CONTINUOUS FLOW MODELING IN REGIONAL SCIENCE

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PREFACE

Although discontinuous methods of regional problem analysis are now widely used, in some cases continuous models might be useful. Professor T. Puu's work is an introduction to the fundamental tools of continuous-flow modeling. Numerous possible implementations of this approach are analyzed in the paper.

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The continuous approach to regional modelling, with which Martin Beckmann and I have been working, should not be regarded as a single model. Rather, it is a set of models, developed for various purposes, or, equivalently, a philosophy about model building. There are many mathematical issues involved; concerning existence, uniqueness, dynamic adjustments etc. which I cannot comment on now. Neither can I discuss the various applications: to productive specialization, road investment planning, natural resource extraction, and water supply management with which we have so far been concerned.

Therefore, I will start by presenting the fundamental tools of continuous flow modelling and finish by presenting a single application.

First, however, I have to say a few words about the reasons for using a continuous approach. The common starting point in discrete spatial analysis is a subdivision of the region considered into a collection of subregions labelled by a single index, something like Figure 1, and to study various interrelations in terms of some matrix $[a_{ij}]$, representing commuting trips, migration, or whatever that can be specified for a pair of subregions.

In view of this approach, we could as well tear the whole picture into pieces and forget the shapes of the various subregions. This geometric information will not be need at all.

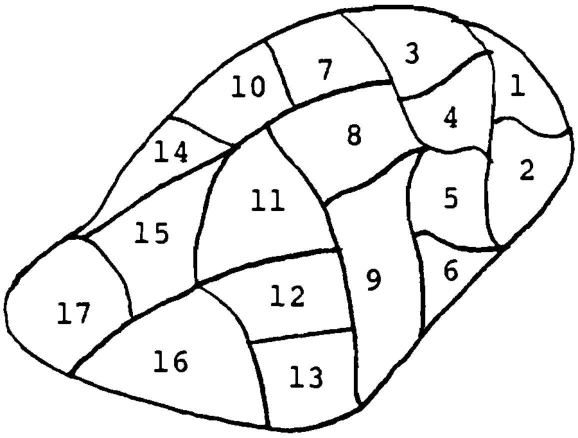

Figure 1. Discrete subdivision of space

As a result, we can never assemble the puzzle again, but this is no problem in the discrete approach.

In our opinion, it is a pity to give up all geometric information at the very outset. By appealing to well-developed concepts from physics and vector analysis we need not do this.

This is the exact background for the continuous model.

Objections could be that: firstly, any empirical data available concern discrete subdivisions, and secondly the design of computation algorithms would again force discretization upon us.

About these objections could be said that the case is not in any way different from the situation in physics. No physicist has yet been able to measure the pressure of a fluid at all points of the flow. He could not, even if he were able to make an infinity of experiments. As a continuum is not denumerable, he could not arrange the experiments in any order, even if he had infinite life-time, and moreover, he would have to carry out all the experiments simultaneously! In despite of this continuous flow modelling has been a valuable tool in hydrodynamics. And the spatial economic phenomena bear a sufficient likeness to physical phenomena to make it profitable to use analagous methods, perfected through ages.

It should be stressed, however, that insights obtainable from this approach are in no way thought as replacements of discrete methods, only as complementary information.

Let me now describe the fundamentals of continuous modelling. In a discrete model, comprised by a set of nodes connected by a set of arcs, we deal with two types of information: densities at the nodes (population, average income, etc), and flows along the arcs (commodities, commuters, messages, etc.).

The information in continuous models is the same. Only, densities vary continuously in space, from one location to another, and flows do the same, changing direction as well as magnitude.

The main principles of continuous modelling are two.

(i) The transportation facilities are not represented by a graph or network. Rather, we fix for each location a cost of movement across it. If transportation facilities are good, the cost is low, if they are bad, the cost is high. Transportation costs between any pair of locations are then obtainable as path integrals of this local transportation cost function and the optimum routing problem can be solved as a well-defined variational problem in terms of an appropriate Euler equation. To make things particularly simple we can let the local cost function be isotropic, i.e., direction-independent. This, naturally, leads to the main departure from real networks, where only a few transit directions are at all possible. But, taking a macroscopic view, the abstraction is no worse than most abstractions we are forced to in science.

