Working Paper

ON THE CHOICE OF MODELS FOR PUBLIC FACILITY LOCATION

Donald Erlenkotter

April 1980
WP-80-47

International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

ON THE CHOICE OF MODELS FOR PUBLIC FACILITY LOCATION

Donald Erlenkotter

April 1980
WP-80-47

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

FOREWORD

The public provision of urban facilities and services often takes the form of a few central supply points serving a large number of spatially dispersed demand points: for example, hospitals, schools, libraries, and emergency services such as fire and police. A fundamental characteristic of such systems is the spatial separation between suppliers and consumers. No market signals exist to identify efficient and inefficient geographical arrangements, thus the location problem is one that arises in both East and West, in planned and in market economies.

This problem is being studied at IIASA by the Public Facility Location Task (formerly the Normative Location Modeling Task) which started in 1979. The expected results of this Task are a comprehensive state-of-the-art survey of current theories and applications, an established network of international contacts among scholars and institutions in different countries, a framework for comparison, unification, and generalization of existing approaches, as well as the formulation of new problems and approaches in the field of optimal location theory.

This paper is a result of collaboration between the Human Settlements and Services Area and the Resources and Environment Area which is hosting Professor Erlenkotter at IIASA. The author argues that for a large class of public sector location problems suitably modified private sector models perform better than typical public sector models.

Oleg Vasiliev
Chairman
Resources and
Environment Area

Andrei Rogers
Chairman
Human Settlements
and Services Area

ABSTRACT

Public sector facility location models have been defined as those that minimize client costs for a given level of service subject to a public budget constraint, whereas private sector models are those that minimize the total costs for meeting fixed client demands. We show that a slight reformulation of a typical public sector location model is both superior to the original model and equivalent to a typical private sector formulation. Thus, for the class of problems considered, a standard model type is appropriate regardless of the institutional context.

CONTENTS

1. INTRODUCTION, 1

2. PUBLIC AND PRIVATE SECTOR MODELS, 2

3. THE CHOICE OF A MODEL, 5

4. AN ILLUSTRATIVE EXAMPLE, 7

5. CONCLUDING REMARKS, 9

REFERENCES, 11

ON THE CHOICE OF MODELS FOR PUBLIC FACILITY LOCATION

D. Erlenkotter

1. INTRODUCTION

Traditionally a distinction has been drawn between public and private facility location models (ReVelle, Marks, and Liebman, 1970; Swain, 1974). According to this differentiation, public sector models typically have the objective of minimizing client costs for a given level of service subject to a public budget constraint, while private sector models seek to minimize the total costs for meeting specified fixed demands. The purpose of this note is to show for a large class of public sector location problems that so-called public sector models are economically and logically inferior to models of the private sector type.

The particular class of location problems that we address involves the public provision of what are essentially "private" goods or services; we do not pretend to have models that address all the aspects of location of "public" goods raised by Teitz (1968), Schuler and Holahan (1977), and Lea (1979). However, as noted by Schuler and Holahan (1977), many public sector location problems and models involve provision of goods or services that are really "private" in nature in that a client travels to a facility location to receive a well-defined "quantity" of a good or service. Thus this category of location problems seems worthy of attention.

2. PUBLIC AND PRIVATE SECTOR MODELS

A standard "public sector" facility location model is:

$$\underset{\substack{x_{ij} \varepsilon \{0,1\} \\ y_j \varepsilon \{0,1\}}}{\text{Minimize}} \quad Z = \sum_{i \varepsilon I} P_i \sum_{j \varepsilon J} a_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{j \varepsilon J} x_{ij} = 1 \quad , \quad i \varepsilon I \tag{2}$$

$$x_{ij} \leq y_j \quad , \quad i \varepsilon I \quad , \quad j \varepsilon J \tag{3}$$

$$\sum_{j \varepsilon J} f_j y_j + \sum_{j \varepsilon J} b_j \sum_{i \varepsilon I} P_i x_{ij} \leq B \tag{4}$$

where

P_i is the population of clients at location $i \varepsilon I$;

a_{ij} is the travel cost per client from location $i \varepsilon I$ to facility location $j \varepsilon J$;

x_{ij} is the fraction of the population of clients at $i \varepsilon I$ that receive service at facility location $j \varepsilon J$;

y_j is 1 if a facility is opened at location $j \varepsilon J$ and 0 otherwise;

f_j is the fixed cost for opening a facility at location $j \varepsilon J$;

b_j is the variable capacity and service cost per client served at facility location $j \varepsilon J$;

B is the total budget for facility costs.

The objective (1) is to minimize total client costs, subject to the constraints that ensure (2) that all clients receive service, (3) that service is provided only from facilities that are open, and (4) that the total budget for facilities is not exceeded.

