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SPATIAL MODELING OF URBAN SYSTEMS: AN ENTROPY APPROACH

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FOREWORD

Changing rates of natural population growth, continuing differential levels of regional economic activity, and shifts in migration patterns are characteristic aspects of many developed countries. In some regions they have combined to bring about population decline of highly urbanized areas, in others they have brought about rapid metropolitan growth. Whether growing or declining, however, the large urban agglomerations share common concerns related to their internal manageability, the costs of spatial interaction, the quality of life, and urban redevelopment issues.

One of the objectives of the Urban Change Task in IIASA's Human Settlements and Services Area is to carry out the international review, assessment, and development of models of intraurban systems.

In this report, Dr. Boris Shmulyian, of the Institute for Systems Studies, USSR Academy of Sciences in Moscow, discusses the structure and applications of several urban models based on the entropy approach. The paper presents a methodology for incorporating detailed prior information concerning trip distribution patterns. It also focuses on applying the findings to planning the location of working places and residences within a large city.

A list of publications in the Urban Change Series appears at the end of this paper.

Andrei Rogers
Chairman
Human Settlements
and Services Area

ABSTRACT

This paper deals with general methodological questions of urban systems modeling. The main feature of the models under discussion is their analogy with models derived by the use of statistics. This analogy is extended and generalized to take into account the structure of *a priori* spatial preferences of urban residents. Abstract systems that have been constructed following this approach can be applied to the analysis and simulation of individual urban subsystems as well as of the urban systems as a whole.

CONTENTS

SPATIAL MODELING OF URBAN SYSTEMS:
AN ENTROPY APPROACH

1. INTRODUCTION

The use of entropy methods in the spatial modeling of urban systems has become increasingly popular in the past decades. This paper considers a number of fundamental methodological questions raised by such an approach. To motivate the discussion, we begin by listing a few examples related to forecasting. All require some prior information, as well as assumptions on the preferences of resident populations of particular regions.

1.1 Examples

Forecasting Shopping Center Attendance

For the forecasting of shopping center attendance we suppose that $j = 1,\ldots,n$ shops (such as supermarkets and department stores) are to be constructed in a region. It is known that they will be attended by residents from $i = 1,\ldots,m$ districts. The capacities of the districts P_i (amount of prospective buyers at the time) are given. The average cost of a purchase in each shop, c_j, and the average cost $\bar{c}$ of all purchases in the center by all the buyers are statistically determined. We assume that the buyers are socially homogeneous, shops are identical, and buyers choose a shop in a random way. It is necessary, then, to determine the distribution of the buyers among the shops.

Forecasting the Population Distribution in a New City

In order to forecast the population distribution for a new city, we suppose that n places of work with capacities Q_j, j = 1,...,n are given; m districts, which may be used for housing construction are given also and the main characteristic of the links between residential districts, i, and the places of work, j, is the length of time necessary for the trip t_{ij}. The average time for home-to-work trips T_m, for all residents, may be determined by using analogies with existing cities of similar types. It is again assumed that people choose working places independently and in a random way. We note that the problem is essentially of a non-optimal kind because T_m is fixed and is not to be minimized. In this case it is necessary to determine the most appropriate allocation of housing construction within the city.

Analyzing and Forecasting Urban Traffic

This task is a classical one and has been investigated since the 1930s. Here the graph of the city transport network is given and zones of trip origins P_i, i = 1,...,m and destinations Q_j, j = 1,...,n are identified. The important feature of this problem is the presence of prior information on f(t), the frequency distribution of the passengers over the trip length between origin-destination pairs (i,j), where t_{ij} is the shortest route. It is necessary to determine interzonal links (trips) x_{ij} and transport network loads. We assume that the passengers choose a trip according to an origin-destination pair (i,j) in a random way consistent with the prior distribution function f(t).

1.2 General Definitions

The above examples show how a *stochastic spatial interaction system* can be defined. We define *a system* as a finite set of N elements without regard to their internal structure. The term *spatial* means that elements of the system belong to a space (in particular, to a geometric space) and that they may be located in some units of that space--the elements belonging to the same unit being indistinguishable.

The term *interaction* means that the system's work consists in transporting elements from one group of units $i = 1,\dots,m$ (origins) to another $j = 1,\dots,n$ (destinations). For each origin-destination pair, a characteristic $h_{ij} \geq 0$ is defined. Finally, the term *stochastic* suggests that the (i,j) pair is chosen in a random way and independently, with a probability ν_{ij}.

It then follows, that a random state of the system--the flow matrix $X = \{x_{ij}\}$--is realized. The complete characterization of random variables is represented by the distribution functions of the variables, therefore we say that distributions over origins $p_i = \frac{1}{N}\sum_j x_{ij}$, destinations $q_j = \frac{1}{N}\sum_i x_{ij}$, and interactions exist. In the last case, we may interpret a random choice of an (i,j) pair as a realization h_{ij} of a random variable H, provided the density distribution $f(h)$ exists. Since h_{ij} takes on a finite number of values, the distribution $f(h)$ is a discrete one; but since this number is large in actual systems, we introduce the distribution density $f(h)$ that clearly may be constructed from the discrete distribution function.

1.3 Assumptions and Mathematical Structure

Each *state* of the system X can be realized in many *ways*, differing only in the parts allocated to the (i,j) pairs when the flows x_{ij} are fixed (Wilson 1974; Imelbaev 1978a). This is the classical stochastic scheme of the multinomial distribution and is presented in Assumption A.

Assumption A. The individuals choose an (i,j) pair in a random and independent way. The probability of occurrence of each state is proportional to the number of ways that it may be realized.

This number, for state X, is given by (Wilson 1974)

$$W(X) = \frac{N!}{\prod_{ij} x_{ij}!} \tag{1}$$

and the probability of this state is

$$p(X) = \alpha \frac{N!}{\prod_{ij} x_{ij}!} \tag{2}$$

If ν_{ij}, representing the prior probability that (i,j) will be preferred to other pairs, is given for all (i,j) (Imelbaev 1978a), we have

$$p(X) = \frac{N!}{\prod_{ij} x_{ij}!} \prod_{ij} \nu_{ij}^{x_{ij}} \tag{3}$$

Expressions (2) and (3) completely determine the random multi-dimensional variable X and allow us to calculate the probability of the realization of every state. Taking algorithms (3) and using the approximation $\ln Z! \approx (\ln Z - 1)Z$, then

$$\phi(X) = \ln p(X) \approx \sum_{ij} x_{ij} \ln \frac{\nu_{ij}}{x_{ij}} + c \tag{4}$$

Or, if ν_{ij} is constant for all (i,j) pairs

$$\phi(X) = -\sum_{ij} x_{ij} \ln x_{ij} + c_1 \tag{4'}$$

The expression (4') corresponds to the system's entropy (Landau 1970). This fact, as well as the similarity of the assumption given above to the assumptions used in statistical mechanics, allows us to introduce the following assumption.

Assumption B. The system tends toward a stable state. This state corresponds to the maximum of the system's entropy (4), (4'), which for state X is

$$\max_X \phi(X)$$

Note. Assumption A, in general is a convenient abstraction and can be confirmed in part by social studies and, eventually, by comparison of the model results with observed data. On the other hand, assumption B holds in physical systems (Kittel 1969) (the second law of thermodynamics) and is highly probable in urban systems.

1.4 Examples

The three problems of urban systems forecasting that were presented in the first part of this paper can now be considered again. Three examples are given in order to describe mathematically the planning problems.

a) A first example describes the flows x_{ij} between the districts of origin and destination giving

$$\max_{x_{ij}} \left(- \sum_{ij} x_{ij} \ln x_{ij} \right)$$

and the initial information corresponds to the exact constraints on x_{ij}

$$\sum_{j} x_{ij} = P_i \qquad , \qquad i = 1,\ldots,m$$

$$\frac{1}{N} \sum_{j} c_j \sum_{i} x_{ij} = \bar{c}$$

$$x_{ij} \geq 0$$

It is convenient to present the cost constraints as

$$\sum_{ij} c_j \, x_{ij} = N \, \bar{c}$$

b) A second example is the problem of determining the flows which correspond to

$$\max_{x_{ij}} \left(- \sum_{ij} x_{ij} \ln x_{ij} \right)$$

under the constraints

$$\sum_i x_{ij} = Q_j \quad , \qquad j = 1,\ldots,n$$

$$\sum_{ij} x_{ij} t_{ij} = N T_m$$

$$x_{ij} \geq 0$$

c) The final example determines flows x_{ij} from

$$\max_{x_{ij}} \sum_{ij} x_{ij} \ln \frac{\nu_{ij}}{x_{ij}}$$

subject to $\sum_j x_{ij} = P_i \quad , \qquad i = 1,\ldots,m$

$$\sum_i x_{ij} = Q_j \quad , \qquad j = 1,\ldots,n$$

where the values ν_{ij} have to correspond with f(t) which is given.

1.5 Types of Prior Information

The problems considered above differ in terms of the data needed on distributions over the interactions, origins, and destinations. Let us introduce a classification of data types.

1. Data is absent.
2. Distribution parameter is given.
3. Distribution is given.

These data generate constraints on the flows x_{ij}, presented in Table 1.1.

Table 1.1 Alternative constraints relating to different data type.

