

# THE ANALYSIS OF CAUSAL LINKAGES IN DEMOGRAPHIC THEORY

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## PREFACE

Since its inception, IIASA has had a keen interest in the methods of demography. At present, demography lacks a method for doing general sensitivity analysis -- a method that yields closed-form analytical expressions for the response of demographic indices to changes in age- and time-specific functions.

This paper develops a method for sensitivity or causal-linkage analysis in demography; this method is then applied to several unsolved problems in the field.



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## THE ANALYSIS OF CAUSAL LINKAGES IN DEMOGRAPHIC THEORY

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### SUMMARY

*Many seemingly different questions that interest demographers can be phrased as the same technical question: how, within a given demographic model, would variable  $y$  change if the age- or time-specific function  $f$  were to change arbitrarily in shape and intensity? At present demography lacks the machinery to answer this question in analytical and general form.*

*This paper suggests a method, based on modern functional calculus, for deriving closed-form expressions for the sensitivity of demographic variables to changes in input functions or schedules. It uses this "causal linkage" method on three bodies of theory: stable population analysis, nonstable or transient population analysis, and techniques for the estimation of incomplete demographic data.*

*In stable theory, closed-form expressions are obtained for the response of the intrinsic growth rate, birth rate, and age composition to arbitrary marginal changes in the age patterns of fertility and mortality.*

*In nonstable theory, expressions are obtained for the transient response of the age composition to time-varying changes in the birth sequence, and to changing age-specific fertility and mortality patterns. The problem of "bias" in period vital rates is also looked at.*

*In incomplete-data analysis, a general format for robustness or error analysis is suggested; this is applied to a standard Brass estimation technique.*

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### INTRODUCTION

Many of the questions that appear and reappear in the demographic literature of this century, while seemingly quite different, are but specific instances of a single and fundamental question: *How do the aggregate measures — numbers, rates, and distribution — of a population change, when its underlying behavior at the individual level changes?*

That this question should arise with regularity in various different guises is hardly surprising. Much of demographic theory aims to translate the events and consequences of

individual lives – the timing and number of children, the choice of area of residence, entry to the work force, time of retirement, age and cause of death – into the grand measures of society itself, its size and growth rate, its distribution by region, its vital rates, its proportions by age, sex, work or other category. And since human behavior over the life cycle, reflecting social habit and environment, is forced to change as society evolves, the demographer in turn is forced to seek analytical ways to translate these shifts in individual behavior into changes in the aggregate population measures he uses.

Mathematical demography bridges the gap between individual behavior and aggregate measures by observing that human behavior and the main events in human life are closely tied to age. It captures and frames these events by means of demographic schedules or functions – statistical summaries of individual behavior along the age and time dimensions. It then uses these as inputs to mathematical models, sometimes simple, sometimes elaborate, the output or end-result being variables that represent the aggregate measures of growth, distribution, number, and rate. We may therefore pose the question of causal linkage between individual behavior and aggregate measures in a more analytical and precise way: *How, within a given demographic model, would arbitrary changes in its age- and time-specific schedules alter the output variables that interest us?*

Within the present body of demographic theory there is no way to answer this question. The obstacle is a technical one. To answer, with generality and precision, how a change in age or time function  $f$  would affect variable  $y$ , the demographer needs to use some form of sensitivity analysis. Were  $f$  a simple variable, or even a vector, the analysis would be straightforward. Elementary calculus could be brought to bear, the derivative  $\partial y/\partial f$  constructed, and the differential change in  $y$  written down as a function of the change in  $f$ . But standard calculus allows us no way of taking derivatives with respect to functions, and thus we reach an impasse. Questions of key interest to demographers – how the age composition responds to an arbitrary change in the mortality pattern, or how period vital rates are affected by changes in the birth sequence – therefore remain without analytical solution. What is needed, for questions of causal linkage in demographic theory, is machinery more powerful than standard calculus.

In the absence of such machinery, demographers have developed several ways to investigate the effects of changes in age schedules. None of these is entirely satisfactory. The simplest possibility, blunt but effective, is to calculate numerically the value of the variable  $y$  before and after the behavioral change in the schedule. But this gives no general expression for an arbitrary change: each case must be calculated separately. A second possibility is to parameterize the age schedule in question, and try to capture changes in its shape by changes in the parameters. This reduces the problem to the standard-calculus procedure of varying parameters. But parameterization can be tedious, and again no general expression results. A third possibility is to look only at special cases, restricting the change in the age pattern to a certain simple shape – a simple increase in intensity, for example. But here again no general insights are guaranteed.

This paper proposes a method of deriving the effect on demographic variables of arbitrary changes in age- and time-specific functions directly as closed-form expressions, without resorting to numerical techniques or to parameterization. It draws on concepts from modern functional analysis to construct a “causal linkage” method suited to demographic problems. If it is true that many open questions in demography call for such a method, then we would expect this method to yield new results. This turns out to be the

case. Some of the results we obtain are quite general; others are for special cases. One or two have been obtained in the literature before; most are new.

The paper is arranged as follows. Section 1 proposes a general linkage method, illustrating it with examples. Sections 2 and 3 apply the method to stable theory, in particular to how arbitrary changes in the age patterns of fertility and mortality affect the intrinsic growth rate, birth rate, and age composition. These results show, among other things, why fertility has more effect on the age composition than does mortality. Section 4 takes up a special case as an illustration: age delays in childbearing. Section 5 turns to nonstable theory, investigating how the age composition responds to changes in the birth sequence and in fertility and mortality patterns. Section 6 looks at the “bias” produced in vital rates by changes in the age composition and birth sequence, and suggests how this may be partially corrected. Section 7 takes up the problem of incomplete data estimation, and proposes a method for assessing errors in the estimates, given underlying assumptions that are not perfectly fulfilled. It illustrates this with an example based on a standard Brass estimation technique. The main sections are largely independent, but the reader is urged to understand the method before turning to any applications.

## 1 THE CAUSAL LINKAGE METHOD

In this section I shall develop both the vocabulary we need and a general method for linkage analysis in demographic theory. I will avoid abstractions, and will assume (at some cost to full mathematical rigor) that the functions dealt with inhabit appropriate, if unspecified, spaces and that they are smooth enough to allow the operations we want. I start by reviewing briefly the familiar standard theory of differential changes, then spend some time extending it to differentials with respect to functions, finally proposing a general method for sensitivity or linkage analysis in demographic theory.

### The Standard Theory

Begin with  $y$  as a simple function of the variable  $x$ :

$$y = f(x) \tag{1}$$

Given that  $x$  is increased an amount  $h$ , the familiar standard calculus tells us that a good approximation to the change in  $y$ , when  $h$  is small, is given by the *differential*  $\delta y$ , defined by

$$\delta y [h] = f'(x)h \tag{2}$$

where  $f'$  is the derivative or gradient taken at  $x$ , and is itself a function of  $x$ . (Since the differential,  $\delta y$ , is a function of the change  $h$ , evaluated at  $x$ , we write it as  $\delta y [x; h]$ ; or, when  $x$  is understood, as  $\delta y [h]$ ; or, when  $h$  is also understood, simply as  $\delta y$ .)

If we merely want the change in  $y$  on going from  $x$  to  $x + h$ , why bother with the differential, an approximation? Why not calculate  $y(x + h) - y(x)$  directly? This, of course, is possible. But the differential has two advantages. First, it applies to all  $x$  in the

domain of the function and to all changes  $h$  — it is more general. Second, and more important, useful qualitative information is usually contained in the expression for  $f'(x)$ ; the connection between  $y$  and  $x$  can therefore be usefully interpreted and studied.

One way to define the differential directly, without resorting to the gradient, is via the limit

$$\delta y [x; h] = \lim_{\alpha \rightarrow 0} \left( \frac{f(x + \alpha h) - f(x)}{\alpha} \right) \quad (3)$$

Thus the change in  $y$  caused by a small step in the direction of  $h$ , divided by the step length, can be shown in the limit to yield the same linear approximation as the differential in  $y$  given in eqn. (2). This fact will be useful below.

Finally, recall that if  $y$  is a function of several variables  $y = f(x_1, x_2, \dots, x_n)$  and if changes  $h_i$  occur in the variables  $x_i$  (with indices  $i$  in the set  $I$ ), the others being held constant, the differential in  $y$  becomes the summation

$$\delta y = \sum_{i \in I} (\partial f / \partial x_i) h_i$$

### Functional Differentials

So much for the standard theory. We now proceed to the case of interest in this paper. This time we begin with a function  $z$ , whose domain in demographic theory is usually age or time.

At the outset a notational difficulty must be cleared up. Elementary textbooks often write the function  $z$  as  $z(a)$ , where they mean the entire function over the range of  $a$ . Since this might be confused with the *value* of  $z$  at point  $a$ , I shall follow modern notation and reserve the label  $z$  for the function itself, using  $z(a)$  for its value at point  $a$ .

Typically, in demographic theory, models are built out of functions (and variables), the simplest possible being

$$y = F(z) \quad (4)$$

Here  $F$  is a rule which assigns a real number  $y$  to a given curve  $z$ , and is called a *functional*. As examples

$$F(z) = \max_{0 \leq t \leq 1} z(t)$$

$$R_0(p, m) = \int_0^\omega p(a)m(a)da$$

are functionals. The first attaches a real value to the curve  $z$ ; the second, a functional of two functions, assigns a real value,  $R_0$ , given the curves  $p$  and  $m$ . Demographers will recognize the second example as the net reproduction rate, where  $p$  and  $m$  are the age schedules of survival and fertility, respectively.

