambda_{\xi,\beta}(t) = -\infty$ while $\lim_{t\to\infty} x_{\xi,\beta}(t) = 1/K$. Due to Theorem 2 such asymptotic behavior does not correspond to an optimal process because in this case $\lim_{t\to\infty} \lambda_{\xi,\beta}(t) x_{\xi,\beta}(t) = -\infty$ that contradicts the necessary condition (42).

2) $\lim_{t\to T_{\xi,\beta}} x_{\xi,\beta}(t) = \infty$ and $\lim_{t\to T_{\xi,\beta}} \lambda_{\xi,\beta}(t) = 0$. In this case $(x_{\xi,\beta}(t), \lambda_{\xi,\beta}(t)) \in \Gamma_{+,+}$ for all sufficiently large times $t < T_{\xi,\beta}$. If $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ corresponds to an optimal pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$ then due to Theorem 2 $x_*(\cdot) \equiv x_{\xi,\beta}(\cdot)$, $T_{\xi,\beta} = \infty$, $\lim_{t\to\infty} x_*(t) = \infty$, and $\lim_{t\to\infty} \lambda_{\xi,\beta}(t) = 0$. Let us put $\lambda_*(\cdot) \equiv \lambda_{\xi,\beta}(\cdot)$ in this case and define the function

$\phi_* \colon [0,\infty) \mapsto \mathbb{R}^1$ as follows:

$$\phi_*(t) = \lambda_*(t)x_*(t), \qquad t \in [0,\infty).$$

By direct differentiation we get (see (41))

$$\begin{aligned}\dot{\phi}_*(t) &\overset{\text{a.e.}}{=} \dot{\lambda}_*(t)x_*(t) + \lambda_*(t)\dot{x}_*(t)\\ &= (\rho+r)\lambda_*(t) + 2 - r\lambda(t)x_*(t) - 1 + a\lambda_*(t) = \rho\phi_*(t) + 1 + a\lambda_*(t), \qquad t \in [0,\infty).\end{aligned}$$

Hence,

$$\phi_*(t) = e^{\rho t}\left[\phi_*(0) + \int_0^t e^{-\rho s}\left(1 + a\lambda_*(s)\right)\, ds\right], \qquad t \in [0,\infty). \tag{48}$$

Since $\lim_{t\to\infty} \lambda_*(t) = 0$ the improper integral $\int_0^\infty e^{-\rho s}\left(1 + a\lambda_*(s)\right)\, ds$ converges, and due to (42) we have $0 > \phi_*(t) = \lambda_*(t)x_*(t) > -1/\rho$ for all $t > 0$. Due to (48) this implies

$$\phi_*(0) = -\int_0^\infty e^{-\rho s}\left(1 + a\lambda_*(s)\right)\, ds = -\frac{1}{\rho} - a\int_0^\infty e^{-\rho s}\lambda_*(s)\, ds.$$

Substituting this expression for $\phi_*(0)$ in (48) we get

$$\phi_*(t) = -\frac{1}{\rho} - ae^{\rho t}\int_t^\infty e^{-\rho s}\lambda_*(s)\, ds, \qquad t \in [0,\infty).$$

Due to the L'Hospital rule we have

$$\lim_{t\to\infty} e^{\rho t}\int_t^\infty e^{-\rho s}\lambda_*(s)\, ds = \lim_{t\to\infty}\frac{\int_t^\infty e^{-\rho s}\lambda_*(s)\, ds}{e^{-\rho t}} = \lim_{t\to\infty}\frac{\lambda_*(t)}{\rho} = 0.$$

Hence,

$$\lim_{t\to\infty} u_*(t) = \lim_{t\to\infty}\frac{-1}{\lambda_*(t)x_*(t)} = \lim_{t\to\infty}\frac{-1}{\phi_*(t)} = \rho.$$

But due to the system (41) and the inequality $r > \rho$ this implies $\lim_{t\to\infty} x_*(t) \leq a < \infty$ that contradicts the equality $\lim_{t\to\infty} x_*(t) = \infty$. Thus, all trajectories of (41) are the blow up trajectories in the case 2). Thus, we conclude that there are not any trajectories of (41) that can correspond to optimal processes in $(P3)$ due to Theorem 2 in the case 2).

3) $\lim_{t\to\infty}(x(t),\lambda(t)) = (\hat{x},\hat{\lambda})$ as $t \to \infty$. In this case, since the equilibrium $(\hat{x},\hat{\lambda})$ is of saddle type, there are only two trajectories of (41) (which are unique up to the shift in time) which tend to the equilibrium point $(\hat{x},\hat{\lambda})$ asymptotically as $t \to \infty$ and lying on the stable manifold of $(\hat{x},\hat{\lambda})$. One such trajectory $(x_1(\cdot),\lambda_1(\cdot))$ approaches the point $(\hat{x},\hat{\lambda})$ from the left from the set $\Gamma_{+,+}$ (we call this trajectory *the left equilibrium trajectory*), while the second trajectory $(x_2(\cdot),\lambda_2(\cdot))$ approaches the point $(\hat{x},\hat{\lambda})$ from the right from the set $\Gamma_{-,-}$ (we call this trajectory *the right equilibrium trajectory*). It is easy to see that both these trajectories are fit to estimate (42) and stationarity condition (45). Hence, $(x_1(\cdot),\lambda_1(\cdot))$, $(x_2(\cdot),\lambda_2(\cdot))$ and the stationary trajectory $(\hat{x}(\cdot),\hat{\lambda}(\cdot))$, $\hat{x}(\cdot) \equiv \hat{x}$, $\hat{\lambda}(\cdot) \equiv \hat{\lambda}$,