Data type	Distribution		
	A-over interactions	B-over origins	C-over destinations
1	$\nu_{ij} = \frac{1}{mn}$	$\sum_{ij} x_{ij} = N$	$\sum_{ij} x_{ij} = N$
2	$\sum_{ij} h_{ij} x_{ij} = N\bar{h}$ $\nu_{ij} = \frac{1}{mn}$	$\sum_{ij} a_i x_{ij} = N\bar{a}$	$\sum_{ij} b_j x_{ij} = N\bar{b}$
3	$\nu_{ij} = \psi\{f(h)\}$	$\sum_{j} x_{ij} = P_i$ $i = 1,\dots,m$	$\sum_{i} x_{ij} = Q_j$ $j = 1,\dots,n$

Let us consider the determination of ν_{ij} under given $f(h)$ more closely. For this purpose subdivide the range of distances of h, from h_{min} to h_{max}, into intervals Δ_r, $r = 1,\dots,k,\dots,l$ and define characteristic functions for any interval Δ_k in this range (Figure 1.1a)

$$\Lambda_k(h) = \begin{cases} 1 & h \in \Delta_k \\ 0 & h \notin \Delta_k \end{cases}$$

Calculate

$$f_k = \int_{\Delta_k} f(h)\,dh$$

where f_k is the probability of choosing links of a length within the interval Δ_k: $h_{ij} \in \Delta_k$ (Figure 1.1b). We note, that since $f(h)$ are aggregate data on the present state X^0 then

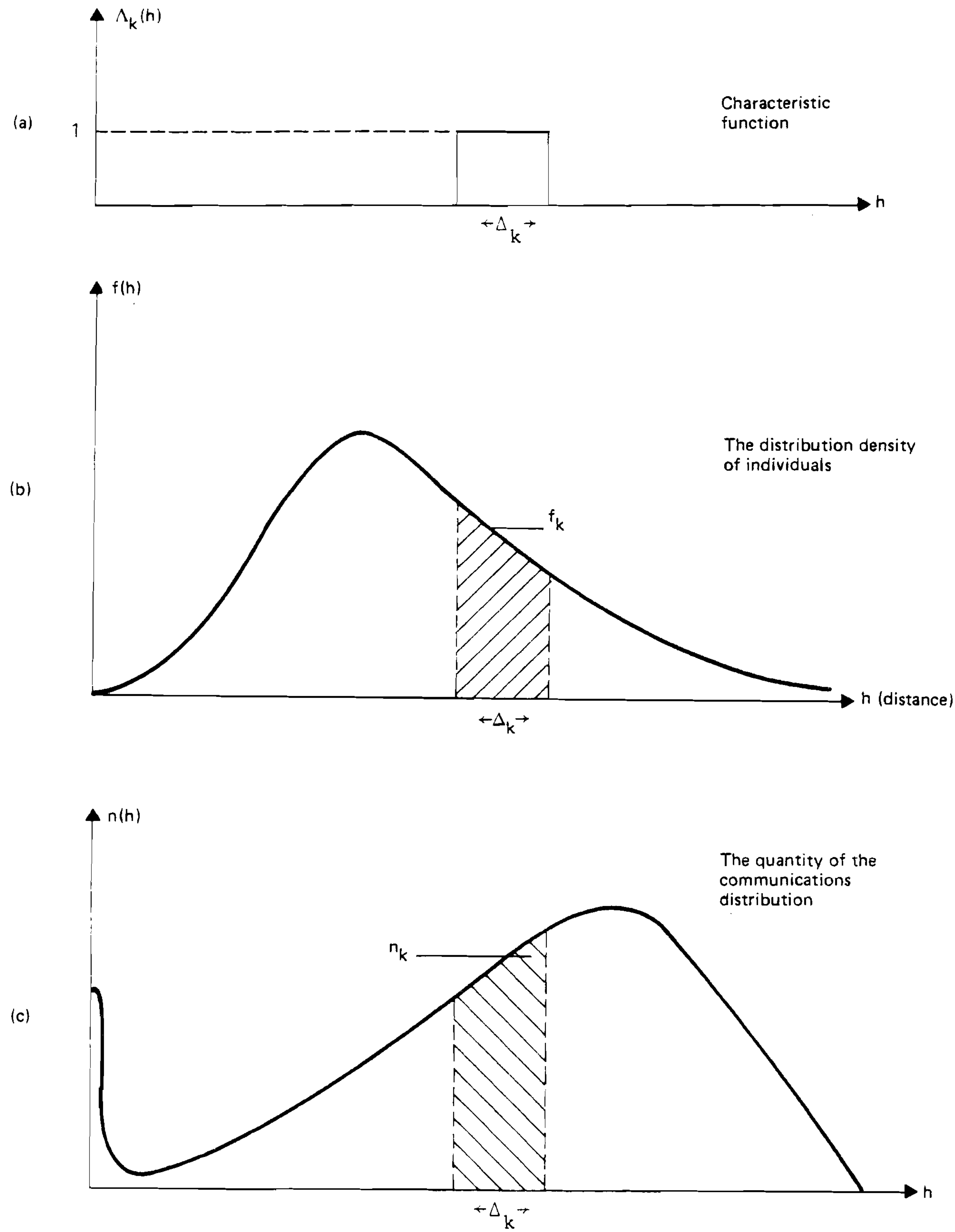


Figure 1.1 Calculation of probabilities ν_{ij} under the given distribution density f(h).

$$\sum_{ij} x_{ij}^0 \Lambda_k(h_{ij}) = N f_k$$

is the number of individuals traveling on trips with lengths Δ_k.

We have no information implying that any one link $h_{ij} \in \Delta_k$ is preferred over any other link in the same distance interval. Therefore, we may define the probabilities ν_{ij} as equal to each other over the same group, i.e.,

$$\nu_{ij} = \frac{f_k}{n_k} \quad , \qquad \forall\ h_{ij} \in \Delta_k \tag{5}$$

where

$$n_k = \sum_{ij} \Lambda_k(h_{ij}) \tag{5'}$$

is the number of links with lengths in the range Δ_k (Figure 1.1c).

This classification allows for the definition of a set of systems models. Designate them by 3-tuples {A B C}, where A,B, C = 1,2,3 are types of data referring to interactions, origins, and destination. Thus the examples of section 1.4 correspond to models {132}, {213}, and {333}.

1.6 Determining the Flows

All the models reduce to constrained optimization problems (where constraints $x_{ij} \geq 0$ are unessential) and are solved by Lagrangian multipliers (see Imelbaev 1978a).

The general form of the problem is

$$\max_{x_{ij}} \phi(x_{ij}) = \sum_{ij} x_{ij} \ln \frac{\nu_{ij}}{x_{ij}} \tag{6}$$

subject to

$$\sum_{ij} a_{ij}^{z} x_{ij} = b^{z} \quad , \quad z = 1,\ldots,S \quad x_{ij} \geq 0 \qquad (6')$$

where b^z, a_{ij}^z, and ν_{ij} are parameters, determined by the problem type.

The Lagrangian function, L, for (6) and (6') is

$$L(x_{ij},\lambda^{z}) = \sum_{ij} x_{ij} \ln \frac{\nu_{ij}}{x_{ij}} + \sum_{z=1}^{S} \lambda^{z}\left(\sum_{ij} a_{ij}^{z} x_{ij} - b^{z}\right) \qquad (7)$$

where λ^z are Lagrangian mulipliers for constraints (6').

According to the general method, the solutions (6) and (6') are determined from the equations

$$\frac{\partial L}{\partial x_{ij}} = \ln \frac{\nu_{ij}}{x_{ij}} - 1 + \sum_{z=1}^{S} \lambda^{z} a_{ij}^{z} = 0$$

and is of the form

$$x_{ij} = \nu_{ij} \exp(-1 + \sum_{z=1}^{S} \lambda^{z} a_{ij}^{z}) \qquad (8)$$

where λ^z are determined from the equations derived by substituting (8) into (6').

The solutions for all problems, from Table 1.1 are given in Imelbaev (1978a). According to the data type (constraints) some problems have closed-form solutions and some may be reduced to 1-3 transcendental equations. Problems {233} and {333}, where information is the most complete, constitute a special case. Here it is necessary to solve a high order system of equations to determine the flows x_{ij}. For example, the solution of problem {333} is [see 1.4(c)]

$$x_{ij} = a_i \, b_j \, \nu_{ij}$$

where $a_i = \exp(-1 - \alpha_i)$, $b_j = \exp(-\beta_j)$; α_i, β_j are Lagrangian multipliers corresponding to the constraints {333}. The parameters a_i, b_j must satisfy the system of equations

$$\begin{aligned} a_i \sum_j b_j \, \nu_{ij} &= P_i \\ b_j \sum_i a_i \, \nu_{ij} &= Q_j \end{aligned} \qquad (9)$$

[The method to solve the problem ("balancing method") was proposed in the 1930s.]

1.7 Reproduction of the Prior Distribution f(h)

As was already noted, models of the {333} type and their solutions have been known for a long time. The distinguishing feature of the models proposed here lie in the method of calculating probabilities. In the models used earlier in this paper (Wilson 1974), it was stated that $\nu_{ij} = \alpha \, f_k$, which equals (5) only when n_k is constant. There are, however, examples (Imelbaev 1978a, and section 2.2 of this paper) in which it is shown that the distribution

$$\psi_k = \frac{1}{N} \sum_{ij} x^*_{ij} \, \Lambda_k(h_{ij}) \qquad (10)$$

where x^*_{ij} is the solution of the corresponding problem, equals the prior distribution function f_k subject to definition (5), and is almost arbitrary under definition (Wilson 1974).

Another way to reproduce the prior distribution is to introduce additional constraints of the (10) type for flows x_{ij}. Since these constraints are nonlinear, let us introduce into the models the third index k - the number of interaction groups with the characteristics $h_{ij} \in \Delta_k$ - to eliminate this difficulty.

Let us introduce matrices with three indices

$$h_{ijk} = \begin{cases} h_{ij} & h_{ij} \in \Delta_k \\ \text{infinite} & h_{ij} \notin \Delta_k \end{cases}$$

$$\nu_{ijk} = \begin{cases} 0 & h_{ij} \notin \Delta_k \\ \nu_{ij} & h_{ij} \in \Delta_k \end{cases} \tag{11}$$

Now the state of the system is determined by the three-index matrix $X = \{x_{ijk}\}$. It is now possible to develop a probability scheme, analogous to (3), for choosing the 3-tuple (i,j,k). This generates the problem

$$\max_{x_{ijk}} \sum_{ijk} x_{ijk} \ln \frac{\nu_{ijk}}{x_{ijk}} \tag{12}$$

with the constraints (5) and (5') where the indices are correspondingly modified. Note that the flows x_{ijk} in the k plane are non-zero only if $h_{ij} \in \Delta_k$ [see (11), Figure 1.2].