Now suppose that the function  $z$  changes shape as in Figure 1, so that it becomes  $z + h$ , where the perturbation  $h$ , itself a function, is small. How much will the value of  $y$

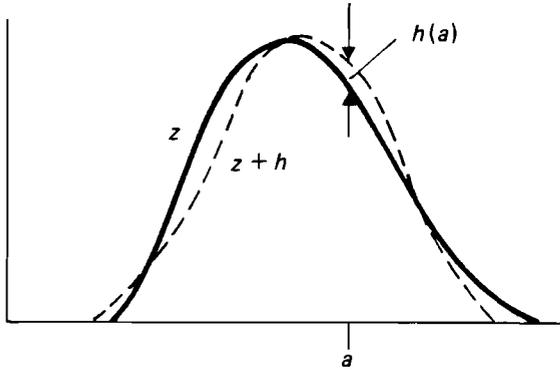


FIGURE 1 A small perturbation  $h$  in function  $z$ .

change? By analogy with eqn. (3), we can simply define the differential (now called a *functional differential*) to be

$$\delta y [z; h] = \lim_{\alpha \rightarrow 0} \left( \frac{F(z + \alpha h) - F(z)}{\alpha} \right) \quad (5)$$

providing, of course, that this limit exists. Just as the simple differential approximates the change in  $y$  when  $x$  changes by an amount  $h$ , the functional differential approximates the change in  $y$  if the function  $z$  is perturbed or changed by a function  $h$ .

As yet this gives no simple way to derive  $\delta y$  without taking limits. However, it is easy to show\* that since  $F(z + \alpha h)$  is a simple function of the parameter  $\alpha$ , eqn. (5) can be rewritten as an ordinary derivative in  $\alpha$ , evaluated at  $\alpha = 0$ :

$$\delta y [z; h] = \frac{d}{d\alpha} F(z + \alpha h) \Big|_{\alpha=0} \quad (6)$$

Usually this gives a convenient way to derive the differential.

*Example 1.* A functional form that often occurs in demography is

$$y(z) = \int_0^{\omega} g(z(a)) da \quad (7)$$

i.e., an integral of a function  $g$  of function  $z$  at age  $a$ . We may write

\*To see this, write the derivative (6) as

$$\lim_{c \rightarrow 0} \frac{F(z + (\alpha + c)h) - F(z + \alpha h)}{c} \Big|_{\alpha=0}$$

This equals

$$\lim_{c \rightarrow 0} \left( \frac{F(z + ch) - F(z)}{c} \right)$$

which is the same as eqn. (5).

$$\begin{aligned}
\delta y [z; h] &= \frac{d}{d\alpha} \int_0^\omega g(z(a) + \alpha h(a)) da \Big|_{\alpha=0} \\
&= \int_0^\omega \frac{d}{d\alpha} g(z(a) + \alpha h(a)) da \Big|_{\alpha=0} \\
&= \int_0^\omega \frac{\partial g}{\partial z(a)} h(a) da
\end{aligned} \tag{8}$$

the last step following from standard calculus.  $\square$

*Example 2.* We may try this formulation on the net reproduction rate example mentioned above.

$$R_0(p, m) = \int_0^\omega p(a)m(a)da \tag{9}$$

Suppose that the fertility schedule  $m$  is perturbed by a function  $\delta m$ , the survival schedule  $p$  remaining fixed; what is the differential in the variable  $R_0$ ? Applying the rule from Example 1:

$$g(m(a)) = p(a)m(a)$$

$$\frac{\partial g}{\partial m(a)} = p(a)$$

whence

$$\delta R_0 [m; \delta m] = \int_0^\omega p(a) \delta m(a) da \tag{10}$$

Knowing the variation in the fertility schedule, we can easily calculate  $\delta R_0$ .  $\square$

Functional differentials obey the same rules as normal differentials:

$$1. \quad y = aF(z) + bG(z) \tag{summation}$$

$$\delta y = a\delta F + b\delta G$$

$$2. \quad y = F(z)G(z) \tag{product}$$

$$\delta y = \delta F G(z) + F(z) \delta G$$

$$3. \quad y = G(F(z)) \tag{composition}$$

$$\delta y = \left( \frac{\partial G}{\partial F} \right) \delta F [\delta z]$$

4.  $y = F(z_1, z_2, x)$  (where  $z_1$  and  $z_2$  are functions and  $x$  is a variable)

$$\delta y = \delta F[\delta z_1] + \delta F[\delta z_2] + \delta F[\delta x]$$

5.  $y = \frac{F(z)}{G(z)}$  (quotient)

$$\delta y = \frac{G(z)\delta F - F(z)\delta G}{G^2(z)}$$

*Example 3.* The survival schedule  $p$  is connected to the force-of-mortality schedule  $\mu$  by the equation

$$p(a) = \exp\left(-\int_0^a \mu(t)dt\right) \tag{11}$$

The value  $p(a)$ , in other words, is a *functional* of  $\mu$ . Given a change  $\delta\mu$  in the function  $\mu$ , caused, say, by a change in the incidence of a certain disease, how will the survival schedule  $p$  change?

Let

$$F(\mu) = -\int_0^a \mu(t)dt$$

so that

$$\delta F[\delta\mu] = -\int_0^a \delta\mu(t)dt$$

Now

$$p(a) = e^F$$

so that from the composition rule above

$$\delta p(a) = \frac{\partial(e^F)}{\partial F} \cdot \delta F[\delta\mu] = -e^F \int_0^a \delta\mu(t)dt \tag{12}$$

$$\delta p(a) = -p(a) \int_0^a \delta\mu(t)dt$$

This gives a rule for the differential change  $\delta p$  in the entire survival function  $p$  caused by an alteration  $\delta\mu$  in the force-of-mortality function  $\mu$ . □

Each of the examples discussed so far proceeds tediously, step by step. With practice however, as in elementary calculus, it is possible to write down expressions for the differential by inspection.

A final piece of vocabulary will be useful. In standard calculus we can write the differential as a product

$$\delta y = f'(x) \cdot \delta x$$

calling the coefficient of the change in  $x$  the derivative. In our examples above we can write the differential in the product form:

$$\delta y = \int_0^\omega F'(z, a) \cdot \delta z(a) da \quad (13)$$

By analogy we may call the function  $F'(z)$ , the “coefficient” of the change in  $z$ , the *functional derivative* (or Fréchet derivative) of  $F$  at function  $z$ , understanding that the product here is an inner product. In Example 3, the derivative of the survival schedule at age  $x$ , with respect to the force-of-mortality function  $\mu$ , is  $-p(x)$ .

In the functional case it is not always possible to write the differential in this product form, hence a derivative does not always exist. But where it does we need only retain the information  $F'$ : by taking the inner product of the function  $F'$  and the change in  $z$  we can summon the differential when needed.

### The General Method

We have now assembled enough machinery to construct a fairly simple method for analyzing causal linkages in demographic theory.

Assume that we have a model that expresses variable  $y$  explicitly in terms of functions  $z_i$  and variables (or parameters)  $x_j$ :

$$y = F(z_1, z_2, \dots, z_m, x_1, x_2, \dots, x_n) \quad (14)$$

If we decide which functions  $z_i$  may change, say those for  $i$  in some set I, and which variables  $x_j$  may change, say those for  $j$  in some set J, we can write the differential change in  $y$  as

$$\delta y = \sum_{i \in I} \delta F[\delta z_i] + \sum_{j \in J} \delta F[\delta x_j] \quad (15)$$

We can derive each differential  $\delta F[\delta z_i]$  separately according to the rules above, and we can derive the differentials  $\delta F[\delta x_j]$  quite simply as  $(\partial F/\partial x_j)\delta x_j$ . We now have the sought-for method.

A variant of this method must be used when the variable of interest  $y$  is contained implicitly in the model. In this case we have the implicit functional model

$$0 \equiv H(y, z_1, z_2, \dots, z_m, x_1, x_2, \dots, x_n) \quad (16)$$

As before, we allow certain functions ( $z_i$ ) and variables ( $x_j$ ) to change. The variable  $y$  will respond by the change  $\delta y$ . To maintain the identity at zero all changes must sum to zero. Hence

$$0 = \delta H[\delta y] + \sum_{i \in I} \delta H[\delta z_i] + \sum_{j \in J} \delta H[\delta x_j]$$

that is,

$$0 = \frac{\partial H}{\partial y} \cdot \delta y + \sum_{i \in I} \delta H [\delta z_i] + \sum_{j \in J} \delta H [\delta x_j]$$

$$\delta y = - \left( \frac{\partial H}{\partial y} \right)^{-1} \left\{ \sum_{i \in I} \delta H [\delta z_i] + \sum_{j \in J} \delta H [\delta x_j] \right\} \quad (17)$$

More generally, there may be several implicit equations  $H$  for several variables  $y$ . In this case, we can interpret  $\delta y$  to be a vector of changes,  $\partial H / \partial y$  to be the (nonsingular) Jacobian matrix of partial derivatives of each  $H$  with respect to each  $y$ , and the differentials  $\delta H$  to be assembled in vector form. The same expression then holds.

*Example 4.* To illustrate this method, let us assess the change in the intrinsic rate  $r$  when both the fertility and survival schedules  $m$  and  $p$  change. The characteristic equation connecting  $r$  to functions  $m$  and  $p$  is:

$$0 \equiv \int_0^\omega e^{-ra} p(a) m(a) da - 1 = H(r, p, m) \quad (18)$$

We calculate

$$\frac{\partial H}{\partial r} = - \int_0^\omega a e^{-ra} p(a) m(a) da$$

recognizing this expression as the average age of childbearing in the stable population,  $A_m$ . For perturbations  $\delta p$  and  $\delta m$  we obtain

$$\delta H [\delta p] = \int_0^\omega e^{-ra} \delta p(a) m(a) da$$

$$\delta H [\delta m] = \int_0^\omega e^{-ra} p(a) \delta m(a) da$$

Using eqn. (17) we may write

$$\delta r = \frac{1}{A_m} \left( \int_0^\omega e^{-ra} (\delta p(a) m(a) + p(a) \delta m(a)) da \right) \quad (19)$$

We thus have a general analytical expression for the response of the intrinsic growth rate to arbitrary small changes in the fertility and mortality patterns.  $\square$

We now turn to specific problems in demographic theory.