$t \geq 0$, are unique trajectories of (41) which correspond to the optimal processes in problem $(P3)$ due to the maximum principle (Theorem 2) depending on initial state x_0.

Due to Theorem 1 for any initial state $x_0 > 0$ an optimal control $u_*(\cdot)$ in problem $(P3)$ exists. Hence, for any initial state $\xi \in (0, \hat{x})$ there is a unique $\beta < 0$ such that the corresponding trajectory $(x_{\xi,\beta}(\cdot))$ coincides (up to a shift in time) with the left equilibrium trajectory $(x_1(\cdot), \lambda_1(\cdot))$ on time interval $[0, \infty)$. Analogously, for any initial state $\xi > \hat{x}$ there is a unique $\beta < 0$ such that the corresponding trajectory $(x_{\xi,\beta}(\cdot))$ coincides (up to a shift in time) with the right equilibrium trajectory $(x_2(\cdot), \lambda_2(\cdot))$ on $[0, \infty)$. The corresponding optimal control is defined uniquely by (43). Thus, for any initial state $x_0 > 0$ the corresponding optimal process $(x_*(\cdot), u_*(\cdot))$ in $(P3)$ is unique, and due to Theorem 2 the corresponding current value adjoint variable $\lambda_*(\cdot)$ is also unique.

Further, to the left of the point $(\hat{x}, \hat{\lambda})$ in the set $\Gamma_{+,+}$, the function $x_1(\cdot)$ monotonically increases. Therefore, while $(x_1(\cdot), \lambda_1(\cdot))$ lies in $\Gamma_{+,+}$, the time can be uniquely expressed in terms of the first coordinate of the trajectory $(x_1(\cdot), \lambda_1(\cdot))$ as a smooth function $t = t_1(x)$, $x \in (0, \hat{x})$. Changing the time variable $t = t_1(x)$ on interval $(0, \hat{x})$, we find that the function $\lambda_-(x) = \lambda_1(t_1(x))$, $x \in (0, \hat{x})$, is a solution to the following differential equation on the interval $(0, \hat{x})$:

$$\frac{d\lambda(x)}{dx} = \frac{d\lambda(t_1(x))}{dt} \times \frac{dt_1(x)}{dx} = \frac{\lambda(x)\left((\rho + r)\lambda(x)x + 2\right)}{x\left(-r\lambda(x)x - 1 + a\lambda(x)\right)} \tag{49}$$

with the boundary condition

$$\lim_{x \to \hat{x}-0} \lambda(x) = \hat{\lambda}. \tag{50}$$

Obviously, the curve $\lambda_- = \{(x, \lambda) \colon \lambda = \lambda_-(x), x \in (0, \hat{x})\}$ corresponds to the region of the stable manifold of $(\hat{x}, \hat{\lambda})$ where $x < \hat{x}$.

Analogously, to the right of the point $(\hat{x}, \hat{\lambda})$ in the set $\Gamma_{-,-}$, while $(x_1(\cdot), \lambda_1(\cdot))$ lies in $\Gamma_{-,-}$,the function $x_1(\cdot)$ monotonically decreases. Hence, the time can be uniquely expressed in terms of the first coordinate of the trajectory $(x_1(\cdot), \lambda_1(\cdot))$ as a smooth function $t = t_2(x)$, $x \in (\hat{x}, \infty)$. Changing the time variable $t = t_2(x)$ on interval $(\hat{x}, \infty)$, we find that the function $\lambda_+(x) = \lambda_2(t_2(x))$, $x > \hat{x}$, is a solution to the differential equation (49) on the interval $(\hat{x}, \infty))$ with the boundary condition