To reproduce the prior distribution f(h) exactly, the modified constraints (10) are added to the models, i.e.,

$$\sum_{ij} x_{ijk} = N f_k \quad , \quad k = 1,\ldots,\ell \tag{13}$$

These three-index models are more general, although their solutions are still rather complex. The models with complete prior information on the distributions are the most complicated. In order to calculate their parameters (Imelbaev 1978a) the three-index balance method has been developed, and its convergence has been proved.

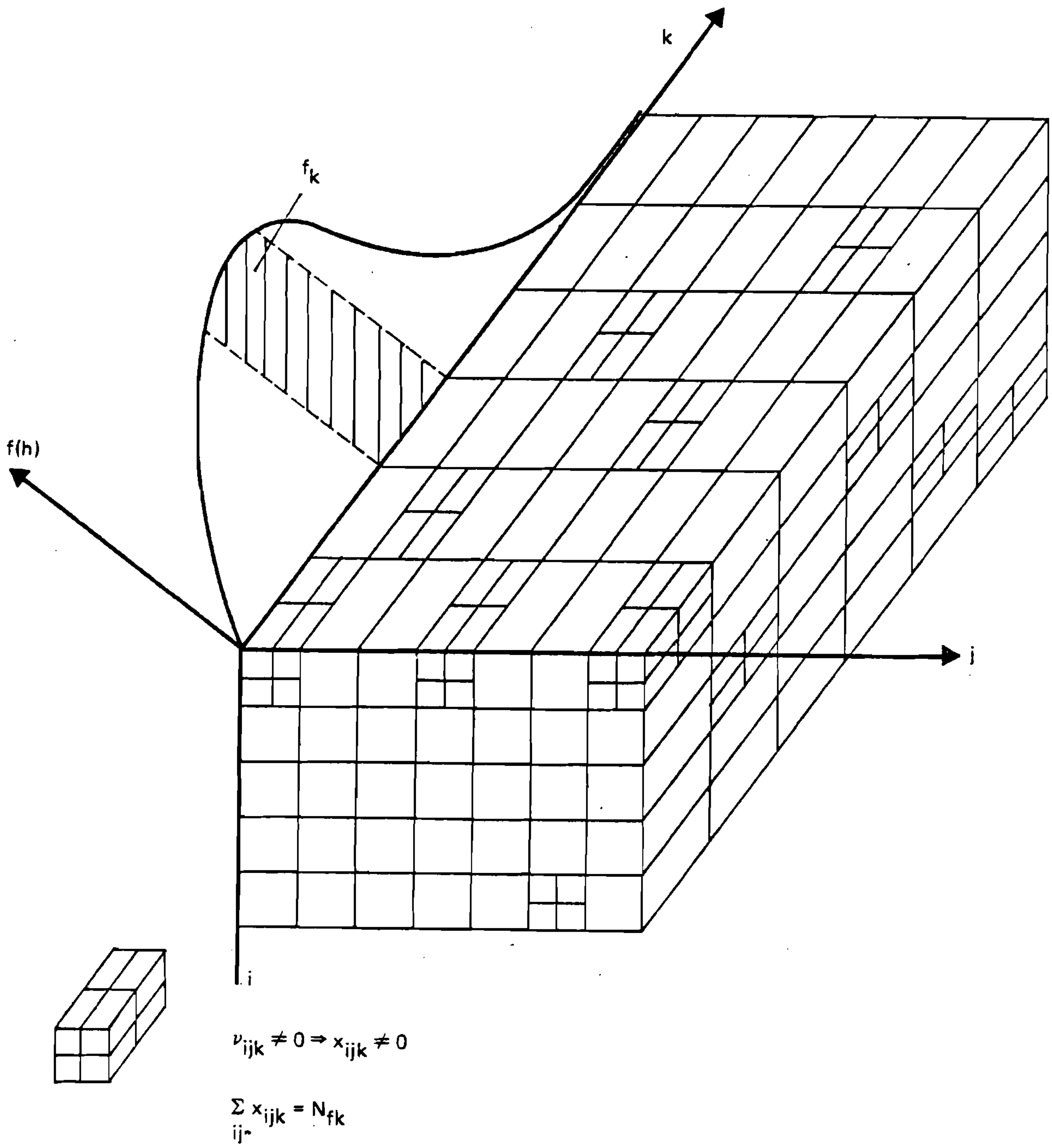


Figure 1.2 Reproduction of prior distribution f(h). A three-index matrix.

1.8 Generalizations

In some cases the assumptions, which are used in the models, may be excessively restrictive. In particular, some parameters for the distributions, $\bar{a}_1,\ldots,\bar{a}_s$, such as the average length of the trip, dispersion, etc., may be known. Alternatively, information on different distributions f(h) for particular groups of origins and destinations may be available. For an urban system this could mean that individual districts may differ in terms of spatial preference of their residents (for example, people living in the outer ring of a city may be accustomed to long-distance commuting or shopping trips while those in the inner-city are not).

These generalizations lead to a generalization in the models and in the corresponding solution methods. This, in turn, leads to some interesting results.

Several Transport Modes for the (i,j) Pairs

Suppose that passengers traveling from i to j can use several routes with characteristics h^s_{ij}, not necessarily the shortest one. Here additional assumptions on the prior probability of choosing a transport mode are needed, because now the choice of an (i,j) pair does not define the mode. In particular the following hypotheses may be proposed:

- The 3-tuple (i,j,k), where k is the number of sequent Δ_k to which h^s_{ij} belongs, is chosen randomly. This hypothesis may be justified for small variations of h^s_{ij} from $\min_s h^s_{ij}$ corresponds to the shortest path.
- The choice of the segment k takes place in accordance with a conditional distribution $\varphi(Z)$ where (i,j) is fixed. Here $Z = h^s_{ij} - \min_s h^s_{ij}$. This means that the probability of passengers choosing longer trips is smaller (but nonzero).

These generalizations generate some nonzero elements in the matrix ν_{ijk} (and consequently in x_{ijk}). The conditions of the balance over the index values, as given in (13), provide an exact reproduction of the prior distribution f(h).

Disaggregation of Residents

For the urban system the disaggregation of residents means an allocation of some social groups that differ in their mobility [distributions $f_s(h)$]. If information on the total number of individuals in each group N_s, s = 1,...,S is available, we can determine the layers of the model indexed by k and calculate the solution of the three-index problem with constraints analogous to (13).

2. MODEL APPLICATION: ANALYSIS OF PASSENGER FLOWS IN A LARGE CITY

The methodology described in section 1 has been used to construct a transport network model and has been realized as a software package. Joint calculation of passenger trips using two transport modes (public transportation and cars) for two kinds of trips (home-to-work and home-to-service) has been carried out.

2.1 The Model

The model and the software package were designed to solve problems connected with alternative forecasts of transport network development. The characteristic feature of the data used in these problems is the inability of obtaining exact travel time frequency distributions f(t) for both transport modes.

Therefore, application of the models with an exact reproduction of the prior distribution [see (7)] is not advisable here. It is more natural to use the model with two layers, indexed by k (either of the two transport modes), and to determine the corresponding ν_{ijk} for each of them. Thus the model is:

$$\max_{x_{ijk}} \sum_{ijk} x_{ijk} \ln \frac{\nu_{ijk}}{x_{ijk}}$$

subject to

$$\begin{aligned} &\sum_{jk} x_{ijk} = P_i \quad , \quad i = 1,\ldots,m \\ &\sum_{ik} x_{ijk} = Q_j \quad , \quad j = 1,\ldots,n \\ &\sum_{ij} x_{ijk} = N_k \quad , \quad k = 1,2 \\ &x_{ijk} \geq 0 \end{aligned} \qquad (14)$$

where P_i, Q_j are capacities of the origins and destinations of passenger trips [see section 1.1(c)], N_k is the volume (numbers) of passengers that move using public (k = 1) transport or cars (k = 2); and $N = N_1 + N_2$. The values ν_{ijk} are determined as in (5), i.e.:

$$\nu_{ijk} = \frac{N_k}{N} \cdot \frac{f_k^\ell}{n_k^\ell} \quad , \quad t_{ij} \in \Delta_\ell \qquad (15)$$

where (N_k/N) are the probabiliites that passengers will choose transport mode k

$$f_k^\ell = \int_{\Delta_\ell} f_k(t)\,dt$$

is the conditional probability of the choice of a group of routes with the trip time $t_{ijk} \in \Delta_\ell$ using given transport mode k;

f_k^ℓ/n_k^ℓ are the same for each route in this group; and

Δ_ℓ, $\ell = 1,\ldots,L$ is a regular subdivision of [0, t max].

Data on the general form of the distributions $f_k(t)$ means that functions $f_k(t, \beta_1, \ldots, \beta_z, \bar{t}_k)$ are given. Then, if $\bar{t}_k$ is known, we can fix values $\beta_1, \ldots, \beta_z$, calculate the values $x_{ijk} = x_{ijk}(\beta_1, \ldots, \beta_z)$, calculate the estimate of the average time

$$\hat{t}_k(\beta_1, \ldots, \beta_z) = \frac{1}{N_k} \sum_{ij} t_{ijk} \cdot x_{ijk}(\beta_1, \ldots, \beta_z)$$

and begin an iterative process that will minimize the difference between $\bar{t}_k$ and $\hat{t}_k$. In this case, parameters $\beta_1, \ldots, \beta_z$, are chosen to minimize:

$$I = \sum_k \left(\hat{t}_k(\beta_1, \ldots, \beta_z) - \bar{t}_k\right)^2 \tag{16}$$

This procedure is simplest for just two unknown parameters β_1, β_2 [according to the number of the constraints in (14)].

This leads to the solution of the system

$$\begin{aligned} \hat{t}_1(\beta_1, \beta_2) &= \bar{t}_1 \\ \hat{t}_2(\beta_1, \beta_2) &= \bar{t}_2 \end{aligned} \tag{17}$$

using, for example, the generalized secant model. In the model developed, a gamma-distribution was used as $f(t)$

$$f_k(t, \alpha_k, \beta_k) = \frac{\beta_k^{\alpha_k}}{\Gamma(\alpha_k)} t^{\alpha_k - 1} \exp(-\beta_k t) \tag{18}$$

Here $\bar{t}_k = \alpha_k/\beta_k$ and the distribution maximum is biased to the left of $\bar{t}_k$ at $1/\beta_k$.