Formulations of the type of (1)-(4) have been addressed in particular by Rojeski and ReVelle (1970), who provide an approximate solution approach based on linear programming, and Hansen and Kaufman (1974, 1976), who develop branch-and-bound approaches that guarantee satisfaction of constraint (4). These authors also include constraints that ensure assignment of clients to a "nearest" open facility with minimum a_{ij}; in the absence of explicit constraints of this type, the influence of the budget limit (4) on the solution may cause violations of such assignments. However, for reasons that will become clear later in our discussion, we shall ignore such constraints here.

As noted by ReVelle and Swain (1970), a particularly simple and widely-studied version of the model (1)-(4) is obtained when $b_j \equiv b$ and $f_j \equiv f > 0$ for all $j \in J$. Constraint (4) then becomes

$$\sum_{j \in J} y_j \leq \left[B - b \sum_{i \in I} P_i \right] / f \qquad (5)$$

which, if we define p as the integer part of the right-hand side of (5), is equivalent to

$$\sum_{j \in J} y_j \leq p \quad . \qquad (6)$$

The model (1)-(3), (6), which seeks to locate a fixed number of facilities, represents the p-median problem of Hakimi (1964, 1965). Here the assignment of clients to the "nearest" open facility occurs naturally and special constraints are not needed. Several computational approaches for this model have been developed (ReVelle and Swain, 1970; Swain, 1974; Cornuejols, Fisher, and Nemhauser, 1977; Narula, Ogbu, and Samuelsson, 1977). Recently an elegant theory of approximate solution methods has been developed that includes this model as a particular case

(Nemhauser and Wolsey, 1978; Nemhauser, Wolsey, and Fisher, 1978), and efficient methods have been developed for solving problems with special structure (Kariv and Hakimi, 1979).

The "private sector" form of the uncapacitated facility location model is:

$$\underset{\substack{x_{ij} \in \{0,1\} \\ y_j \in \{0,1\}}}{\text{Minimize}} \quad Z' = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j y_j \tag{7}$$

subject to

$$\sum_{j \in J} x_{ij} = 1 \quad , \qquad i \in I \tag{8}$$

$$x_{ij} \leq y_j \quad , \qquad i \in I \quad , \quad i \in J \tag{9}$$

where the notation is as before with the addition that

c_{ij} is the total variable capacity, production, and transportation cost for meeting all of location $i \in I$'s demand from facility location $j \in J$.

The objective here is one of minimizing the total costs of meeting fixed demand at the various locations. In such private sector location models, the indices i and j often are interchanged in comparison with the order used here since the flow of goods is from the facility to the client location, but this does not affect the substance of the model.

The formulation (7)-(9) dates back to papers by Stollsteimer (1963), Manne (1964), and Balinski (1965). Properties of approximate solution procedures for this model have been derived by Cornuejols, Fisher, and Nemhauser (1977) and Nemhauser, Wolsey, and Fisher (1978). Erlenkotter (1978) presents a dual-based solution procedure and provides computational evidence of the superiority of this approach over several others.

3. THE CHOICE OF A MODEL

We shall now demonstrate that the model (1)-(4) is an inferior choice for public sector facility location, and that a better choice is equivalent to the private sector model (7)-(9). This argument is based on two characteristics of the model (1)-(4): first, service demands are perfectly inelastic and must be met; and second, the service is essentially a private good for which the quantity and associated cost are well-defined through (4).

Since service demand is inelastic, the quantity of service will not be affected if some price ψ_j is charged for each unit of service provided at location j. These service charges are added to the client costs in the objective (1), which becomes

$$\underset{\substack{x_{ij} \varepsilon \{0,1\} \\ y_j \varepsilon \{0,1\} \\ \psi_j}}{\text{Minimize}} \quad Z'' = \sum_{i \varepsilon I} P_i \sum_{j \varepsilon J} a_{ij} x_{ij} + \sum_{i \varepsilon I} P_i \sum_{j \varepsilon J} \psi_j x_{ij} \tag{10}$$

This requires, of course, that the client costs a_{ij} and the prices ψ_j be commensurable. But even in the simpler p-median model the role of analysis is to provide the decision-maker with trade-off options between user costs and the number of facilities (or size of facility budget), which implies that the issue of commensurability must be faced eventually. Thus there is reason for assuming it in the objective (10), though one certainly could perform a sensitivity analysis of the conversion rate between measures of client travel times and financial costs.

The service charges also are added as revenues to the right-hand side of the budget constraint (4), which becomes

$$\sum_{j \varepsilon J} f_j y_j + \sum_{j \varepsilon J} b_j \sum_{i \varepsilon I} P_i x_{ij} \leq B + \sum_{i \varepsilon I} P_i \sum_{j \varepsilon J} \psi_j x_{ij} \tag{11}$$

The resulting augmented public facility location model is given by the objective (10) and the constraints (2), (3), and (11). Notice that the objective value for the solution to this model is always at least as good as that for the original model (1)-(4), which corresponds to fixing $\psi_j = 0$ for all $j \in J$. Thus the model (10), (2), (3), and (11) is preferable to the original model.