## 2 EFFECT OF THE FERTILITY PATTERN ON STABLE POPULATION PARAMETERS

We begin the investigation of causal linkages in demography with stable population theory. How does the fertility pattern determine the growth and age composition of a stable population?

There have been several attempts to answer this question. Dublin and Lotka (1925) examined the response of the intrinsic growth rate in the special case where the fertility function is multiplied by a constant factor and shifted slightly over age. Some thirty years later, Coale (1956) extended these results to include the effects on age composition. More recently, Demetrius (1969), Goodman (1971), and Keyfitz (1971) derived formulae for the response of various stable parameters to an increase in fertility at a single arbitrary age  $x$ . And in 1977 Keyfitz derived an approximation for the response of the intrinsic growth rate  $r$  to an arbitrary small perturbation in  $m$ , the fertility function, with a result similar to that given below.

### Fertility Change and the Intrinsic Growth Rate\*

A suitable model that connects  $r$  with the fertility function  $m$  is supplied by the familiar characteristic equation

$$1 = \int_0^{\omega} e^{-ra} p(a) m(a) da \quad (20)$$

where  $a$  is age,  $\omega$  is an upper bound on length of life, and  $p$  is the survival function.

Suppose that the fertility function  $m$  changes, to become  $m^1$ , where the difference  $m^1 - m = \delta m$  is itself a function. The survival schedule  $p$  is assumed to be held fixed. How will  $r$  respond?

Write eqn. (20) in the implicit form

$$H(r, m) = \int_0^{\omega} e^{-ra} p(a) m(a) da - 1 \equiv 0 \quad (21)$$

When  $m$  is perturbed an amount  $\delta m$ , the rate  $r$  changes by  $\delta r$ . To maintain the identity at zero these changes must offset each other. Therefore

$$\delta H[\delta m] + \delta H[\delta r] = \delta H[\delta m] + (\partial H/\partial r)\delta r = 0 \quad (22)$$

Evaluating the differential (as in Example 4) and the partial differential yields

$$\int_0^{\omega} e^{-ra} p(a) \delta m(a) da - \delta r \int_0^{\omega} a e^{-ra} p(a) m(a) da = 0$$

And since the second integral is  $A_m$ , the average age of childbearing in the stable population, we obtain our first result:

$$\delta r = \frac{1}{A_m} \int_0^{\omega} e^{-ra} p(a) \delta m(a) da \quad (23)$$

---

\*Although we have already looked at this problem in Example 4, I shall for completeness of this section rederive the result.

or, in demographic terms,

$$\text{Change in intrinsic growth rate} = \frac{\text{Change in reproductive value at age zero}^*}{\text{Average age of childbearing}}$$

We thus have in eqn. (23) a general, closed-form expression for the response of the intrinsic growth rate to an arbitrary small change in the fertility pattern.\*\* This result was arrived at independently by Keyfitz in 1977, by an approximation argument.

We can immediately extract a simple theorem from this result. *In a growing population, any given pattern of marginal reduction in fertility has more effect if it is concentrated at younger ages.* To show this, suppose we consider a certain “bite”, of shape  $\delta m$ , taken from the fertility function. Recall that  $c(a)$ , the age density at  $a$ , is given in stable theory by

$$c(a) = b e^{-ra} p(a) \tag{24}$$

where  $b$  is the intrinsic birth rate. Substituting this into eqn. (23) yields

$$\delta r = \frac{1}{b A_m} \int_0^\omega c(a) \delta m(a) da \tag{25}$$

a second and new form of the above result. Since  $c(a)$  must decline with age in a growing population, and  $b$  and  $A_m$  are positive, any given pattern of reduction  $\delta m$  will lower  $r$  more if it occurs at earlier ages. Other things being equal, a contraceptive method is more effective in reducing population growth if it is adopted by younger women.

A third, and yet more useful form of the above result is possible. The mean level of fertility in the population,  $\bar{m}$ , can be written as

$$\bar{m} = \int_0^\omega c(a) m(a) da$$

since  $c(a)$  describes the density of the population at age  $a$ . Recalling that  $\delta m = m^1 - m$ , we may write eqn. (25) as

$$\delta r = \frac{1}{b A_m} (\bar{m}^1 - \bar{m}) \tag{26}$$

where both means are taken with respect to the original age distribution  $c$ . The change in the intrinsic growth rate, in other words, equals the change in the mean level of fertility in the population, normalized appropriately. We could use this, for example, to estimate how much fertility would have to change to achieve some prescribed reduction in the growth rate.

---

\*Reproductive value at age 0 is defined as  $\int_0^\omega e^{-ra} p(a) m(a) da$ .

\*\*This exercise is one of comparative statics. We must interpret the change in  $r$  derived here as the differential between two stable populations that differ only in fertility function, or alternatively as the long-run difference in growth trends in a population with altered fertility.

### Fertility and Other Stable Population Parameters

One way to extend these results to the effect on the birth rate and the age distribution is to recall that  $b$  is a function of  $r$ , and  $c(a)$  of  $b$  and  $r$  through the equations

$$b = 1 / \int_0^{\omega} e^{-ra} p(a) da \quad (27)$$

$$c(a) = b e^{-ra} p(a) \quad (28)$$

Thus, knowing the change in  $r$ , the changes in  $b$  and  $c(a)$  can be easily derived by ordinary calculus.

A more instructive way to proceed, however, is to use the implicit functional method, as described in the previous section. Set up the system

$$\begin{aligned} H_0(r, m) &= \int_0^{\omega} e^{-ra} p(a) m(a) da - 1 \equiv 0 \\ H_1(r, b) &= b \int_0^{\omega} e^{-ra} p(a) da - 1 \equiv 0 \\ H_2(r, b, c(x)) &= b e^{-rx} p(x) - c(x) \equiv 0 \end{aligned} \quad (29)$$

where  $c(x)$  is the age density at specific age  $x$ . Let  $y$  be the column vector  $[r, b, c(x)]'$  so that the Jacobian matrix  $\partial H / \partial y$  is obtained from eqn. (29) as

$$\frac{\partial H}{\partial y} = \begin{bmatrix} -A_m & 0 & 0 \\ -A_n & 1/b & 0 \\ -xc(x) & c(x)/b & -1 \end{bmatrix} \quad (30)$$

where  $A_n$  is the average age of the population. Given the driving change  $\delta m$ , we then have

$$\delta y = - \left( \frac{\partial H}{\partial y} \right)^{-1} \{ \delta H [\delta m] \} \quad (31)$$

We know that  $\delta H_0 [\delta m] = \int_0^{\omega} e^{-ra} p(a) \delta m(a) da$ , and that the other differentials  $\delta H_1 [\delta m]$  and  $\delta H_2 [\delta m]$  are zero. Inverting  $\partial H / \partial y$  yields

$$\begin{bmatrix} \delta r \\ \delta b \\ \delta c(x) \end{bmatrix} = \begin{bmatrix} 1/A_m & 0 & 0 \\ A_n b/A_m & -b & 0 \\ c(x)(A_n - x)/A_m & -c(x) & 1 \end{bmatrix} \begin{bmatrix} \int_0^{\omega} e^{-ra} p(a) \delta m(a) da \\ 0 \\ 0 \end{bmatrix} \quad (32)$$

So that, multiplying out, we obtain the results

$$\delta r = \frac{1}{A_m} \int_0^{\omega} e^{-ra} p(a) \delta m(a) da \quad (33)$$

$$\frac{\delta b}{b} = \frac{A_n}{A_m} \int_0^\omega e^{-ra} p(a) \delta m(a) da \quad (34)$$

$$\frac{\delta c(x)}{c(x)} = \frac{(A_n - x)}{A_m} \int_0^\omega e^{-ra} p(a) \delta m(a) da \quad (35)$$

We now have closed-form expressions for the proportional change in the birth rate and the age distribution for arbitrary changes in the fertility function.\* Expressions (34) and (35) are believed to be new.

Of great interest is the effect of changes in fertility behavior on the age composition. We see from eqn. (35) that whatever the change in shape of the fertility function, it always has the same type of effect on the age distribution. It “pivots” the distribution around the average age of the population  $A_n$ , a net increase in fertility increasing the proportions younger than the average age and decreasing the proportions older than the average age. This tendency of fertility change to pivot the age distribution has been described before, by Lotka (1939) and particularly by Coale (1956, 1972).

### 3 EFFECT OF THE MORTALITY PATTERN ON STABLE POPULATION PARAMETERS

We now turn to the effect of changes in the age pattern of mortality on stable population parameters. As with fertility, there have been several analyses of this problem, most of them of special cases. Coale (1956, 1972) investigated the effect on stable parameters when the force-of-mortality function underwent certain stylized changes close to those observed in real populations. He further provided some empirical results. Keyfitz (1971) looked at the special case of a change in the force-of-mortality function at an arbitrary single, specific age. And in the most general analysis to date, Preston (1974) derived expressions for the proportional change in  $r$ ,  $b$ , and  $c(x)$  caused by arbitrary changes in the mortality function.\*\*

#### Mortality Change and Stable Parameters

Mortality change can be viewed in two different ways, depending on whether we take the change in the force-of-mortality function  $\mu$  or in the survival schedule  $p$  as the

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\*These results have a straightforward connection with those derived by Keyfitz (1971) in his classic article on the effects of a change in fertility at the single age  $a$ . To obtain Keyfitz's results from ours let  $\delta m(a)$  be a unit increase in  $m$ , sustained over one age unit at age  $a$ . In this case eqn. (35), for example, would yield  $[(A_n - x)/A_m] e^{-ra} p(a)$ , as in Keyfitz. In this paper, however, we consider the case where fertility is changed right across the age dimension. To obtain our results from those of Keyfitz we would need to multiply by the change at age  $a$ ,  $\delta m(a)$ , and integrate over age. Stated another way, above we obtain the functional *differential*; Keyfitz, by other methods, calculates the functional *derivative*. Since we can construct the more general differential easily from the derivative, we could use Keyfitz's list of expressions to calculate the response for other stable-theory parameters not treated here.