Quantitative characteristics of the initial data on transport network are: 10,000 edges, 4,100 nodes, 750 districts. The main calculation steps are the input and verification of the initial data, the calculation of the shortest routes among all the districts for two transport modes, the calculation of matrix of flows x_{ijk}, the estimation of the loads on the network by summing the flows over the shortest routes, and the calculation of transport system characteristics.

2.2 Some Calculation Results

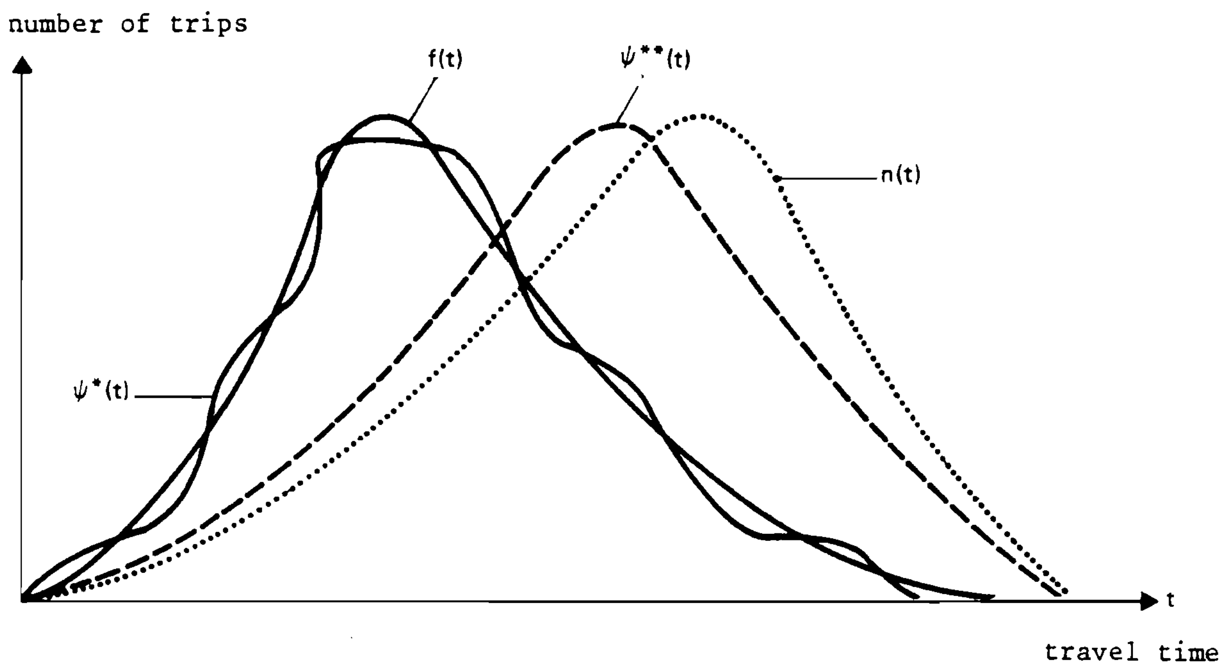
The complete calculation results for several stages of this study were presented to the Institute of Moscow Master-Plan, which has sponsored this research (Bolbot et al. 1978). Let us consider some of them here.

As noted above, the model described in section 2.1 differs from the known ones in its method of determining the probabilities ν_{ij} in equation (5). Comparison of the results by new and old methods has shown that the curves f(t) and ψ(t) using (5) are rather close to each other, which is not the case when the condition (5) is abandoned (Figure 2.1).

In section 1.5 the notion of the interaction distribution [or its discrete equivalent - n_k^ℓ (5')] was presented, which is used to determine ν_{ijk}. Calculations have shown that overload of the transport routes, especially radial ones intended for long-distance trips, takes place if $\bar{t}_k$ is near to $\bar{n}_k$ (the average of n_k^ℓ), see Figure 2.2a. If $\bar{t}_k < \bar{n}_k$ the load on all the highways is lower and concentric highways (for short trips) take a relatively larger load (Figure 2.2b).

As to the question of modal split, only the total number of the passengers on each mode [see section 1.8(b)] was given *a priori* in the model. Since the city of Moscow has a radial-circular structure, the distributions $f_1(t)$ and $f_2(t)$ are close to each other in the central zone (Figure 2.3a), but towards the outskirts $f_2(t)$ is essentially to the left of $f_1(t)$

(Figure 2.3b). With this data it turns out that the proportion of people using cars for traveling to work is lower in the central zone than in the outskirts (Figure 2.3c). Note, that this inference is based on the assumption that passengers consider the length of the trip only and do not take into account the possible social heterogeneity of urban space.



f(t) – prior distribution
ψ*(t) – the calculated distribution accounting for n_k
ψ**(t) – the calculated distribution without accounting for n_k
n(t) – the quantity of the communications distribution

Figure 2.1 Reproduction of prior distribution f(t). Comparison of the methods of calculating ν_{ij}.

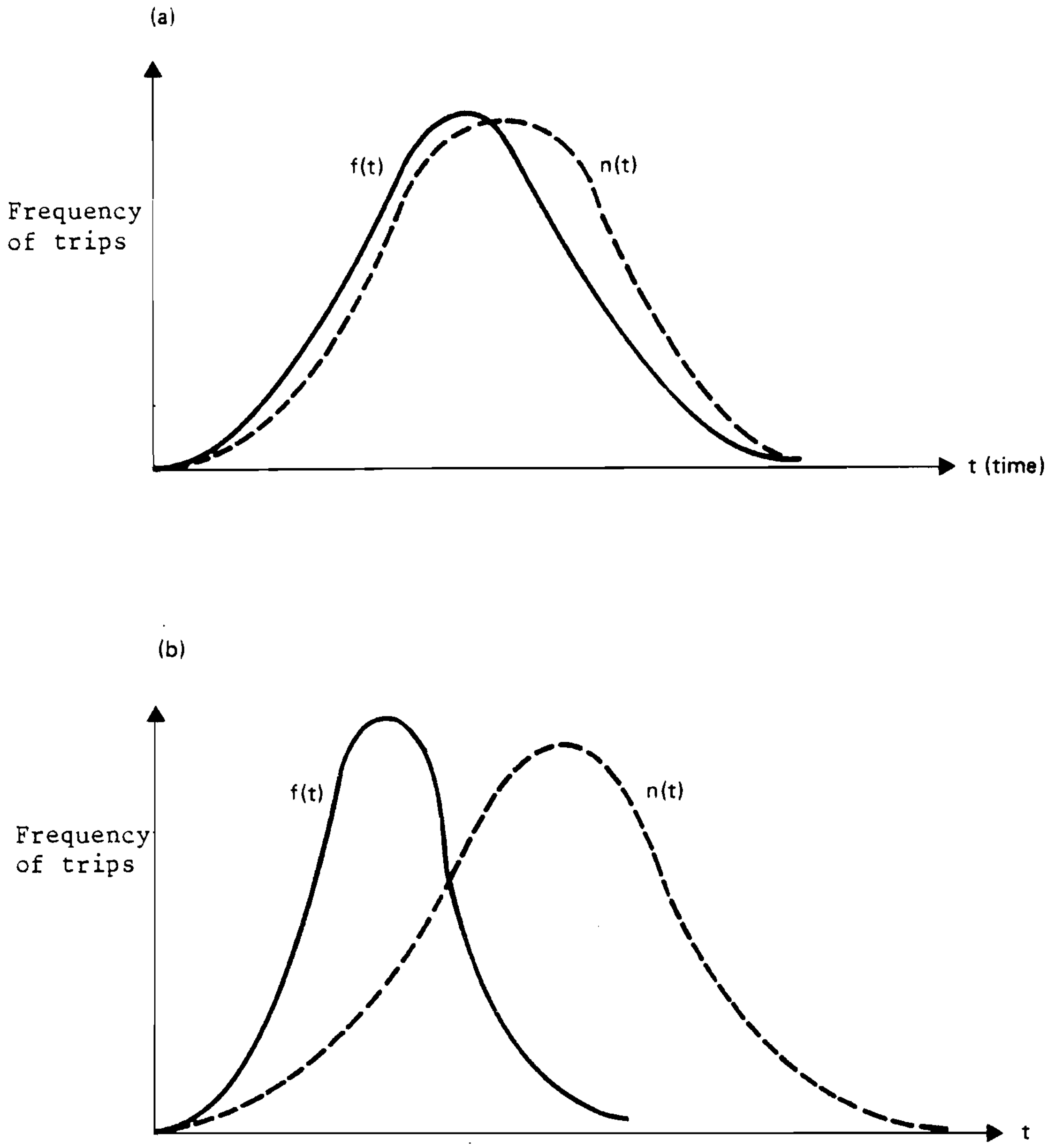


Figure 2.2 Comparison of traffic patterns for two different trip-length preference distributions [f(t)].