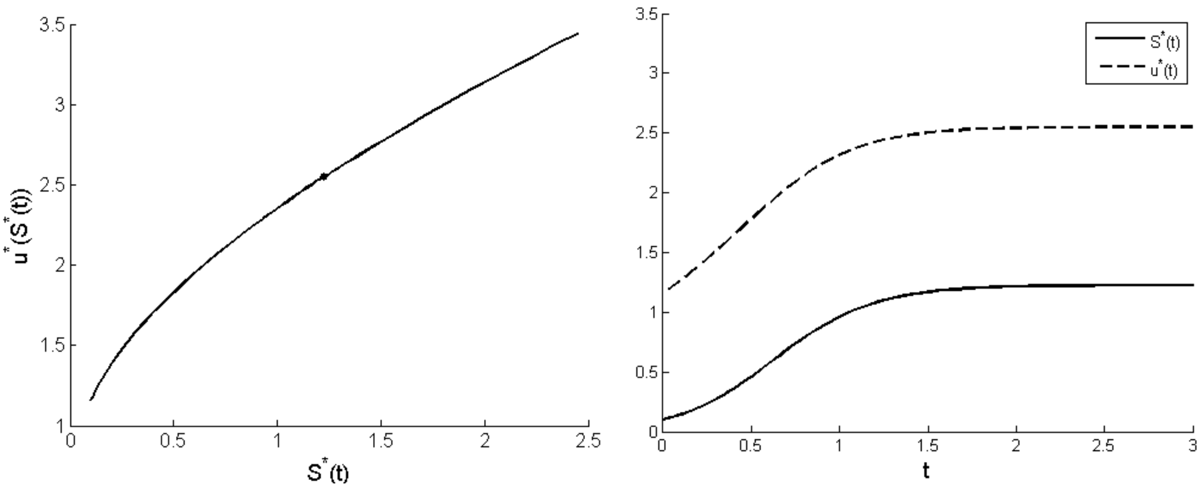
$$\lim_{x \to \hat{x}+0} \lambda(x) = \hat{\lambda}. \tag{51}$$

As above, the curve $\lambda_+ = \{(x, \lambda) \colon \lambda = \lambda_+(x), x \in (\hat{x}, \infty)\}$ corresponds to the region of the stable manifold of $(\hat{x}, \hat{\lambda})$ where $x > \hat{x}$.

Using solutions $\lambda_-(\cdot)$ and $\lambda_+(\cdot)$ of differential equation (49) along with (43) we can get an expression for the optimal feedback law as follows

$$u_*(x) = \begin{cases} -\frac{1}{\lambda_-(x)x}, & \text{if } x < \hat{x}, \\ \frac{\rho + r}{2}, & \text{if } x = \hat{x}, \\ -\frac{1}{\lambda_+(x)x}, & \text{if } x > \hat{x}. \end{cases}$$

This means that in order to find the optimal feedback, we must be able to determine the trajectories $\lambda_-(\cdot)$ and $\lambda_+(\cdot)$ on their domains of definition $(0, \hat{x})$ and $(\hat{x}, \infty)$ respectively.



(a) The optimal feedback law obtained by numerically solving (49).

(b) Solution with the optimal feedback law. Here the initial stock $S(0) = 0.1$.

Figure 2: The optimal feedback law and representative solution in the case $r > \rho$. Here $r = 5$, $\rho = 0.1$ and $K = 2.5$.

An analytical solution to nonlinear differential equation (49) is difficult to obtain. However, it is possible to solve numerically. A graphical depiction of the feedback law in the original variables obtained by numerically solving the above ODE can be seen in Figure 2a. A representative solution of $(P1)$ incorporating this feedback law is also shown in Figure 2b. The trajectories show convergence of the stock and consumption to a steady state equilibrium.

Now, consider the case (ii) when $r \leq \rho$. In this case $y_2(x) > y_1(x)$ for all $x > 1/K$ and hence the nullclines γ_1 and γ_2 do not intersect in Γ. Accordingly, the system (41) does not have an equilibrium point in Γ.

The nullclines γ_1 and γ_2 divide the set Γ in three open regions:

$$\hat{\Gamma}_{-,-} = \Big\{(x,\lambda) \in \Gamma\colon\ \lambda < y_1(x), x > \frac{1}{K}\Big\},$$

$$\hat{\Gamma}_{+,-} = \Big\{(x,\lambda) \in \Gamma\colon\ \lambda < y_2(x), 0 < x \leq \frac{1}{K}\Big\} \bigcup \Big\{(x,\lambda) \in \Gamma\colon\ y_1(x) < \lambda < y_2(x), x > \frac{1}{K}\Big\},$$

$$\hat{\Gamma}_{+,+} = \Big\{(x,\lambda) \in \Gamma\colon\ y_2(x) < \lambda < 0, 0 < x \leq \hat{x}\Big\} \bigcup \Big\{(x,\lambda) \in \Gamma\colon\ y_1(x) < \lambda < 0, \hat{x} < x < \infty\Big\},$$

Obviously,

$$\Gamma = \hat{\Gamma}_{-,-} \bigcup \hat{\Gamma}_{+,-} \bigcup \hat{\Gamma}_{+,+} \bigcup \gamma_1 \bigcup \gamma_2.$$

The behavior of the flows is shown in Figure 3 through the phase portrait.

Any solution $(x(\cdot), \lambda(\cdot))$ of (41) in Γ has the definite signs of derivatives of its (x, λ) coordinates in each set $\hat{\Gamma}_{-.-}$, $\hat{\Gamma}_{+,+}$, and $\hat{\Gamma}_{-,+}$ as indicated by subscript indexes.

