A DECISION MODEL FOR R & D EXPENDITURES: SOME REMARKS

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Manne and Marchetti [1] consider the following decision problem: Given that each line of research has an identical and independently distributed probability of success (1-p) and the cost associated with each line of research is a proportion c of the benefits B accruing if one or more of the lines turn out to be successful, determine the optimum number n^* of lines of research to be undertaken in order to maximize the expected value of benefits less costs. They show that this number is approximately $\frac{\text{Log}\,[c/-\text{Log}\,p]}{\text{Log}\,p}$. They also consider a sequential extension of this model.

It can be seen that n^* approaches zero if (1-p) the probability of success approaches either its <u>lower</u> bound c or its <u>upper</u> bound 1. This is easily established. The expected value of net benefits if n lines of research are simultaneously pursued is given by $f(n) = B[1 - p^n - cn]$. The marginal value of an additional experiment when n experiments are being pursued is $f(n+1) - f(n) = B[p^n(1-p) - c]$. This is a decreasing function of n. The optimum number n^* [for $c < 1-p$][1] is given by $\left[\frac{\text{Log}\left(\frac{c}{1-p}\right)}{\text{Log}\,p}\right]$ where $[x]$ denotes

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[1] The case $c \geq 1-p$ is uninteresting since $f(n) \leq 0$ for $n \geq 1$ and hence the optimum number of experiments is zero.

the largest integer less than or equal to x. [Since for values of p close to 1 we can approximate 1-p by -Log p we get $n^* \sim \frac{\text{Log}\,[c/-\text{Log}\,p]}{\text{Log}\,p}$]. Now as p → upper bound (1-c) it is clear that $n^* \to 0$. As $p \to 1$ also $n^* \to 0$ since $\lim_{p\to 1-0} \frac{\text{Log}\left(\frac{c}{1-p}\right)}{\text{Log}\,p} = 0$. In other words it does not pay to conduct many experiments if the probability of success is either too <u>low</u> relative to costs (the case of $1-p \to c$ from above) or sufficiently high (the case of $1-p \to 1$). However, and this is important to note, in one case the probability of success is very high and in the other very low, even though the expected net benefits are being maximized with few experiments.

The above argument leads on to a consideration of risk and attitudes towards risk. The expected net benefit maximizer is a risk neutral individual. In order to explore non-neutral attitudes to risk, two approaches are outlined here.

In the first one, risk is measured by the probability p^n of none of the lines of research succeeding when n experiments are being pursued. We then draw up a trade-off curve between expected net benefits and risk. Thus, denoting the risk measure p^n by π, we can express the expected net benefits $f(n) = B[1 - p^n - nc]$ as a function of π by writing

$$f(n) = g(\pi) = 1 - \pi - c\,\frac{\text{Log}\,\pi}{\text{Log}\,p} \quad . \tag{1}$$