(ii) The second principle is simply a continuity equation, relating the changes of flow densities to local sources and sinks. This principle will be explained later on in more detail.

While the first principle is borrowed from geometrical optics and corresponds to Fermat's principle, the second one comes from hydrodynamics and corresponds to the continuity equation of an incompressible fluid.

We need a few mathematical concepts to make the ideas more precise. These are:

1) The <u>vector field</u>. A vector field is a mapping

$$\phi = (\phi_1(x,y),\ \phi_2(x,y)). \qquad (1)$$

With each location x,y in the region is associated a vector ϕ, whose direction, the unit vector

$$\phi/|\phi| = (\cos\theta,\ \sin\theta), \qquad (2)$$

represents the <u>direction</u> of flow, and whose magnitude, the euclidean norm

$$|\phi| = \sqrt{\phi_1^2 + \phi_2^2}\ , \qquad (3)$$

represents the <u>intensity</u> of the flow. Hence, the vector field and continuous flow concepts are equivalent.

In Figure 2 is illustrated a vector field in terms of arrows that represent flow direction and magnitude (length) by a set of arrows. The continuous flow lines too are illustrated.

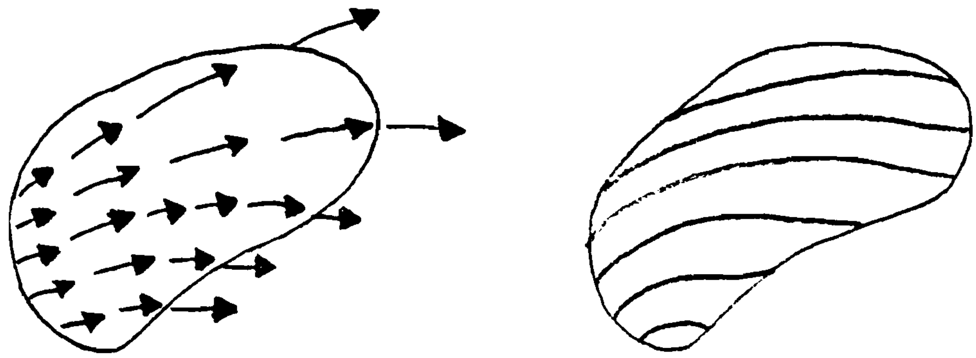

Figure 2. Vector field and flow lines on a a region.

2) We also need the concept of the <u>gradient</u> of a <u>potential</u> function. If we consider the potential p(x,y) as a surface in three-dimensional x,y,p-space, then

$$\text{grad } p = (p_x, p_y) \qquad (4)$$

is a vector in the direction of steepest ascent of this surface, its magnitude being the rate of increase in this direction. The symbols p_x and p_y, of course, denote partial derivatives.

3) For the divergence we can put down the following formal definition:

$$\text{div}\ \phi = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} \quad . \tag{5}$$

The sum of the partial derivatives of the first flow component with respect to the first space co-ordinate and of the second flow component with respect to the second space co-ordinate may seem an arbitrarily defined operator. But, we will immediately understand its meaning by stating:

4) Gauss's divergence theorem. This is probably the most important tool in continuous flow modelling. It states:

$$\iint_S \text{div}\ \phi\ dxdy = \oint_{\partial S} (\phi)n\ ds \quad . \tag{6}$$

The double integral of the divergence of a vector field on a bounded region S hence equals the path integral of the normal component of the field taken along the surrounding boundary ∂S. This normal component is denoted $(\phi)_n$, and is, of course, a scalar measure.

The case is illustrated in figure 3, where we have drawn the actual flow vectors on the boundary and also indicated the normal projections.

Hence, the right hand member of (6) gives net outflow from the region. The formula is of interest both in global and in local form. If we shrink S together to a point, then the left-hand member is the divergence at one single point, and the right-hand member is net outflow at this point. We hence obtain the interpretation:

divergence = net addition to flow.