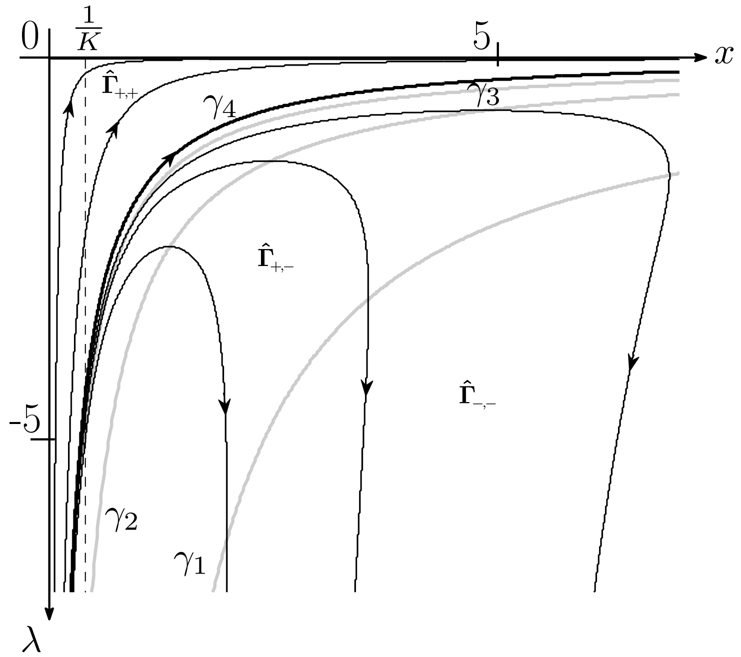


Figure 3: Phase portrait of (41) in the case $r < \rho$. Here $r = 0.1$, $\rho = 0.5$, and $K = 2.5$.

The standard analysis of the behaviors of solutions $(x(\cdot), \lambda(\cdot))$ of system (41) in each of sets $\hat{\Gamma}_{-.-}$, $\hat{\Gamma}_{+.-}$ and $\Gamma_{+,+}$ shows that there are only two types of asymptotic behavior of solutions $(x(\cdot), \lambda(\cdot))$ of (41) that are possible:

1) $\lim_{t\to\infty} x(t) = 1/K$, $\lim_{t\to\infty} \lambda(t) = -\infty$. In this case $(x(t), \lambda(t)) \in \hat{\Gamma}_{-,-}$ for all sufficiently large times $t \geq 0$. Due to Theorem 2 such asymptotic behavior does not correspond to an optimal process because in this case $\lim_{t\to\infty} \lambda(t)x(t) = -\infty$ that contradicts condition (42). Thus this case can be eliminated from the consideration.

2) $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} \lambda(t) = 0$. In this case $(x(t), \lambda(t)) \in \hat{\Gamma}_{+,+}$ for all $t \geq 0$. Since the case 1) can be eliminated from the consideration, we conclude that the case 2) is the only one that can be realized for an optimal process $(x_*(\cdot), u_*(\cdot))$ (which exists) in $(P3)$ due to the maximum principle (Theorem (2)).

Let us consider behavior of trajectory $(x_*(\cdot), \lambda_*(\cdot))$ of system (41) that corresponds to the optimal processes $(x_*(\cdot), u_*(\cdot))$ in the set $\hat{\Gamma}_{+,+}$ in more details.

As in the subcase (b) of case (i) above, define the function $\phi_*\colon [0,\infty) \mapsto \mathbb{R}^1$ as follows:

$$\phi_*(t) = \lambda_*(t)x_*(t), \qquad t \in [0, \infty).$$

Repeating the calculations presented in the subcase (b) of case (i) we get

$$\phi_*(t) = -\frac{1}{\rho} - ae^{\rho t}\int_t^\infty e^{-\rho s}\lambda_*(s)\, ds, \qquad t \in [0, \infty).$$

As in the subcase (b) of case (i) above, due to the L'Hospital rule this implies

$$\lim_{t\to\infty} e^{\rho t}\int_t^\infty e^{-\rho s}\lambda_*(s)\, ds = \lim_{t\to\infty} \frac{\int_t^\infty e^{-\rho s}\lambda_*(s)\, ds}{e^{-\rho t}} = \lim_{t\to\infty} \frac{\lambda_*(t)}{\rho} = 0.$$

Hence,

$$\lim_{t\to\infty} u_*(t) = \lim_{t\to\infty} \frac{-1}{\lambda_*(t)x_*(t)} = \lim_{t\to\infty} \frac{-1}{\phi_*(t)} = \rho.$$

Thus, asymptotically, any optimal admissible control $u_*(\cdot)$ in $(P3)$ satisfies the Hotelling rule [21] of optimal depletion of an exhaustible resource in the case (ii).

Now let us show that the optimal control $u_*(\cdot)$ is defined uniquely by Theorem 2 in the case (ii).