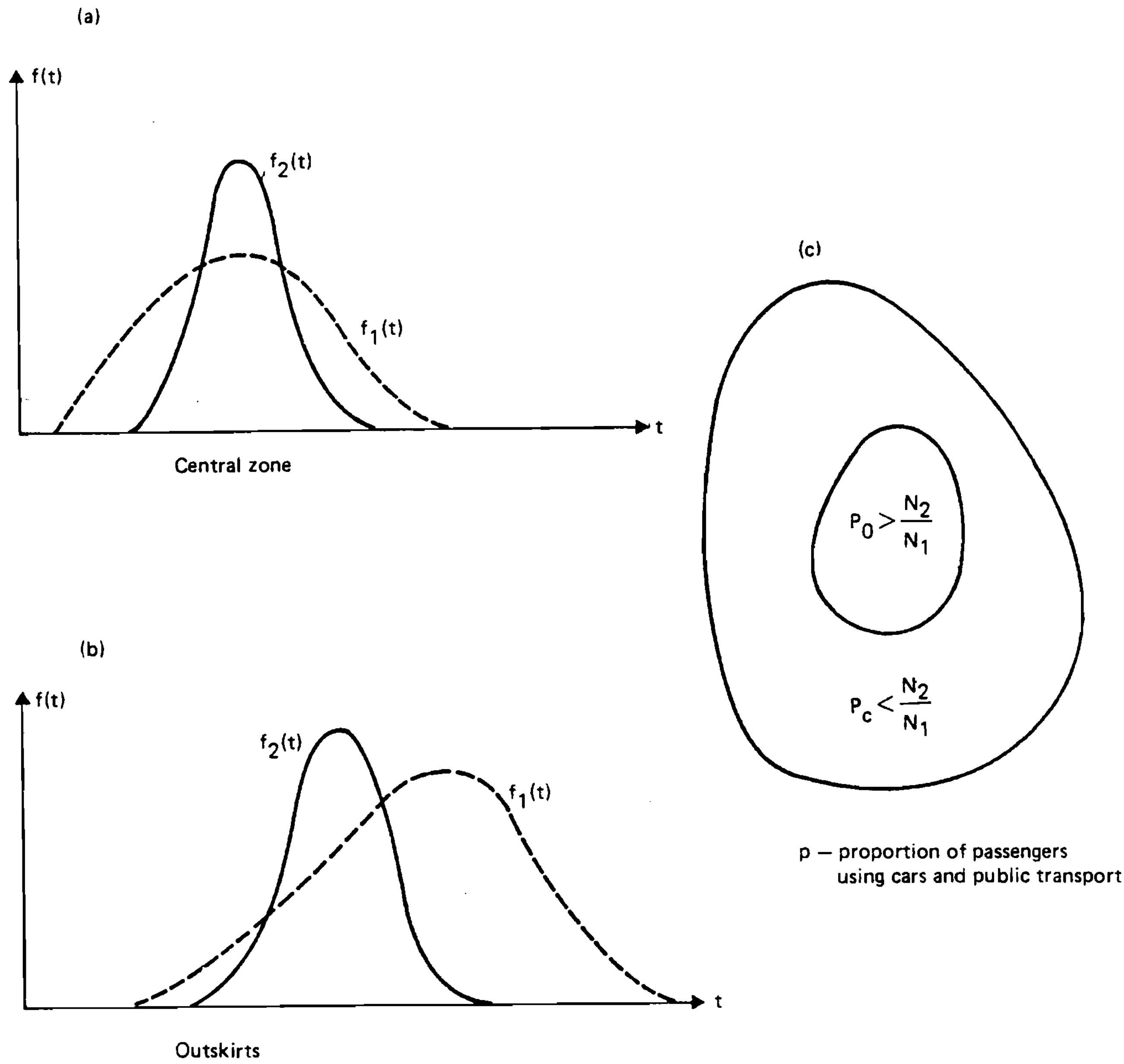


Figure 2.3 Distribution densities of the passengers for public transport $[f_1(t)]$ and cars $[f_2(t)]$ for the central zone and outskirts. Proportion of the passengers using the two types of transportation for the zones.

3. MODEL APPLICATION: ALLOCATION OF PLACES OF RESIDENCE AND OF WORK WITHIN A CITY

3.1 An Outline of the Model

Let us divide the territory of the modeled urban system for several (L) indexed districts and, following Wilson (1974), and Lowry (1964), let us consider the following subsystems.

a) The basic sector (industry, scientific research institutions, administration, etc.) is a subsystem characterized by the basic job distribution over districts given by the basic employment vector, E^B:

$$E^B = \{E_j^B, j = 1,\ldots,L\}$$

b) A second subsystem is the service sector consisting of R types of services (including daily and occasional services such as public entertainment and sports). This subsystem is characterized by the distribution over districts of service employment vectors:

$$E^k = \{E_j^k, j = 1,\ldots,L\} \quad , \quad k = 1,\ldots,R$$

where k refers to types of service employment.

c) Another subsystem is the household sector which is characterized by a distribution of the residents over the districts:

$$N = \{N_j, j = 1,\ldots,L\}$$

We will consider the distribution vector of the basic sector E^B as given, with the distributions of other subsystems, the vectors N and E^k, to be determined.

Interaction Between Subsystems

Let us assume that there are two types of interactions between the subsystems. The first type refers to service employment. The distribution of service E^k depends on the distribution of population N and places of work E^B and E^s where $s = 1,\ldots,R$. Residents working in the service subsystems are (just as any workers) customers of the same subsystem and introduce corrections into the distribution of working places. The second type of interaction is the dependence of the distribution of population N on the distribution of the total number of working places E

$$E = E^B + \sum_k E^k$$

Thus, subsystems N and E^k are dependent upon each other. Let us assume now, that under a fixed distribution of one subsystem [given capacities of origins (i)] the choice of destinations (j) occurs according to the general stochastic scheme of our spatial interaction system [see (2) and (3)]. This assumption allows us to construct a scheme showing the interaction between the two systems (see section 3.2 below).

Note. The interdependence of subsystems accepted in this model is analogous to one used in some earlier papers. In principle, any subsystem considered here may be made more important than another (Popkov 1977). For example, it is possible to assume that the population distribution depends on service and the distribution of places of work depends on the distribution of the population. This issue may be discussed only from the behavioral point of view. In any case, the causal chain followed in the bulk of urban modeling is reflected in the terms "basic" and "service" applied to the major sectors.

With regard to the initial data and constraints used, the following assumptions are made.

a) The distribution over the working places in the basic sector $\{E_j^B, j = 1,\ldots,L\}$ is given.

b) Some of the districts $i \in I^f$ are characterized by a fixed population N_i^f.

These assumptions pertain to the existing parts of the city, which will not undergo reconstruction within the foreseeable future. In the remaining ($i \in I^n$) districts the population N_i is still to be determined.

c) To define the time-cost of interaction between districts we choose the shortest route, that is $h_{ij} \equiv t_{ij}$.

d) Density curves $f(t)$ are given:

$f^{Nk}(t)$, $k = 1,\ldots,R$ - for residents choosing service type k;

$f^{Bk}(t)$, $k = 1,\ldots,R$ - for those working in the basic sector and choosing service type k;

$f^{sk}(t)$, $s,k = 1,\ldots,R$ - for those working in service type s and choosing service type k;

$f^{EN^f}(t)$ - for workers to be settled in the districts with fixed population sizes;

$f^{EN^n}(t)$ - for workers to be settled in the districts where population numbers are not predetermined.

e) The balance indices for entire urban systems are given:

$$\bar{E} = \bar{E}^B + \sum_k \bar{E}^k \text{ - places of work}$$

$$\bar{E}^B = \sum_j \bar{E}^B \text{ - places of work in the basic sector}$$

$$\bar{E}^k = \sum_j E^k \text{ - places of work for service type k}$$

$\bar{N}^f = \sum_{i \in I^f} N_i^f$ - population already present

$\bar{N}^n = \sum_{i \in I^n} N_i$ - population to be resettled

$\bar{N} = \bar{N}^f + \bar{N}^n$ - total population

$\alpha^k, \beta^{Bk}, \beta^{sk}$ - participation rates of customers, of basic sector workers, and of service sector employees, in using services s, s = 1,...,R.

Assuming service employment is proportional to total demand, these coefficients must satisfy the equation

$$\alpha^k \bar{N} + \beta^{Bk} \bar{E}^B + \sum_s \beta^{sk} \bar{E}^s = \bar{E}^k \qquad (19)$$

If c^f and c^n are the labor participation rates of residents in the two types of districts, then

$$c^f \bar{N}^f + c^n \bar{N}^n = \bar{E} \qquad (20)$$

f) In each district the area A_i available for allocating service and population subsystems is given. Therefore location of those subsystems must satisfy the inequalities

$$A_i^N + \sum_k A_i^k \leq A_i \qquad (21)$$

where

$$A_i^N = \frac{N_i}{Z_i^N} \quad , \quad A_i^k = \frac{E_i^k}{Z_i^k}$$

are the areas required for locating N_i and E_i^k accordingly; Z_i^N, Z_i^k are dimensional coefficients (densities) given in advance and generally specific to individual districts.

Let us assume that land required for service activities has an absolute priority over land required for residential purposes.

Then

$$N_i \leq Z_i^N(A_i - \sum_k A_i^k) \qquad (22)$$

In theory, it is possible that service needs may be greater than A_i. Constraints such as (22) must thus be introduced for the service allocation. These constraints are to be modified for fixed-population-size districts, and the population size in corresponding non-fixed districts may be forced to be zero.

3.2 Solution Steps

The assumptions introduced make it possible to construct two submodels for the allocation of urban subsystems subject to constraints. In the first submodel we must determine the stable state of the service subsystem, by maximizing its entropy when a state (allocation) of the population subsystem, as well as allocation of work places are given. In the second submodel we determine the stable state of the population subsystem in an analogous way.

We must then construct an algorithm for computing the required distributions, i.e., to determine operators

$$\begin{aligned} \{E^k, k = 1,\ldots,R\} &= A\{N; E^s, s = 1,\ldots,R\} \\ N &= B\{E^k, k = 1,\ldots,R\} \end{aligned} \qquad (23)$$

Since each of these operators determines a stable state for one subsystem, given fixed levels for all the other subsystems, the iteration process

$$\{E^{k(v+1)}, k = 1,\ldots,R\} = A\{N^{(v)}; E^{s(v)}, s = 1,\ldots,R\}$$

$$N = B\{E^{k(v+1)}, k = 1,\ldots,R\} \qquad (24)$$

$$v = 0,1,2,\ldots$$

with the initial condition $E^{k(0)}$, $N^{(0)}$ can be interpreted as a process of obtaining an equilibrium state for the city system as a whole, taking into account interactions between subsystems.

Finally, after construction of the model and algorithms to compute the model's parameters it seems natural to compare the results with known models. Here the Lowry model (Lowry 1964) is of the utmost interest; the algorithms used constitute a special case for those proposed in this paper (section 3.6 below).

3.3 Models of Subsystems Interactions

These models are constructed on common principles which reduce to formal problems of entropy maximization when the system states are constrained.