\*\*Preston's expressions appear to be quite general, but they hinge partly on a parameter  $A$  that must be separately determined; they are therefore somewhat difficult to interpret.

driving change. The two are connected (see Example 3) by the following one-to-one relation, so we may use them interchangeably:

$$\delta p(a) = -p(a) \int_0^a \delta \mu(t) dt \quad (36)$$

We now proceed as before, obtaining  $\delta r$ ,  $\delta b$ , and  $\delta c(x)$  corresponding to  $\delta p$  (or  $\delta \mu$ ) by the implicit functional method. This time, keeping  $m$  fixed, we have

$$\delta y = -\left(\frac{\partial H}{\partial y}\right)^{-1} \{\delta H[\delta p]\} \quad (37)$$

and since we know  $(\partial H/\partial y)^{-1}$  from the previous section it remains only to evaluate  $\delta H[\delta p]$ .

From eqns. (29),

$$\begin{aligned} \delta H_0[\delta p] &= \int_0^\omega e^{-ra} m(a) \delta p(a) da \\ \delta H_1[\delta p] &= b \int_0^\omega e^{-ra} \delta p(a) da \\ \delta H_2[\delta p] &= b e^{-rx} \delta p(x) \end{aligned} \quad (38)$$

Substituting these into eqn. (37):

$$\begin{bmatrix} \delta r \\ \delta b \\ \delta c(x) \end{bmatrix} = \begin{bmatrix} 1/A_m & 0 & 0 \\ A_n b/A_m & -b & 0 \\ c(x)(A_n - x)/A_m & -c(x) & 1 \end{bmatrix} \begin{bmatrix} \int_0^\omega e^{-ra} m(a) \delta p(a) da \\ b \int_0^\omega e^{-ra} \delta p(a) da \\ b e^{-rx} \delta p(x) \end{bmatrix} \quad (39)$$

Finally, multiplying out,

$$\delta r = \frac{1}{A_m} \int_0^\omega e^{-ra} m(a) \delta p(a) da \quad (40)$$

$$\frac{\delta b}{b} = \frac{A_n}{A_m} \int_0^\omega e^{-ra} m(a) \delta p(a) da - \int_0^\omega e^{-ra} \delta p(a) da \quad (41)$$

$$\frac{\delta c(x)}{c(x)} = \frac{(A_n - x)}{A_m} \int_0^\omega e^{-ra} m(a) \delta p(a) da - \int_0^\omega e^{-ra} \delta p(a) da + \frac{\delta p(x)}{p(x)} \quad (42)$$

We now have general closed-form expressions for the change in the growth rate, the birth rate, and the age distribution, given an arbitrary small change in the life table. These results are believed to be new; some comments on them are in order.

1. The effect of mortality on the growth rate is similar to the effect of fertility. What matters is the numerator in eqn. (40) or eqn. (23) and this is the effect of either fer-

tility or mortality on the reproductive value at age 0. Note that mortality improvements late in life, a cure for cardiovascular diseases say, would have little influence on the reproductive value because the change in survival would come largely after reproductive years were past. Thus  $r$  would show negligible change.

2. A special case of mortality change, “neutral” change, is known to have a particularly simple effect on the growth rate (Preston 1974). We can verify this easily. Suppose

$$\mu(t) \rightarrow \mu(t) - k$$

where  $k$  is a constant, so that

$$\delta\mu = -k$$

$$\delta p(a) = ap(a)k$$

Then

$$\delta r = \frac{k}{A_m} \int_0^\omega a e^{-ra} m(a) p(a) da = \frac{kA_m}{A_m} = k$$

Reducing the mortality function by a constant amount, in other words, increases the growth rate by the same amount.

3. The effect of mortality change on the age distribution is straightforward to analyze, given the above general closed-form expression (42). The first term once again “pivots” the age distribution about  $A_n$ . For any given mortality improvement this term is linear over age, and negatively sloped. The second term is constant and negative, while the third directly reflects the change in the survival function. For the usual pattern of mortality change over time, where survival chances improve significantly at ages under five and over forty, the change in the age distribution has the shape shown in Figure 2.

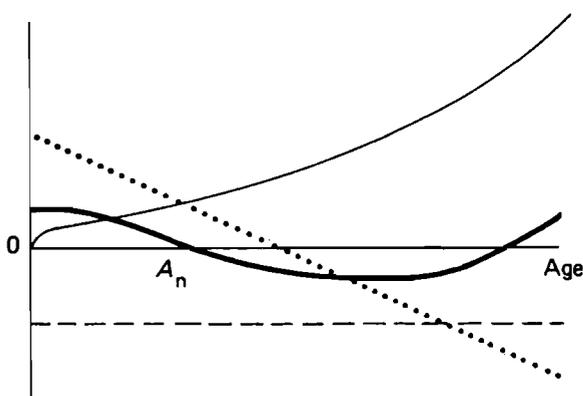


FIGURE 2 Change in the age distribution if the probability of death declines at ages under five and over forty. The dotted line (•••) represents the first term in eqn. (42), the dashed line (- - -) the second term in eqn. (42), the continuous line (—) the third term in eqn. (42), and the bold continuous line (—) the resultant,  $\delta c(x)/c(x)$ .

Again, this confirms Coale's (1972) analysis of the effects of observed changes in the age distribution. "Usual" patterns of mortality change tend to pivot the age distribution clockwise, toward greater proportions in the younger age groups, thus lowering the average age.

4. Survival improvements in the post-reproductive years only, however, pivot the age distribution the other way. The growth effect term disappears and the second term now lowers the age distribution uniformly across the age dimension. The third term again directly reflects the improvement in survival at older ages. The age distribution now pivots anticlockwise, raising the average age of the population.

5. Expressions (35) and (42), taken together, explain the empirical observation that fertility affects the age distribution more than mortality does. Taking fertility and mortality changes that have an equal effect on growth (term on the right-hand side of eqn. (35) equals the first term in eqn. (42)), we see that in contrast to fertility, which acts only through the growth rate, mortality affects the age composition through two other terms. These, in general, tend to offset the first, so that the pivoting of the age distribution is not so pronounced in the mortality case.

6. Finally, the differentials that describe the effects of mortality change and fertility change on any of the stable parameters are additive. We could therefore combine our results to find the net effect of simultaneous fertility and mortality changes, or, working in the opposite direction, decompose a given change in a stable parameter into separate fertility and mortality effects.

*Example 5.* A simple numerical test of the above results is easy to perform. Suppose we start with a stable female population with mortality corresponding to the Coale and Demeny (1966) Model West Level 18 schedule and with a zero growth rate. Can we use the above formulae to "predict" the new growth rate, birth rate, and age composition, if mortality were given instead by Level 20?

The differential function  $\delta p$  is obtained numerically from the tables; it is the difference between survival schedules 20 and 18. The initial Level 18 growth rate and birth rate, the new Level 20 values calculated using eqns. (40) and (41), and the (correct) Level 20 values obtained from the tables are compared in Table 1.

The value calculated for  $r$  via the differential is about 2.5% off in estimating the change, due to the curvature of the functional for  $r$  over its argument function  $p$ .

The age distributions, old and new, calculated from eqn. (42) and interpolated from the Coale–Demeny tables, are compared in Table 2.

There is, of course, a discrepancy since the differential is a first-order approximation to the true response. (Part of this discrepancy may be due to the fact that  $c_{18}$ , used as the input data for  $c(x)$  in eqn. (42), is already rounded to two decimal places in the Coale–Demeny tables.) We can conclude that, in this case, for most purposes the differential brings us tolerably close to the true change. If we wanted the *exact* response of  $r$

TABLE 1 Level 20 growth rate and birth rate calculated via the differential compared with the values taken from the Coale–Demeny tables. The initial (Level 18) values are also given.

Growth rate	Value	Birth rate	Value
$r_{18}$	0.00	$b_{18}$	16.00
$r_{20}$ (calc.)	0.0196	$b_{20}$ (calc.)	15.86
$r_{20}$ (tables)	0.0191	$b_{20}$ (tables)	15.88

TABLE 2 Level 20 age distribution calculated ( $c_{20}^c$ ) via the differential compared with the values interpolated from the Coale - Demeny tables ( $c_{20}^t$ ). The initial (Level 18) distribution is also given.

Age group	$c_{18}$	$c_{20}^c$	$c_{20}^t$	Age group	$c_{18}$	$c_{20}^c$	$c_{20}^t$
0-1	1.53	1.54	1.54	35-39	6.72	6.63	6.63
1-4	5.92	6.00	6.02	40-44	6.56	6.46	6.46
5-9	7.29	7.39	7.41	45-49	6.36	6.25	6.25
10-14	7.24	7.29	7.31	50-54	6.11	6.00	5.99
15-20	7.17	7.18	7.19	55-59	5.76	5.67	5.66
20-24	7.09	7.06	7.07	60-64	5.27	5.21	5.21
25-29	6.98	6.92	6.93	65-69	4.61	4.61	4.60
30-34	6.86	6.78	6.79	70-74	3.71	3.76	3.76
				75-79	2.63	2.74	2.72

and  $c$ , however, for purposes of high-precision tabulation say, we would have to use a more complicated form of the differential -- the functional line integral -- to be touched on briefly in Section 8. □

#### 4 AN ILLUSTRATION: THE EFFECT OF AN AGE SHIFT IN FERTILITY

Thus far our stable theory results contain “ $\delta m$ ” or “ $\delta p$ ” in the expressions, reflecting the fact that we developed them for arbitrary changes in  $m$  and  $p$ . Often, though, we would want to use these results by specifying  $\delta m$  and  $\delta p$  to be particular types of changes in the age pattern. I illustrate a case now.