Define the function $y_3\colon (0,\infty) \mapsto \mathbb{R}^1$ and the curve $\gamma_3 \subset \Gamma$ as follows:

$$y_3(x) = -\frac{1}{\rho x}, \quad x \in (0,\infty), \qquad \gamma_3 = \{(x,\lambda)\colon \lambda = y_3(x), x \in (0,\infty)\}.$$

It is easy to see that $y_3(x) \geq y_2(x)$ for all $x > 0$ and $y_3(x) > y_1(x)$ for all $x > 1/K$ in the case (ii). Hence, the curve γ_3 is located not below γ_2 and strictly above γ_1 in $\hat{\Gamma}_{+,+}$ (see Figure 3). Notice that if $r = \rho$ then γ_3 coincide with γ_2 while if $r < \rho$ then γ_3 lies strictly above γ_2 in $\hat{\Gamma}_{+,+}$.

The curve γ_3 is a boundary of the convex closed set $\operatorname{epi} y_3 = \{(x,\lambda)\colon \lambda \leq y_3(x), x > 0\}$. Let $(\tilde{x},\tilde{\lambda})$ be an arbitrary point on γ_3. Then

$$\mathbf{n}(\tilde{x},\tilde{\lambda}) = \left(\frac{1}{\rho\tilde{x}^2}, -1\right)^*$$

is a normal vector to the set $\operatorname{epi} y_3$ at the point $(\tilde{x},\tilde{\lambda}) \in \gamma_3$. Multiplying $\mathbf{n}(\tilde{x},\tilde{\lambda})$ by vector

$$\mathbf{f}(\tilde{x},\tilde{\lambda}) = \left(-r\tilde{x} - \frac{1}{\tilde{\lambda}} + a, (\rho + r)\tilde{\lambda} + \frac{2}{\tilde{x}}\right)^*$$

of the right hand side of (41) at the point $(\tilde{x},\tilde{\lambda})$ and taking into account that $\tilde{\lambda} = -1/(\rho\tilde{x})$ we get

$$\begin{aligned}\langle \mathbf{n}(\tilde{x},\tilde{\lambda}), \mathbf{f}(\tilde{x},\tilde{\lambda})\rangle &= \frac{1}{\rho\tilde{x}^2}\left(-r\tilde{x} - \frac{1}{\tilde{\lambda}} + a\right) - (\rho + r)\tilde{\lambda} - \frac{2}{\tilde{x}} \\ &= \frac{-r\tilde{x}}{\rho\tilde{x}^2} + \frac{\rho\tilde{x}}{\rho\tilde{x}^2} + \frac{a}{\rho\tilde{x}^2} + \frac{\rho + r}{\rho\tilde{x}} - \frac{2}{\tilde{x}} = \frac{-r + \rho + (\rho + r) - 2\rho}{\rho\tilde{x}} + \frac{a}{\rho\tilde{x}^2} = \frac{a}{\rho\tilde{x}^2} > 0.\end{aligned}$$

This implies that any trajectory $(x(\cdot),\lambda(\cdot))$ of system (41) can intersect curve γ_3 only one time and only in the upward direction.

Due to (42) a trajectory $(x_*(\cdot),\lambda_*(\cdot))$ of system (41) that corresponds to the optimal process $(x_*(\cdot),u_*(\cdot))$ lies strictly above γ_3. Since the system (41) is autonomous by virtue of the theorem on uniqueness of a solution of first-order ordinary differential equation (see [20, Chapter 3]) trajectories of system (41) that lies above γ_3 do not intersect the curve $\gamma_4 = \{(x,\lambda)\colon x = x_*(t), \lambda = \lambda_*(t), t \geq 0\}$ which is the graph of the trajectory $(x_*(\cdot),\lambda_*(\cdot))$.

Further, trajectory $(x_*(\cdot),\lambda_*(\cdot))$ is defined on infinite time interval $[0,\infty)$. This implies that all trajectories $(x_{x_0,\beta}(\cdot),\lambda_{x_0,\beta}(\cdot))$, $\beta \in (-1/(\rho x_0), \lambda_*(0))$, are also defined on the whole infinite time interval $[0,\infty)$, i.e. $T_{x_0,\beta}=\infty$ for all $\beta\in(-1/(\rho x_0),\lambda_*(0))$. Indeed, for arbitrary $\tilde\beta$, $\tilde\beta\in(-1/(\rho x_0),\lambda_*(0))$, consider the corresponding trajectory $\{(x_{x_0,\tilde\beta}(\cdot),\lambda_{x_0,\tilde\beta}(\cdot)\}$ on its maximal time interval of definition $[0,T_{x_0,\tilde\beta})$ with $T_{x_0,\tilde\beta}\le\infty$.