Allocation of Services

For each type of service k we introduce the destinations $j = 1,\ldots,L$ (according to the number of districts) and (2+R) groups of origins $\ell = 1,\ldots,(2+R)L$ (L for each group). The first group corresponds to the population, the second represents the basic sector workers, and the rest represent service workers. Thus, the given capacities of origins (in places of work for the corresponding service type) are equal to

$$\alpha^k N_i \quad , \quad i = 1,\ldots,L$$

for the first group; to

$$\beta^{Bk}\, E_j^B \quad , \quad j = 1,\ldots,L$$

for the second one; and to

$$\beta^{sk}\, E_j^{\,s} \quad , \quad j = 1,\ldots,L \quad ; \quad s = 1,\ldots,R$$

for the rest. For the destinations we assume that their capacity constraints related to (21) are irrelevant.

Probabilities $\nu_{\ell j}^k$ of choosing a given origin-destination pair for service type k are determined according to (5) on the basis of given curves f(t) and a distribution of link lengths n_u. The random event of selecting a pair (ℓ,j) may be conceived of as a product of two random events: Y - selection of one of (2+R) groups of origins and H - selection of destination j by individuals in origin ℓ.

Since these events are assumed to be independent of each other, we have $\nu_{\ell j}^k = p(Y) \cdot p(H)$. The ratio of the number of working places in group ℓ to the total number of working places, $\bar{E}^k$ determines p(Y). The value for p(H) is determined from (5), where we must take into consideration the fact that if some origins have zero capacities, the flows $x_{\ell j}^k$ originating from them must also be zero.

The probability distribution for choosing an interaction link in group u, i.e., of trip time Δ_u, can be considered as uniform over all links in the group u, excluding those generated by the origins with zero capacity. Thus we finally get:

$$\nu^k_{\ell j} = \begin{cases} \alpha^k \cdot \dfrac{\bar{N}}{\bar{E}_k} \cdot \dfrac{f^{Nk}_u}{n^N_u} , & \ell = 1,\dots,L \\[2ex] \beta^{Bk} \cdot \dfrac{\bar{E}^B}{\bar{E}^k} \cdot \dfrac{f^{Bk}_u}{n^B_u} , & \ell = L+1,\dots,2L \\[2ex] \beta^{sk} \cdot \dfrac{\bar{E}^s}{\bar{E}^k} \cdot \dfrac{f^{sk}_u}{n^s_u} , & \ell = (1+s)L+1,\dots,(2+s)L \\ & s = 1,\dots,R \end{cases} \tag{25}$$

where u - the index of trip length is determined from the conditions $t_{\ell j} \in \Delta_u$; f^{Nk}_u, f^{Bk}_u, f^{sk}_u are the probabilities of choosing a trip of length Δ_u, which are obtained from corresponding curves; and n^N_u, n^B_u, n^s_u are the numbers of interactions (trips) in a group u, which are determined when the origins with zero capacity are accounted for.

Thus the allocation of services is reduced to the independent solution of R problems:

$$\max_{x^k_{\ell j}} \sum_{\ell=1}^{(2+R)L} \sum_{j=1}^{L} x^k_{\ell j} \ln \frac{\nu^k_{\ell j}}{x^k_{\ell j}}$$

subject to

$$\sum_j x^k_{\ell j} = P^k_\ell \quad , \quad \ell = 1,\dots,(2+R)L \tag{26}$$

$$x^k_{\ell j} \geq 0$$

where $x^k_{\ell j}$ are flows from introduced origins to destinations; and P^k_ℓ are the capacities of origins of the introduced groups.

The result of solving this problem must be the flows $\overset{o}{x}{}^k_{\ell j}$, which when aggregated over all origins ℓ produce for each j the service allocation

$$E_j^k = \sum_{\ell=1}^{(2+R)L} \overset{o}{x}{}_{\ell j}^k \qquad (26')$$

Note. There is a definite contradiction in this model: in one group of constraints the right-hand side contains values and yet these are the results of solving (26'). We must recall that this submodel is only part of an iteration process (24). Therefore values at the right side (26) must be considered as computed at step v, and (26') corresponds to the step (v+1) of the process (24).

Population Allocation

Here the state of the system is determined by a matrix of flows y_{ji} between the working places with given capacities E_j, j = 1,...,L and the residences i, i = 1,...,L which satisfies the conditions (22). The probabilities μ_{ji} of choosing pairs (j,i) are determined according to the method discussed for service allocations. This is performed separately for fixed districts $i \in I^f$ and nonfixed $i \in I^n$

$$\mu_{ji} = \begin{cases} \dfrac{\bar{N}^f}{\bar{N}} \cdot \dfrac{f_u^f}{n_u^f} \,, & i \in I^f \\[2ex] \dfrac{\bar{N}^n}{\bar{N}} \cdot \dfrac{f_u^n}{n_u^n} \,, & i \in I^n \end{cases}$$

with corresponding determination of u, f_u^f, f_u^n, n_u^f, n_u^n.

Connections between working places and fixed and nonfixed regions are taken into consideration for n_u^f and n_u^n - respectively. Connections with origins of zero capacity and, for n_u^f, connections directed towards the destinations with zero capacity are ignored.

Thus population allocation is reduced to the problem:

$$\max_{y_{ji}} \sum_{j,i=1}^{L} y_{ji} \ln \frac{\mu_{ji}}{y_{ji}}$$

subject to

$$\begin{aligned} &\sum_i y_{ji} = E_j \quad , \quad j = 1,\ldots,L \\ &\sum_j y_{ji} = c^f N^f \quad , \quad i\in I^f \\ &\sum_j y_{ji} \le c^n d_i \quad , \quad i\in I^n \\ &y_{ji} \ge 0 \end{aligned} \tag{27}$$

where d_i is the feasible population of the district i [see (22)]. The result of solving this problem will be the flows $\overset{o}{y}_{ji}$ and the population allocation

$$N_i = \frac{1}{c^n} \sum_j \overset{o}{y}_{ji} \quad , \quad i\in I^n \tag{27'}$$

3.4 Service Allocation Operator

The methods of section 1.6 may be used to solve problem (26). It is easy to see that the solution is of the form

$$\overset{o}{x}{}^k_{\ell j} = \nu^k_{\ell j} \exp\ (-1 - \lambda^k_\ell) \tag{28}$$

with λ^k_ℓ determined by substitution of (28) into the constraint (26), i.e.,

$$\exp\ (-1 - \lambda^k_\ell) \sum_{j=1}^{L} \nu^k_{\ell j} = P^k_\ell \tag{29}$$

From which

$$\overset{o}{x}{}^{k}_{\ell j} = P^{k}_{\ell} \frac{\nu^{k}_{\ell j}}{\sum\limits_{j} \nu^{k}_{\ell j}} \tag{30}$$

According to (26'), the service allocation then becomes

$$E^{k}_{j} = \sum_{\ell=1}^{(2+R)L} P^{k}_{\ell} \frac{\nu^{k}_{\ell j}}{\sum\limits_{j} \nu^{k}_{\ell j}} \quad , \qquad j = 1,\ldots,L \tag{31}$$

The last expression can be rewritten in matrix form if we replace P^{k}_{ℓ} by capacities of corresponding groups of origins (see section 3.3). Then:

$$E^{k} = \alpha^{k} F^{Nk} N + \beta^{Bk} F^{Bk} E^{B} + \sum_{s=1}^{R} \beta^{sK} F^{sk} E^{s} \tag{32}$$

where F are row-normalized square matrices with elements equal to

$$\frac{\nu^{k}_{\ell j}}{\sum\limits_{j} \nu^{k}_{\ell j}}$$

The superscript indices of F in (32) correspond to indices for curves f(t) in (25). Hence (31) and (32) define the operator A of the iteration process (24).

Alternative Iterative Structure

Consider briefly another approach to service allocation. As noted earlier, service allocation $E^{k(v)}$, obtained at the previous stage of the iteration process (24), is given by (32).

Now we construct the second-level process (with index w) at each stage of the process (24)

$$\{E^{k(v,w+1)},\ k = 1,\ldots,R\} = A\{N^{(v)};\ E^{s(v,w)},\ s = 1,\ldots,R\} \quad (33)$$

where we consider the limit value $\{E^{k(v,w)}, k = 1,\ldots,R\}$ at $w \to \infty$ of the process as a limit to service allocation at the stage v+1. The resulting allocation does not depend on $E^{s(v)}$, $s = 1,\ldots,R$, i.e.,

$$\{E^{k(v+1)},\ k = 1,\ldots,R\} = \hat{A}\{N^{(v)}\} \quad (33')$$

We can compute the limit value $E^{k(v+1)}$ by considering (32) as a system of linear equations with RL variables E_j^k, $j = 1,\ldots,L;\ k = 1,\ldots,R$. The value of RL in practical problems is large, therefore this system must be solved by iteration methods.

In the special case $F^{sk} = I$ (the identity matrix) and $\beta^{sk} = \delta^k\ \forall\ s,k$, the solution of the system (32) can be obtained in explicit form. From (32) we have

$$(1 - \delta^k)E^k - \delta^k \sum_{s \neq k} E^s = T^k \quad , \quad k = 1,\ldots,R \quad (34)$$

where

$$T^k = \alpha^k F^{Nk} N + \beta^{Bk} F^{Bk} E^B$$

Then the solution is of the form

$$E^k = g(1 - \sum_{s \neq k} \delta^s)T^k + g\ \delta^k \sum_{s \neq k} T^s \quad (34')$$

$$k = 1,\ldots,R$$

where

$$g = \frac{1}{1 - \sum_s \delta^s}$$

which may be verified by direct substitution into (34).