Now, we observe that the constraint (11) must hold with equality in an optimal solution, since otherwise decreasing some ψ_j corresponding to an $x_{ij} > 0$ would decrease the objective in (10). We therefore may substitute from (11) into (10) to obtain

$$\underset{\substack{x_{ij} \in \{0,1\} \\ y_j \in \{0,1\}}}{\text{Minimize}} \quad Z'' = \sum_{i \in I} \sum_{j \in J} P_i \left[a_{ij} + b_j\right] x_{ij} + \sum_{j \in J} f_j y_j - B \tag{12}$$

The objective in (12) is directly equivalent to that for the private sector model in (7), with $c_{ij} = P_i\left[a_{ij} + b_j\right]$. Since the remaining constraints (2) and (3) in the improved public sector model are identical to (8) and (9) in the private sector model, we have demonstrated that the private sector model (7)-(9) is preferable to the public sector model (1)-(4) for solving public facility location problems.

The role of the prices in the improved public sector model may become clearer if we define the prices in relation to marginal costs as $\psi_j = \psi + b_j$. The effect of the constraint (11) is to remove just one degree of freedom from the price solution, so only a single variable ψ can be determined unambiguously by this constraint. From equality in (11), we find

$$\psi = \frac{\sum_{j \in J} f_j y_j - B}{\sum_{i \in I} P_i} \quad . \tag{13}$$

A positive value for ψ in (13) means a payment towards the fixed facility costs, while a negative payment implies a reimbursement from a budgetary excess. Substituting in (10) for the ψ_j so defined yields (12) directly.

It is clear that the model (12), (2), and (3) will always assign clients to an open facility with lowest value of $a_{ij} + b_j$, which also corresponds to the lowest value of the client's cost $\psi_j + a_{ij} = \psi + a_{ij} + b_j$. Thus there is no need here to impose "nearest facility" assignment constraints of the type discussed for the formulation (1)-(4).

Possible differences in timing between investment in facilities and collection of revenues from clients may be incorporated. The consequences of financing would be to discount the revenue term in the right-hand side of (11) by a factor α, $0 < \alpha \leq 1$. The cost coefficients f_j and b_j in (12) then are divided by α. The effective service price would be $\psi_j = \psi + b_j/\alpha$, where the right-hand side of (13) defining ψ is also divided by α.

Solutions for different values of α, or equivalently the conversion rate between client travel and facility costs, produce different efficient combinations for the two criteria of client costs and facility costs. Such a process is closely related to the use of a Lagrangian relaxation of the constraint (4) to solve the model (1)-(4), where the relaxed problem is equivalent to (7)-(9). As noted by Swain (1974) and Erlenkotter (1978), for some data solutions to (1)-(4) may not be attainable directly by such a procedure. The discussion here shows that such solutions are inherently suboptimal since they do not correspond to solutions of the improved model (12), (2), and (3).

4. AN ILLUSTRATIVE EXAMPLE

To illustrate the points made in the previous discussion, we shall examine a small example. Data for this example are given in Table 1; for simplicity we assume $b_j = 0$ for all $j \in J$. Solutions for the formulation (1)-(4) under different budget levels B are given in Table 2. (For $20 \leq B < 30$, the optimal solution could be taken as any two among the four possible facilities.)

For a budget $B = 20$, the optimal solution with the improved formulation (12), (2), and (3) is to open facilities 1, 2, and 3, with total client costs of 0 and total facility costs of 30. The budget deficit of 10 is made up by charges of $\psi = 10/3$ to each client. The net benefit to the clients is 10 in comparison with the original budget-constrained solution since client costs are reduced from 20 to 0 for an additional facility cost of 10.

Table 1. Data for illustrative example.

$i \backslash j$	a_{ij} 1	2	3	4	p_i
1	0	20	20	10	1
2	20	0	20	10	1
3	20	20	0	10	1
f_j	10	10	10	10	

Table 2. Budget-constrained solutions for example.

Budget range	Optimal facility set	Total client costs	Total facility costs
$10 \leq B < 20$	{4}	30	10
$20 \leq B < 30$	{1,4}	20	20
$30 \leq B$	{1,2,3}	0	30

If we explore the efficient combinations of client costs and facility costs by varying α, we find indifference between the solutions for facility set {4} and facility set {1,2,3} at $f_j/\alpha = 15$. For effective facility fixed charges below 15, the optimal facility set is {1,2,3}; for effective fixed charges above 15, the single facility {4} is opened. Solutions with two open facilities, as for B = 20 in Table 2, are never obtained for this example with the formulation (12), (2), and (3).