As a country develops, individual demographic behavior persists -- a person marries, reproduces, and dies as before -- but it often takes place at different times in the life cycle. Thus the fertility pattern may not change greatly in shape, but may vary in its overall intensity and its location on the age axis. In demographic theory the consequences of changes in intensity (uniform proportional change over the entire age schedule) are easy to analyze. But there has been no easy method to determine the consequences of a simple translation of the schedule along the age axis. Using the results of the previous sections, however, we can now analyze translations in a straightforward way.

##### Translation of the Net Maternity Schedule

We begin by looking at the simplest case: a pure shift in the net maternity schedule,  $\phi = p \cdot m$ , to the right along the age axis. This can be regarded as a *rough* indication of what happens when age at marriage is increased, and is illustrated in Figure 3.

Confining our attention to the stable case, and working from the characteristic equation

$$1 = \int_0^{\omega} e^{-ra} \phi(a) da \tag{43}$$

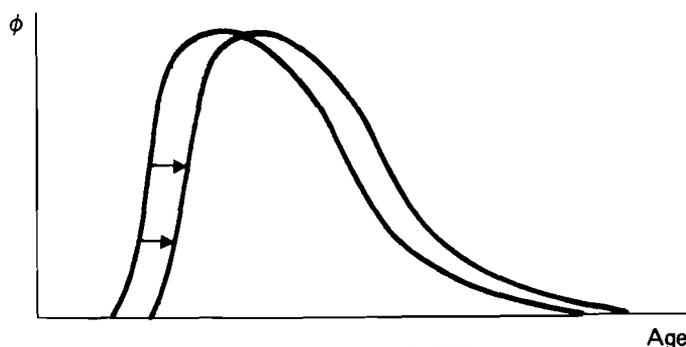


FIGURE 3 An age shift in the pattern of childbearing.

we find as before that

$$\delta r = \frac{1}{A_m} \int_0^{\omega} e^{-ra} \delta \phi(a) da \quad (44)$$

Now, a pure translation of  $\phi$  by  $\tau$  years to the right (later childbearing by  $\tau$  years) means that the change in  $\phi$  is given by

$$\delta \phi(a) = \phi(a - \tau) - \phi(a)$$

so that

$$\begin{aligned} \delta r &= \frac{1}{A_m} \left\{ \int_0^{\omega} e^{-ra} \phi(a - \tau) da - \int_0^{\omega} e^{-ra} \phi(a) da \right\} \\ &= \frac{1}{A_m} \left\{ \int_0^{\omega} e^{-r(a+\tau)} \phi(a) da - \int_0^{\omega} e^{-ra} \phi(a) da \right\} \\ &= \frac{(e^{-r\tau} - 1)}{A_m} \int_0^{\omega} e^{-ra} \phi(a) da \\ \delta r &= (e^{-r\tau} - 1)/A_m \quad (45) \end{aligned}$$

This result tells us that if  $r$  is positive, later childbearing ( $\tau > 0$ ) necessarily decreases  $r$ . If  $r$  is negative, on the other hand,  $\delta r$  is positive, so that later childbearing actually speeds population growth. The reason for this seemingly paradoxical result is that a delay in childbearing means that the next generation arrives later. Since this new generation is smaller than the last and the decline in numbers is spread over a longer time, the rate of decline is therefore not so rapid.

The above result offers a useful rule of thumb for age shifts in childbearing. Dividing by  $r$  we have

$$\frac{\delta r}{r} = \frac{(e^{-r\tau} - 1)}{rA_m}$$

and expanding the exponential term and dropping terms of second order and upward (permissible since  $r$  is small) we obtain

$$\frac{\delta r}{r} = \frac{-r\tau}{rA_m} = \frac{-\tau}{A_m} = - \frac{\text{Increase in average age of childbearing}}{\text{Average age of childbearing}} \quad (46)$$

Thus the proportional fall in the growth rate equals the proportional rise in the mean age of childbearing. Since  $A_m$  is usually about 27 or 28 years, or thereabouts, a year's shift in childbearing would cause a change in the growth rate of  $-(1/27.5)$  or about  $-3.6\%$ . It would therefore take a delay of more than five years in childbearing to cause a 20% decline in the growth rate, or to take 8 points off a crude birth rate of 40.

### The Fertility Schedule: Age Shift and Increase in Intensity

We now look at a slightly more difficult case, assuming that the fertility function shifts by  $\tau$  years, and also increases by a factor  $1 + k$  in intensity. Since differentials are additive we can treat the two changes separately.

Here I make an approximation. Assume that, over the childbearing years, the survival curve declines linearly with slope  $\rho$ . That is,

$$p(a + t) = (1 - \rho t)p(a) \quad (47)$$

From eqn. (23), with the differential  $\delta m = m(a - \tau) - m(a)$

$$\begin{aligned} \delta r &= \frac{1}{A_m} \int_0^\omega e^{-ra} (m(a - \tau) - m(a)) p(a) da \\ &= \frac{1}{A_m} \left\{ \int_0^\omega e^{-r(a+\tau)} m(a) p(a + \tau) da - \int_0^\omega e^{-ra} m(a) p(a) da \right\} \end{aligned}$$

So that, using eqn. (43)

$$\delta r = \frac{1}{A_m} \left\{ e^{-r\tau} (1 - \rho\tau) - 1 \right\} \quad (48)$$

Neglecting terms of order  $r^2$  and upward, we obtain

$$\delta r = \frac{-\tau(\rho + r)}{A_m} \quad (49)$$

Equation (49) summarizes the effect of the age shift on the growth rate.\* Now we analyze

---

\*Equations (48) and (49) may be contrasted with the approximation of Dublin and Lotka (1925):

$$\delta r = \frac{1}{(A_m + \tau)} \left\{ \ln \left[ \frac{p(a + \tau)}{p(a)} \right] - r\tau \right\}$$

Where  $\tau$  is small, Dublin and Lotka's expression coincides with these results.

the effect of the second change,  $\delta m = km(a)$ . Substitution into eqn. (44) shows that this time

$$\delta r = \frac{k}{A_m}$$

Adding the two differentials, we obtain

$$\delta r = \frac{k - \tau(\rho + r)}{A_m} \quad (50)$$

This tells us how much the intrinsic growth rate changes if fertility increases by a factor  $1 + k$ , and is shifted along the age axis by  $\tau$  years, as may happen in the course of development. From eqns. (33) and (35) we can easily write the change in the age distribution as

$$\frac{\delta c(x)}{c(x)} = \frac{(A_n - x)}{A_m} (k - \tau(\rho + r)) \quad (51)$$

*Example 6.* In a well-known paper, Coale and Tye (1961) present an example where the 1956–1958 fertility patterns of two ethnic groups in Singapore, the Malays and the Chinese, resemble each other closely in shape. The mean age of childbearing for the Chinese is 29.1 years, about 3 years higher than that for the Malays, 26.4. On the other hand the survival schedules and overall fertility levels differ only slightly between the two groups. The Chinese intrinsic growth rate is 8.3% lower than the Malaysian rate. How much difference does the age shift in childbearing make, compared to the other factors? Using eqn. (49), with the Malaysian figures as a base, where  $\rho = 0.0013$  and  $r = 0.040$ , we find

$$\frac{\delta r}{r} = \frac{-(29.1 - 26.4)}{26.4} \times \frac{(0.0013 + 0.040)}{0.040} = -10.5\%$$

The higher age of childbearing of the Chinese lowers their growth rate by 10.5%. The Chinese have higher fertility, however, and a slightly different survival schedule, so that the real difference is not quite so great – it is 8.3%. The age shift difference of 10.5% (or 10.15% if we include second-order terms in the approximation) agrees well with Coale and Tye, who calculate 10% due to this effect. □

## 5 CAUSAL LINKAGES IN NONSTABLE THEORY

In the previous sections we looked at the response of the age composition and of intrinsic rates to life-cycle changes within a stable-population model. For nonstable populations the theory is no more difficult. But the nonstable case, being more general, has less mathematical structure than the stable case, and for this reason it is not always possible to obtain closed-form mathematical solutions.

This section takes up two problems. It derives the transient response of the age composition to underlying changes in the birth sequence and to temporal changes in the

fertility pattern. And it examines the more difficult problem of the transient response of the age composition to temporal changes in mortality.

### Response of the Age Composition to Changes in the Birth Sequence

The age composition of the population,  $c(a, t)$ , is given in the general, nonstable case by

$$c(a, t) = \frac{N(a, t)}{N(t)} = \frac{B(t-a)p(a, t)}{\int_0^\omega B(t-a)p(a, t)da} \quad (52)$$

where  $N(a, t)$  is the population density at age  $a$  at time  $t$ ,  $N(t)$  is the total population at time  $t$ , and  $p(a, t)$  is the probability that a person (born at  $t-a$ ) survives to be aged  $a$  at time  $t$ .

Suppose there is a given nominal birth sequence,  $B(t)$ , which may be an arbitrary function of time, or may follow some particular form, such as exponential growth. How will the age composition respond, over time, to an arbitrary change in this function – a baby boom, for example, or a deviation from exponential growth?