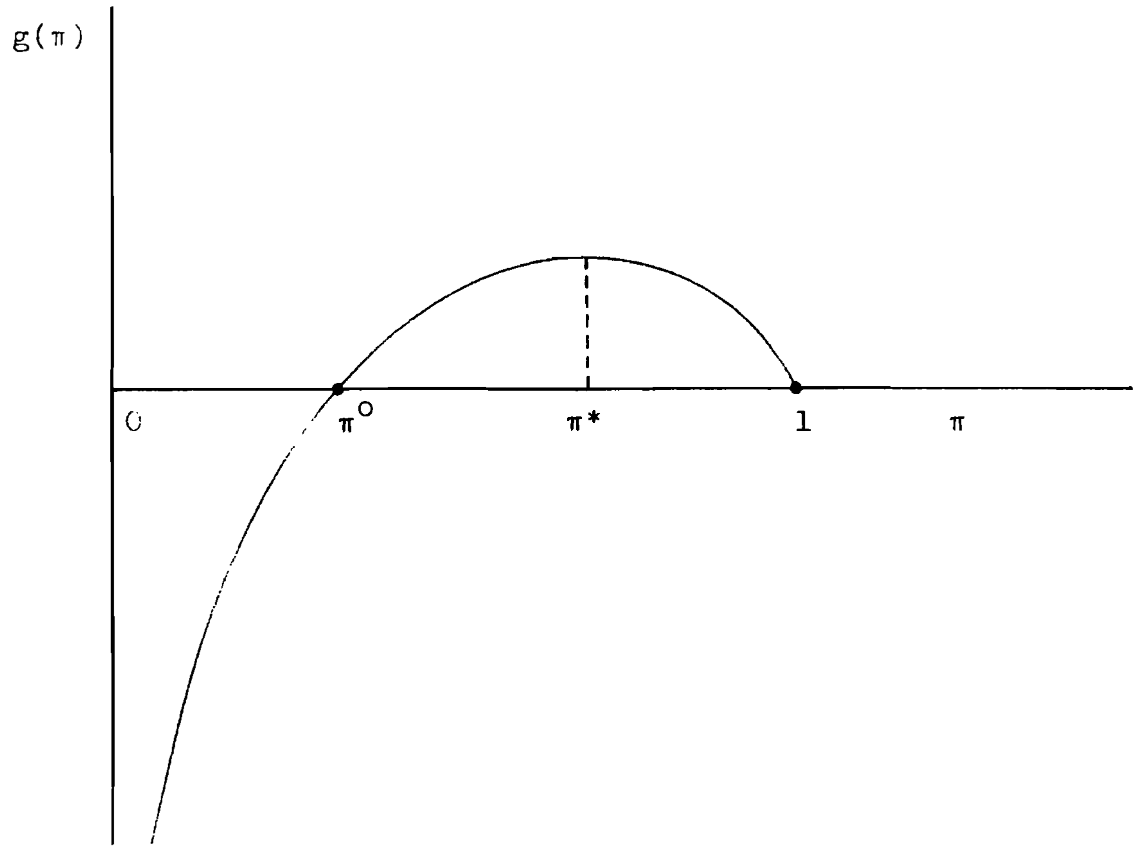

Figure 1

In Figure 1 we have drawn the graph $g(\pi)$ (which is concave in π) as a function π for the case $-\text{Log } p > c$ (corresponding to the condition $c < 1-p$). [If $-\text{Log } p \leq c$, the curve $g(\pi)$ never rises above the horizontal axis and as such expected net benefits are negative as long as any experimentation is undertaken at all!] The point $\{\pi^*, g(\pi^*)\}$ corresponds to expected benefit maximization while the point $\{\pi^o, 0\}$ corresponds to risk minimization subject to the condition that the expected net benefits are non-negative. There is a trade-off between risk and expected net benefit in the region (π^o, π^*). As long as expected net benefits are required to be non-negative and the utility function of the individual is non-decreasing in expected net benefits and non-increasing in risk, his choice is restricted to the interval (π_o, π^*). Any choice of $\pi < \pi^*$ will mean more experiments than n^* being conducted.

In the second approach we consider an individual whose current income is Y_o and utility function $U(Y)$. The case of linear $U(Y)$ corresponds to a risk neutral individual. A strictly concave (convex) U will correspond to risk averse (loving) individual. We confine ourselves here to a risk averse individual, i.e., $U(Y)$ is strictly concave in Y with positive marginal utilities. His problem now is to maximize his expected utility. His utility will be $U[Y_o - Bcn]$ if none of the n experiments succeed and $U[Y_o + B - Bcn]$ if at least one succeeds. Given the probabilities p^n and $1-p^n$ respectively of no success and at least one success, we get the expected utility as

$$EU = p^n U[Y_o - Bcn] + (1-p^n)\, U[Y_o + B - Bcn] \quad . \tag{2}$$

Treating n as a non-negative real number rather than a non-negative integer and differentiating we get the first order condition for maximization (for an interior solution) of EU as

$$\frac{dE}{dn} = -Bc\left[p^n U'(Z_n) + (1-p^n)\, U'(Z_n + B)\right] + p^n \operatorname{Log} p \left[U(Z_n) - U(Z_n + B)\right] = 0 \quad , \tag{3}$$

where $Z_n = Y_o - Bcn$. It can be varified that $\frac{d^2E}{dn^2} < 0$ when $\frac{dE}{dn} = 0$ so that we do indeed get a maximizing (in fact unique) solution with optimum $n > 0$ provided $\frac{dE}{dn} > 0$ at $n = 0$. This will hold as long as $-\text{Log } p > c\left[\frac{BU'(Y_o)}{U(Y_o + B) - U(Y_o)}\right] > c$.