The trajectory $\{(x_{x_0,\tilde\beta}(\cdot),\lambda_{x_0,\tilde\beta}(\cdot))\}$ lies strictly below the curve γ_4. Hence, due to estimate (28) for all $t\ge 0$ such that $x_{x_0,\tilde\beta}(t)>1/K$ we have

$$-\frac{1}{\lambda_{x_0,\tilde\beta}(t)x_{x_0,\tilde\beta}(t)}\le 2(\rho+r).$$

Hence, (see (41))

$$\begin{aligned}\dot x_{x_0,\tilde\beta}(t)&\overset{\text{a.e.}}{=}-rx_{x_0,\tilde\beta}(t)-\frac{1}{x_{x_0,\tilde\beta}(t),\lambda_{x_0,\tilde\beta}(t)}+a\\&\le -rx_{x_0,\tilde\beta}(t)+2(r+\rho)x_{x_0,\tilde\beta}(t)+a=(r+2\rho)x_{x_0,\tilde\beta}(t)+a,\qquad t\in[0,T_{x_0,\tilde\beta}).\end{aligned}$$

This implies

$$\lim_{t\to T_{x_0,\tilde\beta}}x_{x_0,\tilde\beta}(t)\le e^{(r+2\rho)T_{x_0,\tilde\beta}}\left(x_0+\frac{a}{(r+2\rho)}\left[1-e^{-(r+2\rho)T_{x_0,\tilde\beta}}\right]\right).$$

If $T_{x_0,\tilde\beta}<\infty$ then we should have

$$\lim_{t\to T_{x_0,\tilde\beta}}\lambda_{x_0,\tilde\beta}(t)=0.$$

But this contradicts to the fact that trajectory $\{(x_{x_0,\tilde\beta}(\cdot),\lambda_{x_0,\tilde\beta}(\cdot))\}$ does not intersect the curve γ_4 on $[0,T_{x_0,\tilde\beta})$. So, $T_{x_0,\tilde\beta}=\infty$.

Thus, we have proved that there is a nonempty set (a continuum) of trajectories $\{(x_{x_0,\beta}(\cdot),\lambda_{x_0,\beta}(\cdot))\}$, $\beta\in(-1/(\rho x_0),\lambda_*(0))$, $t\in[0,\infty)$, of system (41) lying strictly between the curves γ_3 and γ_4. All these trajectories are defined on the whole infinite time interval $[0,\infty)$ and, hence, all of them correspond to some admissible pairs $\{(x_{x_0,\beta}(\cdot),u_{x_0,\beta}(\cdot))\}$. Since these trajectories are located above γ_3 they satisfy also the estimate (42).

Consider the current value Hamiltonian $M(\cdot,\cdot)$ for (x,λ) lying above γ_3 in $\hat\Gamma_{+,+}$ (see (44)):

$$\begin{aligned}M(x,\lambda)&=\sup_{u\ge\rho}\{u\lambda x+\ln u\}+(a-rx)\lambda-lnx\\&=-1-\ln(-\lambda x)+(a-rx)\lambda-\ln x,\qquad -\frac{1}{\rho x}<\lambda<0.\end{aligned}\tag{52}$$

For any trajectory $(x_{x_0,\beta}(\cdot),\lambda_{x_0,\beta}(\cdot))$ of system (41) lying above γ_3 in $\hat\Gamma_{+,+}$ we have

$$x_{x_0,\beta}(t)\ge e^{(\rho-r)t}x_0,\quad t\ge 0.$$

On the other hand for any trajectory $(x_{x_0,\beta}(\cdot),\lambda_{x_0,\beta}(\cdot))$ of system (41) lying between γ_3 and γ_4 in $\hat\Gamma_{+,+}$ we have

$$\frac{1}{2(r+\rho)}<-\lambda_{x_0,\beta}(t)x_{x_0,\beta}(t)<\frac{1}{\rho}\quad\text{if}\quad x_{x_0,\beta}(t)>\frac{1}{K}.$$

These imply that for any trajectory $(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))$ of system (41) lying between γ_3 and γ_4 in $\hat{\Gamma}_{+,+}$ and for corresponding adjoint variable $\psi_{x_0,\beta}(\cdot)$, $\psi_{x_0,\beta}(t) = e^{-\rho t}\lambda_{x_0,\beta}(t)$, $t \geq 0$, we have

$$\lim_{t\to\infty} H(t, x_{x_0,\beta}(t), \psi_{x_0,\beta}(t)) = \lim_{t\to\infty} \left\{ e^{-\rho t} M(x_{x_0,\beta}(t), \lambda_{x_0,\beta}(t)) \right\} = 0.$$

Hence, for any such trajectory $(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))$ of system (41) we have (see (45))

$$M(x_{x_0,\beta}(t), \lambda_{x_0,\beta}(t)) = \rho e^{\rho t} \int_t^\infty e^{-\rho s} g(x_{x_0,\beta}(t), \lambda_{x_0,\beta}(t))\, ds, \qquad t \geq 0.$$

Let $u_{x_0,\beta}(\cdot)$ be the control corresponding to $x_{x_0,\beta}(\cdot)$, i.e. $u_{x_0,\beta}(t) = -1/(x_{x_0,\beta}(t)\lambda_{x_0,\beta}(t))$. Then taking in the last equality $t = 0$ we get

$$J(x_{x_0,\beta}(\cdot), u_{x_0,\beta}(\cdot)) = \int_0^\infty e^{-\rho s} g(x_{x_0,\beta}(t), \lambda_{x_0,\beta}(t))\, ds = \frac{1}{\rho} M(x_{x_0,\beta}(0), \lambda_{x_0,\beta}(0)).$$