Let  $\delta B(t)$  be the given perturbation in the birth sequence (see Figure 4). The response of  $c(a, t)$  to the alteration in the birth sequence  $\delta B$  is obtained from eqn. (52) by the quotient rule:

$$\begin{aligned} \delta c(a, t)[\delta B] &= (N(t)\delta N(a, t) - N(a, t)\delta N(t))/N(t)^2 \\ \delta c(a, t) &= (\delta B(t-a)p(a, t) - c(a, t)\int_0^\omega \delta B(t-a)p(a, t)da)/N(t) \end{aligned} \quad (53)$$

Dividing through by  $c(a, t)$ , this becomes our first nonstable-theory result:

$$\frac{\delta c(a, t)}{c(a, t)} = \frac{\delta B(t-a)}{B(t-a)} - \frac{\int_0^\omega \delta B(t-a)p(a, t)da}{N(t)} \quad (54)$$

or

$$\frac{\delta c(a, t)}{c(a, t)} = \frac{\delta B(t-a)}{B(t-a)} - \frac{\delta N(t)}{N(t)} \quad (55)$$

Thus the proportional change in the age composition at time  $t$  equals the proportional change in the cohort aged  $a$ , less the proportional change in the total population. We thus see the transient response of the age composition to a sequence of cohorts larger than normal as a bulge that passes through the age composition over time.

The response of the age composition to temporal changes in fertility behavior is now easy to obtain. The standard Lotka equation links the birth sequence  $B$  with the fertility function  $m$ :

$$B(t) = \int_0^\omega B(t-a)p(a, t)m(a, t)da \quad (56)$$

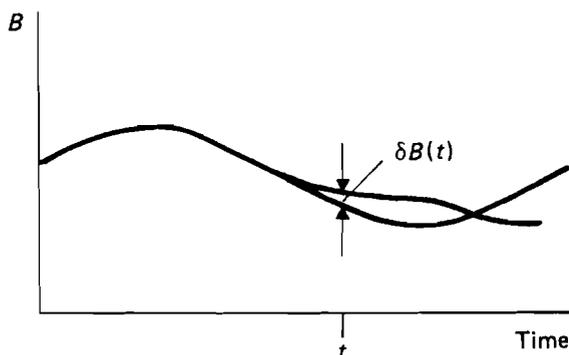


FIGURE 4 A perturbation  $\delta B(t)$  in the birth sequence over time.

with the initial birth sequence given. For a temporal change in the fertility pattern,  $\delta m(a, t)$ , the birth sequence is therefore perturbed by an amount  $\delta B(t)$ , where

$$\delta B(t) = \int_0^{\omega} \delta B(t-a)p(a, t)m(a, t)da + \int_0^{\omega} B(t-a)p(a, t)\delta m(a, t)da \quad (57)$$

with the initial sequence  $\delta B$  zero before the change occurs. Changes in fertility behavior thus affect the birth sequence both directly (second term on the right) and indirectly, through the “echo effect” of the change itself (first term on the right). The perturbation  $\delta B$  is thus given by a Lotka-type renewal equation with a nonhomogeneous or forcing term.\*

Temporal changes in the age pattern of fertility, we can conclude, change the age composition in two stages. They alter first the birth sequence, according to eqn. (57); this then alters the age composition, as in eqns. (54) and (55).

### Response of the Age Composition to Changes in Mortality Rates

Temporal changes in the age pattern of mortality affect the age composition through a more complex mechanism: they alter both the birth sequence  $B$  and the survival function  $p$  in eqn. (52). Here I take the force-of-mortality function  $\mu$  as the first link in the causal chain; improvements in public health, the incidence of epidemics, of wars and natural disasters, the eradication of certain diseases and the shift between one cause of death and another are most directly seen as acting either temporarily or permanently on  $\mu$ , the probability of death at a given age in a given year. I split the analysis into two questions: first, how does a temporal change in the force-of-mortality function  $\mu$  affect the survival function  $p$  over time? Second, how does the resulting transient change in  $p$  affect the age composition over time?

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\*Here we run up against an inherent limitation of differential analysis. The term  $\delta B$  is a first-order approximation to the actual change in the birth sequence, and since it also appears on the right-hand side of the equation any error compounds over time. Hence we may think of eqn. (57) as being “valid” only over the space of two or three generations.

The force-of-mortality, or probability of death per unit time (conditioned on survival until that time), for the cohort aged  $x$  at time  $t_0$  is denoted by  $\mu(x, t_0)$ . It determines the probability of survival  $p(a, t)$  to age  $a$  at time  $t$ , for the cohort born at time  $t - a$ , through the relation

$$p(a, t) = \exp\left(-\int_{t-a}^t \mu(a - (t - \tau), \tau) d\tau\right)$$

The survival probability  $p$ , in other words, is a function of the cohort's force-of-mortality history at previous times  $\tau$  over its life span up to age  $a$ .

Let us now suppose that the force of mortality is altered over the age and time dimensions by an amount  $\delta\mu(x, \tau)$ , at age  $x$  and time  $\tau$ . Then, as in Example 3, the transient response in the survival function  $p$  is obtained as

$$\delta p(a, t) = -p(a, t) \int_{t-a}^t \delta\mu(a - (t - \tau), \tau) d\tau \quad (58)$$

The survival probability is therefore altered over time to a degree proportional to itself multiplied by the sum of all the force-of-mortality changes experienced by the cohort in question in all its previous years from birth onward. To gain some insight into how this linkage works, consider a change in  $\mu$  for one year only at some past time  $\tau_0$  for the cohort then aged  $a_0$ . At future time  $t$  this cohort will be aged  $a = a_0 + (t - \tau_0)$  and from eqn. (58) we obtain the alteration in its survival probability as

$$\delta p(a_0 + t - \tau_0, t) = -p(a_0 + t - \tau_0, \tau_0) \delta\mu(a_0, \tau_0) \quad (59)$$

other changes in the survival function being zero. Thus the one-year alteration in the force of mortality affects one cohort only and it ripples along the survival function with this cohort as it ages. More generally, the effect of a temporary or sustained change over the entire age dimension in the force of mortality on the survival function is the sum of such cohort ripple effects.

We now take the alteration in the survival function over time,  $\delta p(a, t)$ , as given or determined. This change affects the birth sequence over time as follows:

$$\delta B(t) = \int_0^\omega \delta B(t - a) p(a, t) m(a, t) da + \int_0^\omega B(t - a) \delta p(a, t) m(a, t) da \quad (60)$$

with  $\delta B$  zero before the change in survival happens. It also alters the total population,

$$N(t) = \int_0^\omega B(t - a) p(a, t) da$$

over time by an amount

$$\delta N(t) = \int_0^\omega \delta B(t - a) p(a, t) da + \int_0^\omega B(t - a) \delta p(a, t) da \quad (61)$$

Finally, using the quotient rule, we obtain the change in the age composition as

$$\delta c(a, t) = \frac{1}{N(t)^2} \{N(t)B(t - a)\delta p(a, t) + N(t)\delta B(t - a)p(a, t) - B(t - a)p(a, t)\delta N(t)\}$$

Dividing through by  $c(a, t)$ , we have the result:

$$\frac{\delta c(a, t)}{c(a, t)} = \frac{\delta p(a, t)}{p(a, t)} + \frac{\delta B(t-a)}{B(t-a)} - \frac{\delta N(t)}{N(t)} \quad (62)$$

Thus the proportional change in the age composition at age  $a$  and time  $t$  equals the proportional change in the survival function at that age and time, plus the proportional change, if any, in the numbers at birth of the cohort aged  $a$  at time  $t$ , less an adjustment term for the proportional change in the total population. To sum up the causal sequence, changes in death rates, seen as changes in  $\mu$ , affect the survival function through eqn. (58). This in turn affects both the birth sequence through eqn. (60) and the total population size through eqn. (61). The response of the age composition is given in eqn. (62) as the sum of these effects.

Mortality changes that affect only post-reproductive ages allow a closed-form result. In this case the change in the birth sequence,  $\delta B$ , is zero and eqn. (62) reduces to

$$\begin{aligned} \frac{\delta c(a, t)}{c(a, t)} &= \frac{\delta p(a, t)}{p(a, t)} - \frac{\delta N(t)}{N(t)} \\ &= - \frac{p(a, t) \int_{t-a}^t -\delta\mu(a-(t-\tau), \tau) d\tau}{p(a, t)} - \frac{\int_0^\omega B(t-a) \delta p(a, t) da}{\int_0^\omega B(t-a) p(a, t) da} \\ \frac{\delta c(a, t)}{c(a, t)} &= \int_{t-a}^t -\delta\mu(a-(t-\tau), \tau) d\tau - \int_0^\omega c(a, t) \int_{t-a}^t -\delta\mu(a-(t-\tau), \tau) d\tau da \quad (63) \end{aligned}$$

Thus at time  $t$  the proportional change in the age composition equals the accumulated net mortality improvements for the cohort aged  $a$  at that time, less the accumulated mortality improvements averaged over the population as a whole. Since the second term is constant with respect to age and mortality improvements build up with cohort exposure and hence with age, the usual effect of lower mortality in post-reproductive years is to pivot the age composition anticlockwise over time (cf. the analogous stable population case discussed on p. 16 of this paper).

*Example 7.* To illustrate the transient response of the age composition to a change in mortality probabilities, suppose that cardiovascular diseases had been abruptly eliminated as a cause of death in the United States in 1966, and that this improvement in mortality was sustained in all subsequent years (see Table 3).<sup>\*</sup> Using the Keyfitz and Flieger (1971) US 1966 male life table and population projections, how would the survival function and age composition respond over time to this sudden but sustained improvement in mortality?

We can compute the response in the survival function using eqn. (58); the results are shown in Table 4. The projected age compositions, without the mortality improvement, are obtained from Keyfitz and Flieger (p. 335) and listed in Table 5. From eqn.

<sup>\*</sup>This change in  $\mu$  is obtained from Preston *et al.* (1972, p. 768) by eliminating cardiovascular diseases as a cause of death in the US 1964 male tables and making corrections for the effect of competing risks.

(63) and the initial (unchanged) age compositions in Table 5, we can compute the changes in these projections due to the mortality improvement. These results are also given in Table 5.