3.5 Population Allocation Operator

In the problem (27), the objective function is convex, and in addition constraints $y_{ji} \geq 0$ are redundant, except for $i \in I^n$. As a result, the necessary and sufficient condition of optimality is (according to Kuhn-Tucker's theorem) the zero-value of the generalized Lagrangian gradient. The Lagrangian is:

$$L = \sum_{j,i=1}^{L} y_{ji} \ln \frac{\mu_{ji}}{y_{ji}} + \sum_{j=1}^{L} \theta_j \left(E_j - \sum_{i=1}^{L} y_{ji} \right)$$

$$+ \sum_{i \in I^f} \xi_i \left(c^f N_i^f - \sum_{j=1}^{L} y_{ji} \right) \qquad (35)$$

$$+ \sum_{i \in I^n} \gamma_i \left(c^n d_i - \sum_{j=1}^{L} y_{ji} \right)$$

The Kuhn-Tucker optimality conditions are:

$$\frac{\partial L}{\partial y_{ji}} = \ln \frac{\mu_{ji}}{y_{ji}} - 1 - \theta_j - \begin{cases} \xi_i , & i \in I^f \\ \gamma_i , & i \in I^n \end{cases} \qquad (35')$$

with additional conditions, when $i \in I^n$:

$$\gamma_i (c^n d_i - \sum_j y_{ji}) = 0 \quad ; \quad \gamma_i \geq 0 \qquad (35'')$$

Generalized Balancing Algorithm

To solve problems similar to (27), with exact constraints on the rows and columns of matrix y_{ji} only, the "balancing algorithm" has been widely used. This consists of step-by-step normalization of the groups of constraints [see, for example, Imelbaev (1978a) and also section 1.6]. Now let us formulate a *generalized balancing algorithm*. First, transform the expression for y_{ji} (35') into a form:

$$y_{ji} = \begin{cases} a_j\, b_i\, \mu_{ji} \quad , & i \in I^f \\ a_j\, c_i\, \mu_{ji} \quad , & i \in I^n \end{cases} \tag{36}$$

where

$$a_j = \exp(-1 - \theta_j) \quad , \quad b_i = \exp(-\varepsilon_i) \quad , \quad c_i = \exp(-\gamma_i)$$

The conditions (35") take the form

$$\begin{aligned} c_i &= 1 \qquad \text{if} \qquad \sum_j y_{ji} < c^n d_i \\ c_i &< 1 \qquad \text{if} \qquad \sum_j y_{ji} = c^n d_i \end{aligned} \tag{36'}$$

It follows from (36) and (36') that any multiplier c_i is equal to one (does not effect the solution) if its corresponding constraint is not binding. Otherwise the value of c_i must be reduced in order to normalize row i.

The multipliers a_j, b_i, c_i have to satisfy the constraints in (27)

$$a_j\left(\sum_{i\in I^f} b_i\,\mu_{ji} + \sum_{i\in I^n} c_i\,\mu_{ji}\right) = E_j \quad , \quad j = 1,\ldots,L$$

$$b_i \sum_j a_j\,\mu_{ji} = c^f\,N_i^f \quad , \quad i\in I^f \qquad (37)$$

$$c_i \sum_j a_j\,\mu_{ji} \le c^n\,d_i \quad , \quad i\in I^n$$

The algorithm solving the problem (27) follows from (37) and (36'):

$$a_j^w = \frac{E_j}{\sum\limits_{i\in I^f} b_i^{w-1}\,\mu_{ji} + \sum\limits_{i\in I^n} c_i^{w-1}\,\mu_{ji}} \quad , \quad j = 1,\ldots,L$$

$$b_i^w = \frac{c^f\,N^f}{\sum\limits_j a_j^w\,\mu_{ji}} \quad , \quad i\in I^f \qquad (38)$$

$$c_i^w = \min\left(1, \frac{c^n\,d_i}{\sum\limits_j a_j^w\,\mu_{ji}}\right) \quad , \quad i\in I^n$$

Here, set $b_j^0 = c_i^0 = 1$. The superscript $w = 1,2,\ldots,$ refers to the iteration step.

The following theorem can be proved.

Theorem 1. If the feasible set for the problem (27) is not empty, the algorithm converges to the values a_j^*, b_i^*, c_i^* corresponding to $\overset{o}{y}_{ji}$, the only solution of (27).

Proof of this theorem is based on the reduction of (27) to the conditions of the more general theorem (Movshovitch 1976) on convergence algorithms of this form.

The values N_i, $i\in I^n$ which are computed from $\overset{o}{y}_{ji}$ according to (27'), define the operator $N = B(E)$ of the process (24).

3.6 Aggregate Operators of Subsystem Allocations: The Link with the Lowry Model

Recall the main characteristics of the Lowry model. Its basic differences from those of the model presented here consist in: absence of the districts with populations fixed *a priori*, and a "gravity" principle applied to interaction between individual subsystems.

The first level of the Lowry model consists of a "linear" distribution of the subsystems, i.e.,

$$E_j^k = \alpha^k \sum_i N_i \frac{f^{EN}(c_{ij})}{\sum_j f^{Nk}(c_{ij})} + \delta^k E_j \tag{39}$$

$$N_i = \frac{1}{c} \sum_j E_j \frac{f^{EN}(c_{ij})}{\sum_j f^{EN}(c_{ij})} \tag{39'}$$

where

$$f^{Nk}(c_{ij}) \qquad , \qquad f^{EN}(c_{ij})$$

are functions, analogous to those discussed in sections 1 and 3.1, dependent on generalized costs c_{ij} of the trips between districts and normalized respectively along rows and columns; α^k, δ^k, c are balancing coefficients.

Note that (39) is similar to expressions (31) and (32) for service allocation, but differs from them in the method of computing ν_{ij}. The influence of basic sector work places on the "service" subsystem in the Lowry model corresponds to the *special case* outlined in section 3.4. As shown there, the values E_j^k can be computed from (39) in the explicit form (34').

The second level of the Lowry model accounts for constraints (22). For this purpose the values N_i, computed from (39), are changed according to a heuristic algorithm (see below).

We shall now outline some forms for computing operators of the model which are more convenient than those obtained in sections 3.4 and 3.5. These will be compared with the Lowry model algorithms.

Aggregate Operator of Population Allocation

Solution of problem (27) is derived by the exact algorithm (38). Note that the solution for the flows y_{ji}^0, is used at the end to compute values N_i, $i \in I^n$ only. Let us formulate a two-stage algorithm to solve two problems aggregated from (27) and compute these values in another way.

To formulate the *first problem* we consider the same origins and destinations, as in (27), but with only two destination constraints, derived by summation (aggregation) of the constraint groups over I^f and I^n:

$$\max_{u_{ji}} \sum_{i,j=1}^{L} u_{ji} \ln \frac{\mu_{ji}}{u_{ji}}$$

subject to

$$\sum_{i \in I^n} u_{ji} + \sum_{i \in I^f} u_{ji} = E_j \quad , \qquad j = 1,\ldots,L$$

$$\sum_{i \in I^f} \sum_{j=1}^{L} u_{ji} = c^f \bar{N}^f$$

$$\sum_{i \in I^n} \sum_{j=1}^{L} u_{ji} = c^n \bar{N}^n \tag{40}$$

$$u_{ji} \geq 0$$

In general the flows $\overset{o}{u}_{ji}$ computed as the result of solving (40) will not satisfy the separate constraint equalities and inequalities at each destination i.

We shall interpret the values

$$T_i = \frac{1}{c^n} \sum_{j=1}^{L} \overset{o}{u}_{ji} \quad , \qquad i \in I^n \tag{40'}$$

as non-normalized prior probabilities ("preferences") of population allocation in the districts $i \in I^n$ without taking into account the constraint-inequalities.

Now, excluding districts $i \in I^f$, we can formulate *the second problem* of reallocating residents from one composite origin with capacity $\bar{N}^n$ to destinations with limited capacities d_i. Removing the origin index j, we have

$$\max_{N^i} \sum_{i \in I^n} N_i \ln \frac{T_i}{N_i}$$

subject to

$$\sum_{i \in I^n} N_i = \bar{N}^n \tag{41}$$

$$0 \le N_i \le d_i$$

The values N_i^* computed as the result of optimizing (40) constitute the objective for population allocation.

The algorithms of the problems (40) and (41) correspond to the stages of the aggregate operator.

Solution of *the first problem* is computed analogously to (9) using the expression

$$u_{ji} = \begin{cases} a_j \; g^f \; \mu_{ji} & , \quad i \in I^f \\ a_j \; g^n \; \mu_{ji} & , \quad i \in I^n \end{cases} \tag{42}$$

where a_j, g^f, g^n correspond to Lagrange multipliers calculated from the system of equations. These are obtained by the substitution of (42) into (40)

$$a_j (g^f \rho_j^f + g^n \rho_j^n) = E_j \quad , \quad j = 1,\ldots,L$$

$$g^f \sum_j a_j \rho_j^f = c^f \bar{N}^f \tag{42'}$$

$$g^n \sum_j a_j \rho_j^n = c^n \bar{N}^n$$

where

$$\rho_j^f = \sum_{i \in I^f} \mu_{ji} \quad \text{and} \quad \rho_j^n = \sum_{i \in I^n} \mu_{ji}$$

The solution to (42') is computed by the standard balancing method mentioned above. The relatively low dimension of (42') reduces the computational difficulties significantly.

Substituting (42) into (40') and using (42') we have

$$T_i = \frac{g^n}{c^n} \sum_j E_j \frac{\mu_{ji}}{g^f \rho_j^f + g^n \rho_j^n} \tag{42''}$$

Note. If no districts have predetermined population sizes, i.e.:

$$I^f = \phi \quad , \quad I^n = \{1,\ldots,L\}$$

$$\rho_j^f = 0 \quad , \quad \rho_j^n = \sum_{i=1}^{L} \mu_{ji}$$

the expression (42") coincides with (39'), although the method of calculating μ_{ji} differs and therefore corresponds to the first level of the Lowry model.

Now let us move to the solution of *the second problem* taking into account the land-constraints algorithm in the Lowry model. According to this algorithm

$$N_i^* = \begin{cases} \psi^* T_i & , \quad i \in I_1(\psi^*) \\ d_i & , \quad i \in I_2(\psi^*) \end{cases} \tag{43}$$

where $I_1(\psi)$ and $I_2(\psi)$ are subsets of indices from I^n, which depend on the parameter ψ:

$$\begin{aligned} I_1(\psi) &= \{i : i \in I^n, \ \psi T_i \le d_i\} \\ I_2(\psi) &= \{i : i \in I^n, \ \psi T_i > d_i\} \end{aligned} \tag{43'}$$

and ψ^* is computed by a finite iteration process

$$\psi_{w+1} = \frac{\overline{N}^n - \sum\limits_{i \in I_2(\psi_w)} d_i}{\sum\limits_{i \in I_1(\psi)} T_i} \tag{43"}$$

The following theorem can be proved (see Appendix):

Theorem 2. Algorithm (43), (43') and (43") solves (41) if a solution exists.