For any $t \geq 0$ function $M(x_*(t), \cdot)$ (see (52)) increases on $\{\lambda\colon -1/(\rho x_*(t)) < \lambda < 0\}$. Hence, $M(x_*(t), \cdot)$ reaches its maximal value in λ on the set $\{\lambda\colon -1/(\rho x) < \lambda \leq \lambda_*(t)\}$ at the point $\lambda_*(t)$ that correspond to the optimal path $x_*(\cdot)$. Thus, all trajectories $(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))$ of system (41) lying between γ_3 and γ_4 in $\hat{\Gamma}_{+,+}$ do not correspond to optimal processes in $(P3)$.

From this we can also conclude that all trajectories $(x(\cdot), \lambda(\cdot))$ of system (41) lying above γ_4 also do not correspond to optimal processes in $(P3)$. Indeed, if such trajectory $(x(\cdot), \lambda(\cdot))$ corresponds to an optimal process $(x(\cdot), u(\cdot))$ in $(P3)$ then it must satisfy to condition (45). But in this case we have $\lambda(0) > \lambda_*(0)$ and

$$J(x(\cdot), u(\cdot)) = \frac{1}{\rho} M(x_0, \lambda(0)) = \frac{1}{\rho} M(x_0, \lambda_*(0)) = J(x_*(\cdot), \lambda_*(\cdot)),$$

that contradicts the fact that function $M(x_0, \cdot)$ increases on $\{\lambda\colon -1/(\rho x) < \lambda < 0\}$.

Thus, for any initial state x_0 there is a unique optimal process $(x_*(\cdot), u_*(\cdot))$ in $(P3)$ in the case (ii). The corresponding current value adjoint variable $\lambda_*(\cdot)$ is also defined uniquely as the maximal negative solution to equation (see (41))

$$\dot{\lambda}(t) = (\rho + r)\lambda(t) + \frac{2}{x_*(t)} \tag{53}$$

on the whole infinite time interval $[0, \infty)$. The solution $(x_*(\cdot), \lambda_*(\cdot))$ of system (41) lies in the set $\hat{\Gamma}_{+,+}$ in this case.

The function $x_*(\cdot)$ monotonically increases on $[0, \infty)$. Therefore, the time can be uniquely expressed in terms of the trajectory $x_*(\cdot)$ as a smooth function $t = t_*(x)$, $x \in (0, \infty)$. Changing the time variable $t = t_*(x)$ on interval $(0, \infty)$, we find that the function $\lambda_0(x) = \lambda_*(t_*(x))$ is solution to the differential equation (49) on the infinite interval $(0, \infty)$.

Using solution $\lambda_0(\cdot)$ of differential equation (49) along with (43) we can get an expression for the optimal feedback law as follows

$$u_*(x) = -\frac{1}{\lambda_0(x) x}, \qquad x > 0.$$

This means that in order to find the optimal feedback, we must be able for an initial state $x_0 > 0$ to determine the corresponding initial state $\lambda_0 < 0$ such that solution $(x_*(\cdot), \lambda_*(\cdot))$ of system (41) with initial conditions $x(0) = x_0$ and $\lambda(0) = \lambda_0$ exists on $[0, \infty)$ and $\lambda_*(\cdot)$ is the maximal negative function among all such solutions.

Let us summarize the results obtained in this section in the following theorem.

Theorem 3. *For any initial state $x_0 > 0$ there is a unique optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ in problem $(P3)$, and there is a unique adjoint variable $\psi(\cdot)$ that corresponds $(x_*(\cdot), u_*(\cdot))$ due to the maximum principle (Theorem 2).*

If $r > \rho$ then there is a unique equilibrium $(\hat{x}, \hat{\lambda})$ (see (46)) in the corresponding current value Hamiltonian system (41) and the optimal synthesis in problem $(P3)$ is defined as follows

$$u_*(x) = \begin{cases} -\frac{1}{\lambda_-(x)x}, & \text{if } x < \hat{x}, \\ \frac{r+\rho}{2}, & \text{if } x = \hat{x}, \\ -\frac{1}{\lambda_+(x)x}, & \text{if } x > \hat{x}, \end{cases}$$

where $\lambda_-(\cdot)$ and $\lambda_+(\cdot)$ are the unique solutions of (49) that satisfy the boundary conditions (50) and (51) respectively. In this case optimal path $x_*(\cdot)$ is either monotonically decreasing, or increasing or $x_*(t) \equiv \hat{x}$, $t \geq 0$, depending on the initial state x_0. For any optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ we have $\lim_{t\to\infty} x_*(t) = \hat{x}$ and $\lim_{t\to\infty} u_*(t) = \hat{u}$ (see (47)).