TABLE 3 Differential due to eliminating cardiovascular diseases as a cause of death among US males in 1966.

Age	$-\delta\mu$	Age	$-\delta\mu$
45	0.00346	65	0.02445
50	0.00606	70	0.03564
55	0.01004	75	0.05254
60	0.01543	80	0.08196

TABLE 4 Response of the survival function to the elimination of cardiovascular diseases as a cause of death among US males in 1966.

Age	Survival function			
	1966	1971	1976	1981
45	0.89404	0.90951	0.91778	0.92162
50	0.86081	0.88689	0.90178	0.90975
55	0.80954	0.85018	0.87471	0.88871
60	0.73493	0.79170	0.82860	0.85087
65	0.63747	0.71540	0.76468	0.79665
70	0.51314	0.60458	0.66731	0.70695
75	0.37576	0.47447	0.54143	0.58737
80	0.24400	0.34399	0.40809	0.45157

TABLE 5 Projected age composition (assuming no improvement in mortality) and change in this projection if cardiovascular diseases were eliminated as a cause of death among US males in 1966.

Age group	Age composition (%)			Change in age composition		
	1971	1976	1981	1971	1976	1981
50-54	5.34	5.24	4.61	0.035	0.043	0.043
55-59	4.78	4.68	4.58	0.144	0.225	0.274
60-64	3.97	4.02	3.92	0.263	0.433	0.532
65-69	3.13	3.16	3.18	0.365	0.612	0.781
70-74	2.23	2.29	2.29	0.416	0.721	0.923
75-79	1.56	1.47	1.50	0.473	0.734	0.957

Notice that the influence on the age composition grows through time, as cohorts are exposed to the mortality improvement over progressively longer periods of their life span. The response would stabilize about forty years or so after the onset of the improvement: all cohorts in this case would be subject to the entire new mortality function over their life history.  $\square$

## 6 DEMOGRAPHIC CHANGE AND VITAL RATES

The easiest demographic measures to obtain for a population are its vital rates – the number of occurrences of a vital phenomenon in year  $t$ , divided by the total population.

Standard measures of this type are the crude death rate,  $DR$ , and crude birth rate,  $BR$ :

$$DR(t) = \frac{D(t)}{N(t)} \quad BR(t) = \frac{B(t)}{N(t)} \quad (64)$$

where  $D(t)$  and  $B(t)$  are the total number of deaths and births, respectively, in year  $t$ . Other behavioral rates, such as the crime rate in the population, may be similarly defined and measured.

In general we consider some age- and time-related phenomenon, with an age-specific rate  $g(a, t)$  for the cohort aged  $a$  at time  $t$ . Summing over all cohorts we obtain the vital rate for this phenomenon at the benchmark time,  $t_0$ , as

$$G(t_0) = \frac{1}{N(t_0)} \int_0^\omega g(a, t_0) B(t_0 - a) p(a, t_0) da = \int_0^\omega g(a, t_0) c(a, t_0) da \quad (65)$$

If we measure this rate again at some later time,  $t$ , we obtain

$$G(t) = \int_0^\omega g(a, t) c(a, t) da \quad (66)$$

The rate will have changed on two counts: first the function  $g$  itself will have changed with time and this is what we hope to measure, and second the age composition  $c$  will have changed. Ideally we would like to measure the overall change in  $G$  without the bias introduced by underlying changes in the age composition.

Suppose that we know the change in the age composition between  $t_0$  and  $t$ , either directly, or by evaluating it as in the previous section from changes in the birth sequence or from changes in fertility or mortality behavior. Expanding  $G(t)$  about  $t_0$ , to first order, by Taylor series we obtain

$$G(t) = G(t_0) + \int_0^\omega g(a, t_0) \delta c(a, t_0) da + \int_0^\omega \delta g(a, t_0) c(a, t_0) da \quad (67)$$

The second integral on the right is what we seek: it is the change in the phenomenon itself, averaged over the population. Denoting this by  $\Delta G$ , we can write it as

$$\Delta G = G(t) - G(t_0) - \int_0^\omega g(a, t_0) \delta c(a, t_0) da \quad (68)$$

We have now obtained the result we need. The “true” change in the vital phenomenon is given by the measured change in the rate less a correction for the change in the age composition. This correction factor in general may not be easy to compute; some knowledge of the age-specific phenomenon and the change in the age composition is necessary. One or both of these may have to be approximated on an incomplete data basis: this however is a standard task in demographic work.

*Example 8.* We can use a similar type of analysis to find how period rates differ from the “correct” cohort rates, given fluctuations in the birth sequence.\* To look at the question

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\*Preston (1972) analyzed a related special case: the response of the crude death rate to exponential growth in the birth sequence.

with precision, suppose (i) a stationary population, with  $N(t) = N$  and  $B(t) = B$ , and (ii) mortality functions  $p$  and  $\mu$  that are constant over time. By virtue of (ii), all birth cohorts face the same life table, the same mortality experience, regardless of the birth sequence. And by virtue of (i), in the absence of perturbations in  $B$ , the crude death rate  $DR$  will equal the “correct” cohort rate  $d$ ; it will show no bias.

Now, the crude death rate

$$DR(t) = \frac{D(t)}{N(t)} = \frac{1}{N} \int_0^{\omega} B(t-a)p(a)\mu(a)da = d$$

responds to an arbitrary perturbation  $\delta B$  in the birth sequence by a change  $\delta DR(t)$ , where

$$\delta DR(t) = \frac{1}{N} \int_0^{\omega} \delta B(t-a)p(a)\mu(a)da - \frac{1}{N} \int_0^{\omega} \delta B(t-a)p(a)da \quad (69)$$

This expression gives the deviation of  $DR$  from the cohort rate  $d$ , given a fluctuation in the birth sequence. We can gain further insight by specifying the perturbation  $\delta B$  to be a single-year “boom” in the birth sequence, such that the cohort born in year  $t_0$  is larger by  $\delta B$  than the usual birth sequence  $B$ .\* We now have

$$\delta B(t) = \begin{cases} \delta B & t_0 \leq t \leq t_0 + 1 \\ 0 & \text{otherwise} \end{cases}$$

And for this particular change, eqn. (69), on integrating out, becomes

$$\delta DR(t) = \frac{\delta B}{N} \cdot p(t-t_0)\mu(t-t_0) - d \cdot \frac{\delta B}{N} \cdot p(t-t_0)$$

Writing  $\delta B/B$  as  $\beta$ , and  $B/N$  as  $b$ , we find

$$\delta DR(t) = b\beta p(t-t_0)(\mu(t-t_0) - d) \quad (70)$$

We have thus found an expression for the bias in the crude death rate caused by an additional “pulse” of births at  $t_0$ ; these changes are illustrated in Figure 5. The bias in the crude death rate is strongly positive just after  $t_0$ , then negative as time progresses further, becoming positive again at about  $t_0 + 60$ , and finally fading to zero. The short pulse of births, in other words, causes the crude death rate to be more positive than the cohort rate just after  $t_0$  due to high mortality in the infant years. Then, in its middle years, it swells population numbers but shows few deaths and the crude death rate falls below the “true” cohort rate. Finally, the crude death rate is biased positively again as the “pulse” reaches the older, high-mortality years, the effect fading as the cohort passes out of the population.

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\*Since we are free to examine the response to any perturbation in  $B$  as an analytical experiment, we shall simplify the example by ignoring echo effects.

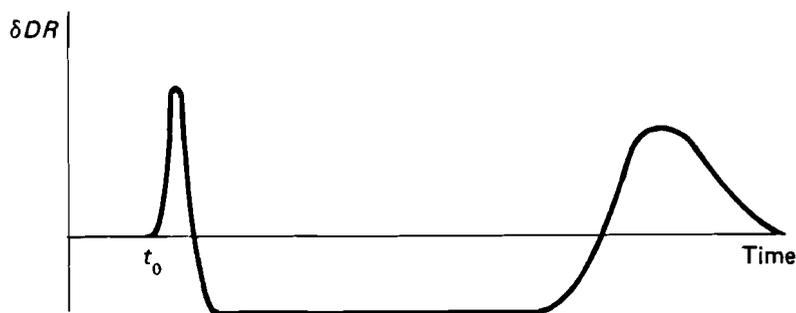


FIGURE 5 Bias in the crude death rate caused by a single-year "baby boom" at time  $t_0$ .

## 7 TOWARD A THEORY OF ROBUSTNESS FOR DEMOGRAPHIC ESTIMATES\*

In the last fifteen years or so, since the seminal work of Brass and Coale (1968), demographers have become highly skilled at estimating vital rates from census data that are fragmentary or incomplete. It is normally impossible to base an estimation technique on data that are largely missing, but in demography a fortuitous circumstance makes this possible. By and large, demographic behavior follows highly regular age patterns. The demographer need only use the available data to select an approximate age pattern from a standard and known family of such patterns. Knowing the approximate pattern he can then fill in the blanks, as it were, and calculate the desired rates or parameters.

It is useful to view this procedure in the abstract. The demographer begins with a standard schedule, of mortality or fertility say, which can be varied by one or two parameters to create a family of model schedules. His observed data tell him how to adjust these parameters to transform the standard schedule into one that approximates the "true" but unknown schedule in the population under study. Thus any particular estimation procedure may be viewed as mapping certain observations, plus one or more standard schedules, into the real numbers to produce the desired estimate. More precisely, then, the estimate can be viewed as a functional of the standard schedules and a function of the observed data:

$$\hat{q} = F(O_1, O_2, \dots, O_n, L_1^s, L_2^s, \dots, L_m^s)$$

where  $q$  is the parameter to be estimated,  $\hat{q}$  the estimate,  $O$  the observed data, and  $L^s$  the standard age schedules. The particular functional form of  $F$  of course depends on the demographic identities on which the estimation procedure is based, and these in turn depend on the demographic assumptions that underlie the procedure.