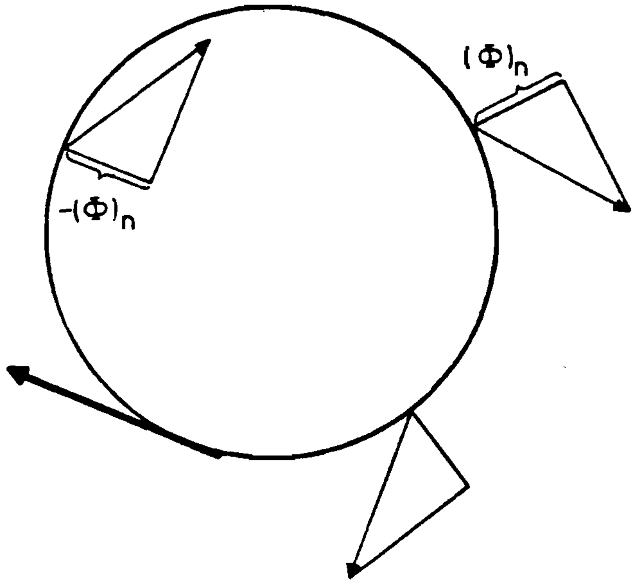

Figure 3. Flow vectors and normal projections on the boundary.

Hence, Gauss's theorem in the local version gives a neat interpretation to the divergence operator. We should also state that there is nothing mysterious about the divergence theorem. If we define $f(x) = dF(x)/dx$, then the fundamental theorem of calculus states that $\int_a^b f(x)dx = F(b) - F(a)$. So, the definite integral of a derivative is related to the value of the primitive function at the boundary. In our case, the integral is taken on an area, not an interval, and the boundary is a curve, not a pair of points. Otherwise, the theorems are equivalent and the divergence is, indeed, a sort of derivative.

Having these preliminaries in mind, we can state here fundamental principles of continuous flow modelling. They are:

A) The "divergence law". Let k,l,m be local employment of capital, labour and land service inputs and $f(k,l,m)$ be a production function. If a f denotes local consumption then the considerations above say that excess supply $f(k,l,m) - q$ enters the flow, if positive, or is withdrawn from it, if negative, by the divergence. Formally:

$$f(k,l,m) - q = \text{div}\ \phi \quad . \tag{7}$$

This states the continuity principle. We also have:

B) The "gradient law". Let the aforementioned local cost of transportation be h(x,y). Then:

$$h \frac{\phi}{|\phi|} = \text{grad}\ p \quad , \tag{8}$$

where p is product price. This equation tells two things:

a) flows move in the direction of steepest price increase,

and

b) in this direction price increases by transportation cost. (The latter is obvious as (8) implies $|\text{grad}\ p| = h$.)

Intuitively, this gradient law makes good economic sense. In a formal way we can derive it as a solution to the following optimization problem:

Local shipment quantity being $|\phi|$, and the cost per unit being h, we arrive at the local transportation cost $h|\phi|$. Total transportation costs obtained by aggregation over localities are:

$$\iint_S h|\phi|\ dxdy \quad . \tag{10}$$

Suppose these costs are minimized subject to the constraint (7). We associate a Lagrangean multiplier p with the constraint and put up the expression

$$\iint_S (h|\phi| + p(f-q-\text{div}\ \phi))dxdy \quad . \tag{11}$$

The appropriate Euler equation for (11) to be optimal with respect to the choice of the flow field ϕ is exactly equation (8), i.e. our "gradient law".

We will now illustrate the method by treating a production planning problem. Suppose we maximize

$$\iint U(q_1,\ldots,q_n,x,y)\,dxdy \quad , \qquad (12)$$

where $q_1,\ldots,q_n$ are different consumers' goods including housing services, but not transportation. The explicit inclusion of the space co-ordinates in the local utility function make it possible to put various weights due, for example, to various population densities.