To see why two-facility solutions are inherently non-optimal here, consider a case where the fixed charges f_j are 20 for each $j \in J$ and the budget B is 40, just sufficient to open two facilities. The solution to formulation (1)-(4) will open two facilities, with total client costs of 20 and total facility costs of 40. The solution for (12), (2), and (3) will be to open just facility 4 with total client costs of 30 and facility costs of 20. The charge to each client is $\psi = -20/3$, i.e., a refund of excessive budget. The clients have gained a savings in facility costs of 20 at the expense of 10 in additional travel costs. Similar results are obtained for any $f_j \geq 15$ and corresponding budget amounts, whereas for $f_j \leq 15$ the results are as in the original example.

5. CONCLUDING REMARKS

As noted above, our demonstration of the superiority of the "private-sector" model for the solution of public facility location problems depends upon two key characteristics: the inelasticity of demand, and the private nature of the service or good provided. The latter condition certainly is limiting, but it seems appropriate for a significant number of problems. At present it appears too early to predict the ultimate outcome of the emerging development of location models for goods of a more "public" nature, and how these models will relate to those, either public or private, examined here.

The assumption of totally inelastic demand is of more concern. However, it is doubtful that relaxing this assumption would tilt the scales more towards models similar to the formulation (1)-(4) since, as discussed above, this formulation has no inherent capability for balancing client benefits against facility costs or providing guidance as to the prices for public goods or services. In fact, those public-sector models developed thus far that do incorporate price-elastic demands also can be converted to an equivalent "private-sector" formulation (7)-(9) (Wagner and Falkson, 1975; Erlenkotter, 1977).

REFERENCES

Balinski, M.L. (1965) Integer programming: methods, uses, computation. *Management Science* 12:253-313.

Cornuejols, G., M.L. Fisher, and G.L. Nemhauser (1977) Location of bank accounts to optimize float: an analytic study of exact and approximate algorithms. *Management Science* 23:789-810.

Erlenkotter, D. (1977) Facility location with price-sensitive demands: private, public, and quasi-public. *Management Science* 24:378-386.

Erlenkotter, D. (1978) A dual-based procedure for uncapacitated facility location. *Operations Research* 26:992-1009.

Hakimi, S.L. (1964) Optimum locations of switching centers and the absolute centers and medians of a graph. *Operations Research* 12:450-459.

Hakimi, S.L. (1965) Optimum distribution of switching centers in a communication network and some related graph-theoretic problems. *Operations Research* 13:462-475.

Hansen, P. and L. Kaufman (1974) An algorithm for the location of central facilities. *Mathematical Programs for Activity Analysis*, edited by P. Van Moeseke. Amsterdam: North-Holland Publishing Company.

Hansen, P. and L. Kaufman (1976) Public facilities location under an investment constraint. *Operational Research '75*, edited by K.B. Haley. Amsterdam: North-Holland Publishing Company.

Kariv, O. and S.L. Hakimi (1979) An algorithmic approach to network location problems. II: The p-medians. *SIAM Journal on Applied Mathematics* 37:539-560.

Lea, A.C. (1979) Welfare theory, public goods, and public facility location. *Geographical Analysis* 11:217-239.

Manne, A.S. (1964) Plant location under economies-of-scale - decentralization and computation. *Management Science* 11: 213-235.

Narula, S.C. , U.I. Ogbu, and H.M. Samuelsson (1977) An algorithm for the p-median problem. *Operations Research* 25:709-712.

Nemhauser, G.L. and L.A. Wolsey (1978) Best algorithms for approximating the maximum of a submodular set function. *Mathematics of Operations Research* 3:177-188.

Nemhauser, G.L., L.A. Wolsey, and M.L. Fisher (1978) An analysis of approximations for maximizing submodular set functions - I. *Mathematical Programming* 14:265-294.

ReVelle, C., D. Marks, and J.C. Liebman (1970) An analysis of private and public sector location models. *Management Science* 16:692-707.

ReVelle, C. and R.W. Swain (1970) Central facilities location. *Geographical Analysis* 2:30-42.

Rojeski, P. and C. ReVelle (1970) Central facilities location under an investment constraint. *Geographical Analysis* 2:343-360.

Schuler, R.E. and W.L. Holahan (1977) Optimal size and spacing of public facilities in metropolitan areas. *Papers of the Regional Science Association* 39:137-156.

Stollsteimer, J.F. (1963) A working model for plant numbers and location. *Journal of Farm Economics* 43:631-645.

Swain, R.W. (1974) A parametric decomposition approach for the solution of uncapacitated location problems. *Management Science* 21:189-198.

Teitz, M.B. (1968) Toward a theory of public facility location. *Papers of the Regional Science Association* 21:35-51.

Wagner, J.L. and L.M. Falkson (1975) The optimal nodal location of public facilities with price-sensitive demand. *Geographical Analysis* 7:69-83.