If $r \leq \rho$ then there is no equilibrium in the corresponding current value Hamiltonian system (41). For any initial state x_0 the corresponding optimal path $x_*(\cdot)$ in problem $(P3)$ is an increasing function, $\lim_{t\to\infty} x_*(t) = \infty$, and the corresponding optimal control $u_*(\cdot)$ satisfies asymptotically to the Hotelling rule of optimal depletion of an exhaustible resource [21], i.e. $\lim_{t\to\infty} u_*(t) = \rho$ in this case. The corresponding current value adjoint variable $\lambda_*(\cdot)$ is defined uniquely as the maximal negative solution to equation (53) on $[0, \infty)$. The corresponding optimal synthesis is defined as

$$u_*(x) = -\frac{1}{\lambda_0(x)x}, \qquad x > 0,$$

where $\lambda_0(x) = \lambda_*(t_*(x))$ is the corresponding solution of (49).

In the next section we discuss the issue of sustainability of optimal paths for different values of the parameters in the model.

5 Conclusion

Following Solow [28] we assume that the knowledge stock $A(\cdot)$ grows exponentially, i.e. $A(t) = A_0 e^{\mu t}$, $t \geq 0$, where $\mu \geq 0$ and $A_0 > 0$ are a constant growth rate and an initial knowledge stock respectively.

Similar to Valente [29] we say that a process $(S(\cdot), u(\cdot))$ is *sustainable* in our model if the corresponding instantaneous utility $\ln Y(\cdot)$ (see (1) and (2)) is a non-decreasing function of time in the long run, i.e.

$$\lim_{T\to\infty}\inf_{t\geq T}\frac{d}{dt}\ln Y(t)=\lim_{T\to\infty}\inf_{t\geq T}\frac{\dot{Y}(t)}{Y(t)}\geq 0.$$

Substituting $Y(t) = A(t)\,(u(t)S(t))^{\alpha}$, $A(t) = A_0 e^{\mu t}$, $t \geq 0$, in the last inequality we get the following characterization of sustainability of the process $(S(\cdot), u(\cdot))$:

$$\mu+\alpha\lim_{T\to\infty}\inf_{t\geq T}\left[\frac{\dot{u}(t)}{u(t)}+\frac{\dot{S}(t)}{S(t)}\right]\geq 0. \tag{54}$$

We call a process $(S(\cdot), u(\cdot))$ *strongly sustainable* if it is sustainable and, moreover, the resource stock $S(\cdot)$ is non vanishing in the long run, i.e.

$$\lim_{T\to\infty}\inf_{t\geq T} S(t)=S_{\infty}>0. \tag{55}$$

Consider case (i) when $r > \rho$. In this case there is a unique optimal equilibrium process (see (46) and (47)) in the problem:

$$\hat{u}(t)\equiv\hat{u}=\frac{r+\rho}{2},\quad \hat{S}(t)\equiv\hat{S}=\frac{(r-\rho)K}{2r}>0,\qquad t\geq 0.$$

Obviously, condition (54) holds for the optimal equilibrium process $(\hat{S}(\cdot), \hat{u}(\cdot))$ in this case. Due to Theorem 3 for any initial state S_0 the corresponding optimal path $S_*(\cdot)$ approaches asymptotically to the optimal equilibrium state $\hat{S}$ (from the left or from the right) while the corresponding optimal exploitation rate $u_*(\cdot)$ approaches asymptotically to the optimal equilibrium control $\hat{u}$. Hence, both conditions (54) and (55) are satisfied. Thus any optimal process $(S_*(\cdot), u_*(\cdot))$ is strongly sustainable in our model in this case.

Consider case (ii) when $r \leq \rho$. In this case due to Theorem 3 for any initial state S_0 the corresponding optimal control $u_*(\cdot)$ asymptotically satisfies the Hotelling rule of optimal depletion of an exhaustible resource [21], i.e. $\lim_{t\to\infty} u_*(t) = \rho$. The corresponding optimal path $S_*(\cdot)$ is asymptotically vanishing, and we have

$$\lim_{t\to\infty}\frac{\dot{S}_*(t)}{S_*(t)}=\lim_{t\to\infty}\left\{r-\rho-\frac{rS_*(t)}{K}\right\}=r-\rho.$$

Hence,

$$\lim_{t\to\infty}\frac{d}{dt}\ln\,Y(t)=\mu+\alpha\lim_{t\to\infty}\left\{\frac{\dot{u}_*(t)}{u_*(t)}+\frac{\dot{S}_*(t)}{S_*(t)}\right\}=\mu+\alpha(r-\rho).$$

Thus, in the case (ii) the sustainability condition (54) takes the following form:

$$\frac{\mu}{\alpha}+r\geq\rho. \tag{56}$$

Notice that in the case $\alpha = 1$ condition (56) coincides with Valente's necessary condition for sustainability in his capital-resource model with a renewable resource growing exponentially (see [29]).

Since in the case (i) condition (56) holds automatically we conclude that the inequality $\mu/\alpha + r \geq \rho$ is a necessary and sufficient condition (a criterion) for sustainability of the optimal process $(S_*(\cdot), u_*(\cdot))$ in our model while the stronger inequality $r > \rho$ gives a criterion of its strong sustainability.

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