From this the aggregate population allocation operator $N = \hat{B}(E)$ can now be defined. It can also be proved that the operator is the same as the one used in the Lowry model and leads the population subsystem to a stable state (see section 3.2), which differs from the state defined by operator $N = B(E)$ of section 3.5.

Service Allocation Accounting of Constraints

The constraints $E_j^k \le \varepsilon_j^k$ can arise in the service allocation problem [see remarks to equations (3) and (4)]. In addition, if only a small number of service work places is required within a district, i.e., if $E_j^k < \varepsilon_j^k$, then we must set $E_j^k = 0$. Therefore the exact formulation of the problem is

$$\max_{J_3,\, x_{\ell j}^k} \sum_{\ell=1}^{(2+R)L} \sum_{j \in J_3} x_{\ell j}^k \ln \frac{\nu_{\ell j}^k}{x_{\ell j}^k} \qquad (44)$$

subject to

$$\sum_{j \in J_3} x_{\ell j}^k = P_\ell^k \quad , \qquad \ell = 1,\ldots,(2+R)L$$

$$\varepsilon_j^k \le \sum_{\ell=1}^{(2+R)L} x_{\ell j}^k \le \hat{\varepsilon}_j^k \quad , \qquad j \in J_e$$

$$x_{\ell j}^k \ge 0$$

where the index set $J_3 \subset \{1,\ldots,L\}$ corresponds to the destinations for which

$$E_j^k = \sum_\ell x_{\ell j}^k \ge \varepsilon_j^k$$

For all other destinations j, $x_{\ell j}^k = 0$.

In the problem (44), in addition to flows $x_{\ell j}^k$ it is required to compute the set J_3. Therefore, (44) is a mixed-integer programming problem. If J_3 is fixed the solution may be computed by a slight change in the algorithm for (27), but computation of optimal *outflows* is unlikely.

In practice approximate solution of this task can be achieved only through the aggregate Lowry algorithm. At its first stage the values $\hat{E}_j^k$ are calculated without taking into account the constraints (32) or (34'). At the second stage they constitute a base for computing the set J_3 and the values $\hat{\hat{E}}_j^k \geq \varepsilon_j$, $j \in J_3$. At the third stage the values E_j^k are computed analogously to algorithm (43), (43') and (43").

4. CONCLUSION

The models discussed in this paper are characterized by assumptions of homogeneity with regard to the behavior of residents, within the framework of the observed patterns. The homogeneity assumptions allow a unified formal definition of both the processes of daily population movement (using the transport system) and global migration (development of an entire urban system). But one should emphasize that only *models* of population behavior are constructed here. These models are only intended to provide aggregate estimates as to the consequences of alternative city-planning decisions.

APPENDIX: PROOF OF THEOREM 2

1. The problem (41) may be reduced to:

$$\max_{N^i} \left\{ \sum_{i \in I^n} N_i \ln \frac{T_i}{N_i} + \lambda \left(\sum_{i \in I^n} N_i - \bar{N}^n \right) \right\} \tag{A1}$$

$$0 \le N_i \le d_i \quad , \quad i \in I^n$$

where λ is the appropriate Lagrange multiplier from (41). The unconditional maximum of the function is at the point

$$\begin{aligned} \hat{N}_i &= \exp(-1 + \lambda) T_i \\ &= \psi \; T_i \end{aligned} \tag{A2}$$

If some values $T_i = 0$ then the dimension of the space of function values may be reduced.

2. The objective function and constraints of the problem (A1) are separable and therefore the maximum of the function corresponds to the maximum of each term

$$f(N_i) = N_i \left(\ln \frac{T_i}{N_i} + \lambda \right)$$

under the constraint $0 \leq N_i \leq d_i$. As $f(0) = 0$ and $f(N_i)$ increases monotonically when N_i increases from zero to $\hat{N}_i$ (A2, see, for example, Imelbaev 1978b) the maximum is reached at the point

$$N_i(\psi) = \begin{cases} \hat{N}_i = \psi\, T_i & \text{if} \quad \psi\, T_i \leq d_i \\ d_i & \text{if} \quad \psi\, T_i > d_i \end{cases} \qquad \text{(A3)}$$

Introduce the function

$$U(\psi) = \sum_{i \in I^n} N_i(\psi) \qquad \text{(A4)}$$

where $N_i(\psi)$ are determined from (A3). The constraint-equality in (41), i.e.,

$$U(\psi) = \overline{N}^n \qquad \text{(A5)}$$

is intended for computing ψ^* and is, therefore, the desired distribution $N_i(\psi^*)$.

4. Let us analyze the function $U(\psi)$ and use the definition (43') of index subsets $I_1(\psi)$ and $I_2(\psi)$. Then

$$I_1(\psi) \cap I_2(\psi) = \phi$$

$$I_2(\psi) \cup I_2(\psi) = I^n$$

and if ψ increases from zero to $\overline{\psi}$

$$\overline{\psi} = \max_i \frac{d_i}{T_i}$$

then all the indices i, pass step by step from $I_1(\psi)$ to $I_2(\psi)$. Using (A3), (A4):

$$U(\psi) = \sum_{i \in I_2(\psi)} d_i + \psi \sum_{i \in I_1(\psi)} T_i \qquad \text{(A6)}$$

Therefore $U(\psi)$ for $0 \le \psi \le \overline{\psi}$ is a continuous piecewise-linear monotonically increasing function (Figure A1).

5. A unique solution of equation (A5) exists for

$$\sum_{i \in I^n} d_i \ge \overline{N}^n$$

Any standard method for identifying the root of a function of one variable may be used to solve for this. Let us consider Newton's method. The method leads to the iterative process, for some equation $\varphi(x) = 0$;

$$x_{w+1} = x_w - \frac{\varphi(x_w)}{\varphi'(x_w)}$$

where $\varphi'(x_w)$ is the derivative of $\varphi(x)$ at the point x_w. In our problem after elementary transformations using (A6) we have expression (42"), which was to be demonstrated.

Note. It is convenient to put $\psi_0 = 0$; followed by $\psi_1 = 1$, and the sequence ψ_w increases up to the value ψ^* corresponding to the solution of the equation (A6).

The verification of the constraints $\psi_w T_i \le d_i$, and transition of those indices satisfying these constraints into subset I_2, take place at each stage of the process (43"). In the same way as the index set is finite, the process (43") is also finite. The condition for termination of the process is:

$$I_s(\psi_w) = I_s(\psi_{w+1}) \quad ; \quad s = 1,2$$

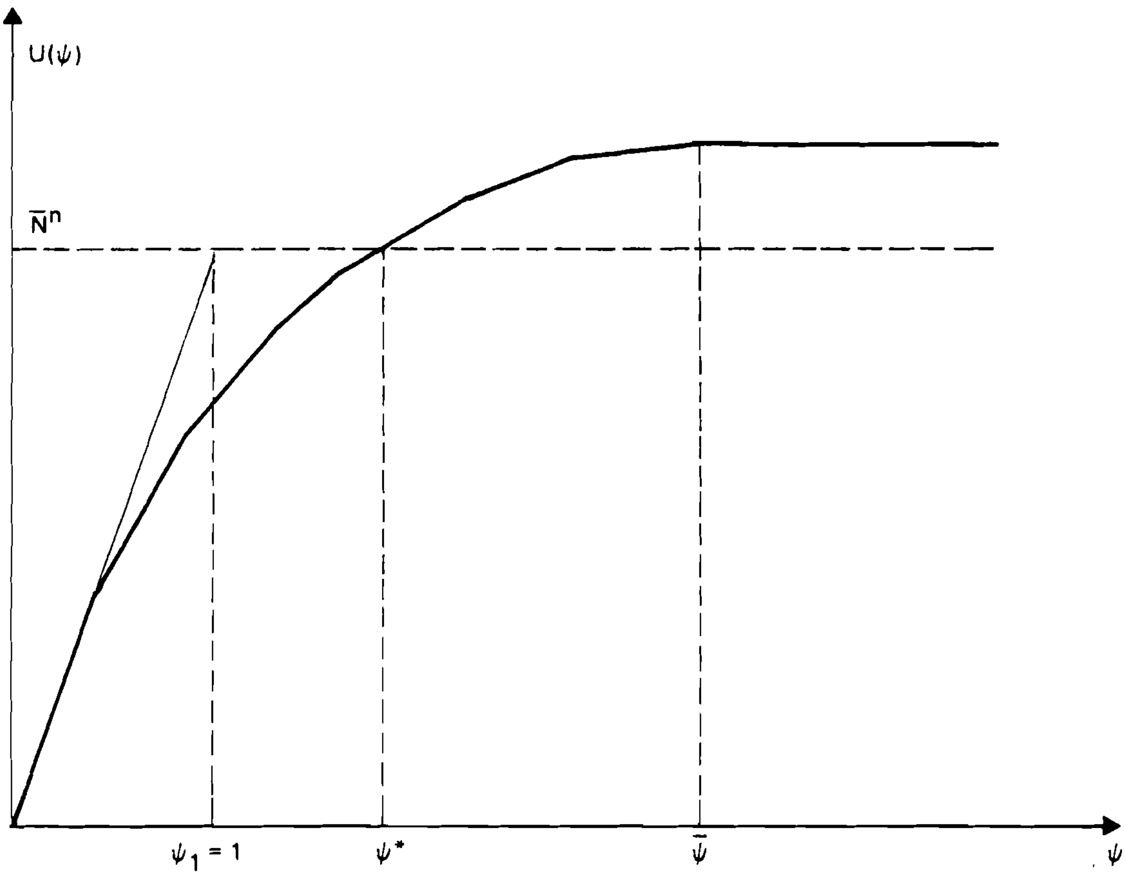


Figure A1. The form of $U(\psi)$.

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