Production is possible by a set of production functions $f^i(k_i,l_i,m_i)$, $i = 1,\ldots,n$. We do not include the space co-ordinates as arguments, so the same production opportunities are open everywhere. If local consumption is subtracted from local production we arrive, in analogy to (7), to

$$f^i(k_i,l_i m_i) - q_i = \text{div}\,\phi_i \quad , \qquad \text{for } i = 1,\ldots,n. \qquad (13)$$

Suppose transportation, which is the only kind of activity not represented in the production functions, uses up κ_i units of capital and λ_i units of labor for each unit of flow intensity of the i:th commodity. These "fixed" coefficients should be taken as functions of the space co-ordinates, the dependence reflecting local transportation facilities.

If given aggregates of labor and capital are available, we arrive at the constraints

$$\iint_S \sum_{i=1}^{n} (k_i + \kappa_i|\phi|)\,dxdy = K \quad , \qquad (14)$$

$$\iint_S \sum_{i=1}^{n} (l_i+\lambda_i|\phi_i|)\,dxdy = L \quad . \qquad (15)$$

In contrast to capital and labor, land cannot be moved around and hence the remaining constraint is in local, not integral, form:

$$\sum_{i=1}^{n} m_i = m \quad . \tag{16}$$

We are hence given aggregates of capital and labor that can be distributed among locations and among activities. Land can only be allocated among activities. We are also given production and transportation possibilities. The objective is to maximize welfare given these technological and resource availability constraints.

Mathematically, we maximize (12) subject to the constraints (13), (14), (15) and (16). The optimum conditions are:

$$P_i \, f_k^i = r \quad , \tag{17}$$

$$P_i \, f_l^i = w \quad , \tag{18}$$

$$P_i \, f_m^i = g \quad , \tag{19}$$

for production,

$$(r\kappa_i + w\lambda_i) \frac{\phi_i}{|\phi_i|} = \text{grad } P_i \quad , \tag{20}$$

for transportation, and

$$U_i = P_i \quad , \tag{21}$$

for consumption. The P_i, r, w and g are now Lagrangean multipliers for the constraints. In optimization they receive the interpretations of output and input prices.

We also note that, r and w, being Lagrangean multipliers for the constraints (14) and (15) which are in integral form, must be constants with respect to location.

The solution is a solution to a <u>planning problem</u>, but most of the conditions have an interpretation in terms of market equilibrium.

So, (17)-(19) are profit maximum conditions for private firms as they tell that the marginal value productivity for each input must everywhere equal its local price. The constancy of capital rent and wages can be so interpreted that in the long run, by capital accumulation and labor migration, these inputs go where their rewards are highest.

Moreover, (20) tells us that all commodity flows take the directions of steepest price increases, and that in these directions prices increase by transportation cost. The later is now equal to $(r\kappa_i + w\lambda_i)$ as we have specified transportation costs in terms of primary input requirements. This condition too makes sense in a competitive equilibrium context. That private transporters take goods in the direction where their prices increase most, and that, by competition among transporters, the prices increase at the rate of transportation costs is acceptable in that context.

The only condition, the fulfillment of which is not guaranteed in a competitive equilibrium is (21).

It is, however, interesting to note that we, by repeated use of Gauss' theorem, can derive the condition:

$$\iint_S \sum_{i=1}^{n} p_i q_i \, dxdy = rK + wL + \iint_S g \, dxdy \quad . \tag{22}$$

The derivation is lengthy, and is therefore not reproduced. The only assumptions we have made is that (i) production functions are linearly homogenous and (ii) any imports of goods are financed by exports of other goods so that trade with the exterior balances.

What (22) tells us is that the lefthand member, which is aggregate consumption value of all goods evaluated at local prices, equals the righthand member, which is the sum of capital rents, wages, and incomes of the local landlords. Hence, in aggregate, a "budget constraint" is fulfilled. As in this aggregate constraint everything is evaluated at local prices we concluded that it is possible to design an intraregional income transfer policy which makes it possible to achieve the fulfillment of (21) keeping autonomy of the consumers behaving according to their preferences within the limits set by the local budget constraints.

This demonstrated the equivalence of planning and competitive equilibrium in the continuous space market.

It should be kept in mind that this is only one out of many possible applications of the continuous flow model to regional planning and equilibrium. The purpose of my presentation was not to survey the possible applications, but rather to present the paradigm by means of an application.

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