Defining the failure probability p^n corresponding to the solution for n from the above equation as π^{**} and recalling that the expected net benefit maximizing failure probability π^* equals $\frac{c}{-\text{Log } p}$ we get on re-arranging $\frac{dE}{dn} = 0$

$$\pi^{**} = \pi^*\left[\frac{BU'(Z_n + B)}{\{U(Z_n + B) - U(Z_n)\} - B\pi^*\{U'(Z_n) - U'(Z_n + B)\}}\right] \quad . \qquad (4)$$

Unfortunately,[1/] even with the assumption of concavity of U, it is not possible in general to conclude anything about the relative magnitudes of π^{**} and π^*. However, the expectation that a risk averter will, in his optimum, choose a larger number of experiments (i.e., lower π^{**}) than the number n^* (and failure probability π^*) chosen by a risk neutral individual, is borne out if a quadratic approximation of $U(Z_n + B)$ at Z_n is good enough. In other words let

$$U(Z_n + B) \sim U(Z_n) + BU'(Z_n) + \frac{B^2}{2} U''(Z_n) \quad ,$$

$$U'(Z_n + B) \sim U'(Z_n) + BU''(Z_n) \quad .$$

[1/] By assuming an exponential utility function Jean-Pierre Ponssard is able to show that $\pi^{**} < \pi^*$. See Ponssard [2].

Then

$$\pi^{**} = \pi^{*}\left[\frac{U'(Z_n) + BU''(Z_n)}{U'(Z_n) + BU''(Z_n) + BU''(Z_n)(\pi^{*} - 0.5)}\right] . \qquad (5)$$

Under the reasonable assumption that $\pi^{*} < 0.5$, we see that $\pi^{**} < \pi^{*}$ or the risk averter will pursue more lines of research than a risk neutral individual.

We can go a little further without making further assumptions about U. We noted earlier that a risk neutral individual will undertake experimentation if and only if $\infty > -\text{Log } p > c$ whereas for a risk averse individual these inequalities turn out to be $\infty > -\text{Log } p > c\left\{\frac{BU'(Y_o)}{U(Y_o + B) - U(Y_o)}\right\} > c$. Thus if $c\left\{\frac{BU'(Y_o)}{U(Y_o + B) - U(Y_o)}\right\} \geq -\text{Log } p > c$, while a risk neutral individual will undertake some experiments, the risk averse one will not. Thus, for values of p close to its upper bound e^{-c}, the risk averter will conduct <u>fewer</u> experiments than the risk neutral individual.

Now as p tends to its lower bound namely zero, the optimum number of experiments chosen by both types of individuals tends to zero as is to be expected since with p close to zero the probability of success of a single experiment is close to 1. We have established this result for the risk neutral case already. For the case of risk avert individual, let us first note that his choice of π for any given p is restricted to $(\pi_o, 1)$ where π_o is that value of $\pi < 1$ which yields

$EU = U(Y_o)$, i.e., his choice of π (and hence the number of experiments n since $\pi = p^n$) should make him no worse off as compared to a situation in which he conducts no experiments and continues to enjoy his income of Y_o. Given that p is less than e^{-c}, it can be easily shown that a unique π_o less than unity yields $EU = U(Y_o)$.

Now

$$\frac{\partial EU}{\partial p} = p^n \text{ Log } p \left[U(Y_o - Bcn) - U(Y_o + B - Bcn)\right] > 0 \quad .$$

Hence as p decreases to zero, π_o increases to 1. This implies that π^{**} which lies between π_o and 1 tends to 1 as p tends to zero or the optimal number of experiments n^{**} tends to zero as p tends to zero.

Now from (4) we know

$$\frac{\pi^{**}}{\pi^*} = \frac{B}{\left\{\frac{U(Z_n + B) - U(Z_n)}{U'(Z_n + B)}\right\} - \pi^*\left\{\frac{U'(Z_n)}{U'(Z_n + B)} - 1\right\}} \quad .$$

Given strict concavity U, $\frac{U(Z_n + B) - U(Z_n)}{U'(Z_n + B)} > B$. As $p \to 0$, $\pi^* = \frac{-c}{\text{Log } p} \to 0$. Hence, provided $\frac{U'(Z_n)}{U'(Z_n + B)}$ is bounded above, $\frac{\pi^{**}}{\pi^*} < 1$ for values of p close to zero. Thus, for values of p close to zero, the optimal number of experiments conducted by a risk averter will be larger than the number conducted by a risk neutral individual.

The Manne-Marchetti model assumes that the failure probability of each line of research was the <u>same</u> and <u>independent</u> of others. It is perhaps more realistic to assume that there is some ordering of possible lines of research according to their (researcher's) subjective probability of success. Thus if n experiments are to be performed, then the first n experiments in the ordered set of possible experiments will be chosen. Let us maintain the independence assumption and postulate that the probability of <u>failure</u> of the k^{th} experiments in the ordered set is

$$p_k = 1 - (1-p)(1-\alpha)^{k-1} \quad , \qquad k = 1,2,\ldots$$

where $0 < p < 1$ and $0 < \alpha < 1$. With the independence assumption, the probability of none of the experiments succeeding when n experiments are performed is

$$\pi_n = \prod_{k=1}^{n} p_k \quad .$$

It is easily seen that $\alpha = 0$ corresponds to the Manne-Marchetti model. As can be varified $\lim_{k\to\infty} p_k = 1$ while $\lim_{n\to\infty} \pi_n = 0$ so that the probability of at least one experiment succeeding can be made arbitrarily close to 1 by choosing a sufficiently large n.

To keep matters simple let us confine ourselves to the case of a risk-neutral researcher. If cost per experiment is a constant proportion c of benefits B then he maximizes expected net benefits as given by

$$H(n) = B[1 - \pi_n - cn] \quad . \tag{6}$$

Now

$$H(n+1) - H(n) = B[\pi_n - \pi_{n+1} - c] \quad ,$$

$$\pi_n - \pi_{n+1} = \pi_n(1-p_{n+1}) = \pi_n(1-p)(1-\alpha)^n \quad . \tag{7}$$

Since π_n and $(1-\alpha)^n$ decrease as n increases, $H(n+1) - H(n)$ is a decreasing function of n. It is clear that for the optimal number of experiments to be at least one, $H(1) > 0$, that is $c < 1-p$, a condition identical to a similar condition in the Manne-Marchetti model. Assuming this to hold, the optimal number of experiments is given by $\hat{n}$ where

$$H(\hat{n}) - H(\hat{n}-1) \geqq 0 \tag{8}$$

and

$$H(\hat{n}+1) - H(\hat{n}) < 0 \quad . \tag{9}$$

It is easily seen that $\hat{n}$ is approximately the solution of

$$H(n) - H(n-1) = 0 \tag{10}$$

or

$$\pi_{(\hat{n}-1)}(1-p)(1-\alpha)^{\hat{n}-1} = c \quad . \tag{11}$$

We remarked earlier that $\alpha = 0$ corresponds to the Manne-Marchetti model. The effect of positive α on $\hat{n}$ is easily seen. For, an increase in α decreases both $(1-\alpha)^n$ and π_n for any <u>given</u> n. As such $\hat{n}$, the solution of $\pi_{\hat{n}-1}(1-p)(1-\alpha)^{\hat{n}-1} = c$ must <u>decrease</u> as α increases. This is to be expected since with an increase in α, the failure probability of every experiment other than the first in the ordered set is increased.

R E F E R E N C E S

[1] Manne, Alan S. and Marchetti, Cesare. "Hydrogen: Mechanisms and Strategies of Market Penetration." Presented at The Hydrogen Economy, Miami Energy Conference, March 1974. IIASA Research Report RR-74-4, Laxenburg, Austria.

[2] Ponssard, Jean-Pierre. "Research and Development Modeled as a Sampling Process: Some Further Considerations." IIASA Working Paper, July 1974, Laxenburg, Austria.