The statistician interested in such estimation procedures might well ask two questions. First, how robust is the estimation procedure, given that not all the necessary

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\*Here I thank Griffith Feeney, who suggested that causal linkage analysis might be used to estimate errors in these techniques. Collaborative work with Michael Stoto has also helped clarify my thinking in this section. A more complete and precise account of the notions explored here appears in a paper by Arthur and Stoto (1981).

assumptions can be perfectly fulfilled? Second, how might we “correct” the estimate, given different and known sources of error in the technique? Mathematically, a theory of error or of robustness for demographic estimates can be based on analysis of the linkages between the estimate  $\hat{q}$  and the standard functions on which it depends. To illustrate this, I carry out such an analysis on a well-known incomplete-data technique – the Brass child survivorship technique. I choose this technique because it uses a minimum of notation. Other Brass techniques are much the same in structure – the type of analysis applied here and some of the general conclusions would just as well apply to them.

### The Brass Child Survivorship Technique

Suppose that we want to estimate the mortality parameter  $q(M)$ , the probability of death between birth and age  $M$ . In many countries where births and childhood deaths are poorly recorded it is not possible to evaluate  $q(M)$  by direct counting. To overcome this problem, Brass (1975) suggests the following estimation technique:

1. Question mothers, aged  $x$ , on the proportion of their children who have failed to survive,  $D_x$ . This is the only observation used; but notice that it is already a rough indicator of mortality in the childhood years.
2. Choose model schedules of mortality  $q^*$  and fertility  $m^*$ , and estimate  $q(M)$ , the probability of death before age  $M$ , using the expression

$$\hat{q}(M) = \frac{q^*(M) \int_0^x m^*(y) dy}{\int_0^x m^*(y) q^*(x-y) dy} \cdot D_x \quad (71)$$

There are several ways of explaining why this procedure should work. One way is to suppose we have chosen the model schedule  $q^*$  fairly well as regards its shape, but are unsure as to what level to set it at to read off  $q(M)$ . Now,  $D_x$ , the proportion of children dead for mothers aged  $x$ , is given by

$$D_x = \frac{\int_0^x m(y) q(x-y) dy}{\int_0^x m(y) dy} \quad (72)$$

where  $m(y)$  is the true fertility rate for mothers aged  $y$ , and  $q(x-y)$  is the true probability that a child will die during the interval  $x-y$ , between the mother’s age  $y$  when it was born and her age  $x$  at the time of the interview. The estimation procedure given in eqn. (71) therefore corrects the guessed or model level  $q^*(M)$  by a factor

$$\frac{D_x \int_0^x m^*(y) dy}{\int_0^x m^*(y) q^*(x-y) dy} = \frac{\text{Observed proportion dead}}{\text{Model proportion dead}}$$

which, if the model schedules  $m^*$  and  $q^*$  have the correct shape, equals the observed or actual mortality divided by the model mortality. The estimation therefore “adjusts”  $q^*(M)$  to a level that corresponds with the observed mortality.

Before proceeding further, it would help to normalize the estimation formula. Let  $m^*(y)/\int_0^x m^*(y)dy$  be  $f^*(y)$ , the (model) probability density of childbearing at age  $y$ , less than age  $x$ . Similarly, define  $f(y)$  as the true density of childbearing at age  $y$ , less than age  $x$ . We may then write the estimate as

$$\hat{q}(M) = \frac{q^*(M)D_x}{\int_0^x f^*(y)q^*(x-y)dy} \quad (73)$$

where, if the observation  $D_x$  has been measured correctly, we have

$$D_x = \int_0^x f(y)q(x-y)dy \quad (74)$$

Finally, following Brass (1975), we can express eqn. (74) in a useful approximate form as

$$\int_0^x f(y)q(x-y)dy \cong q(x-A) \quad (75)$$

where  $A$  is the average age of childbearing up to age  $x$  for mothers aged  $x$ , that is, the average age of childbearing under the density function  $f$ .<sup>†</sup> Thus  $D_x$  approximately measures the probability of death at the average age,  $x - A$ , of the children being indirectly sampled. (Age here means time elapsed since birth and is unaffected by the death of the child.)

### Error Theory

Where the model schedules  $q^*$  and  $f^*$  coincide with the true functions  $q$  and  $f$ , and where  $D_x$  has been measured correctly so that it conforms with eqn. (74),  $\hat{q}(M)$  estimates  $q(M)$  exactly:

$$\hat{q}(M) = \frac{q(M) \int_0^x f(y)q(x-y)dy}{\int_0^x f(y)q(x-y)dy} = q(M) \quad (76)$$

Errors can arise from three and only three sources. First, and most likely, the observation  $D_x$  will be in error, due to sampling bias, age misreporting, and underreporting of deaths. These errors have a directly proportional effect on the estimate and we have no need to consider them here. We will assume that  $D_x$  has been correctly measured. Second, the model schedule  $f^*$ , which must be guessed, will be in error. It will deviate from the true function  $f$  by the function  $\delta f = f^* - f$ . And third, the model schedule  $q^*$ , which must also be guessed, will be in error. It will deviate from the true function  $q$  by the function  $\delta q = q^* - q$ . We may take these last two sources of error separately, for they are additive in differential form.

<sup>†</sup>To see this, expand  $q(x-y)$  by Taylor series around  $q(x-A)$ :  $q(x-y) = q(x-A) + (y-A) \times q'(x-A) + O^2$ . This gives us  $\int f(y)q(x-y)dy = q(x-A) \int f(y)dy + q'(x-A) \int (y-A)f(y)dy + \int O^2 dy \cong q(x-A)$ , since the second term is zero. Where  $q$  is relatively linear, this approximation is good.

*Error in f*

Assume for the moment that the mortality function has been chosen correctly, that is, that  $q^* = q$ . If  $f^*$  deviates from the true fertility function  $f$  by  $\delta f$ , then the deviation in the estimate,  $\hat{q}(M) - q(M)$ , may be approximated by the differential  $\delta\hat{q}(M)[\delta f]$ , evaluated at the function  $f$ . Applying the quotient rule to eqn. (73), we can write the differential, at  $f$ , as

$$\delta\hat{q}(M) = - \frac{q(M) \int_0^x \delta f(y)q(x-y)dy}{\int_0^x f(y)q(x-y)dy} \quad (77)$$

Writing  $\delta\hat{q}/q$ , the relative error, as  $\text{err } \hat{q}(M)$ , we have the result:

$$\text{err } \hat{q}(M) = - \frac{\int_0^x \delta f(y)q(x-y)dy}{\int_0^x f(y)q(x-y)dy} \quad (78)$$

Knowing the form of  $\delta f$ , as we would in a given application, we could use eqn. (78) to calculate error bounds on the estimate.

*Error in q*

Now assume that the model fertility schedule has been guessed correctly, so that  $f^* = f$ , but that the model mortality schedule  $q^*$  deviates from the true schedule  $q$  by  $\delta q$ . Applying the quotient rule to eqn. (73) once again, we obtain the differential change in the estimate,  $\delta\hat{q}(M)[\delta q]$ , as

$$\delta\hat{q}(M) = \delta q(M) - q(M) \frac{\int_0^x f(y)\delta q(x-y)dy}{\int_0^x f(y)q(x-y)dy} \quad (79)$$

so that, in this case

$$\text{err } \hat{q}(M) = \frac{\delta q(M)}{q(M)} - \frac{\int_0^x f(y)\delta q(x-y)dy}{\int_0^x f(y)q(x-y)dy} \quad (80)$$

Using the average age approximation:

$$\text{err } \hat{q}(M) \cong \frac{\delta q(M)}{q(M)} - \frac{\delta q(x-A)}{q(x-A)} \quad (81)$$

We can draw some general conclusions from this error analysis. Taking the estimation error caused by imperfect choice of the model mortality schedule  $q^*$  first, we see from eqn. (80) that it is to some extent self-cancelling: the two terms in the error expression offset each other to some degree. From eqns. (80) and (81) we can conclude that the technique is robust against errors in the choice of model mortality schedule to the extent

that (i) the model mortality schedule  $q^*$  and the true mortality schedule  $q$  have the same shape, or that (ii) the estimation age  $M$  does not differ significantly from the average age,  $x - A$ , of the children being “sampled”.

Error caused by imperfect choice of the model density  $f^*$  (or the model fertility function  $m^*$ ) is more problematical. Equation (78) shows that such error is not self-cancelling as in the mortality case, and there is no parameter such as  $M$  which can be adjusted to minimize the error. All that can be done is to fit  $f^*$  as accurately as possible. For this reason  $f^*$  (or  $m^*$ ) is usually selected on the basis of ancillary information from parity ratios – this information greatly improves the closeness of fit.

We can go much further with this type of analysis. The Brass technique rests on several specific assumptions – for example, that the true mortality and fertility experience of the population has not changed significantly in the years preceding the survey. Violation of implicit assumptions such as this causes characteristic errors in the choice of model schedules  $f^*$  and  $q^*$  – in other words, it causes  $\delta f$  and  $\delta q$  to assume specific forms. Analytical expressions can thus be found for the error due to violated assumptions (see Arthur and Stoto 1981) using the error formulae derived in this section.

## 8 THREE FORMS OF LINKAGE ANALYSIS: SOME FURTHER REMARKS

Throughout this paper I have deliberately presented the various linkage expressions in the form of functional differentials. This is not the usual practice in sensitivity analysis and I owe the reader some explanation for this. There are three forms in which one can write the response to a change in a function, each with a different purpose: the derivative, the differential, and the line integral.

For the models that interest us in demography, recall from eqn. (13) that the differential can usually be written as an inner product of some expression  $F'(z)$  with the driving perturbation  $\delta z$ :

$$\delta y = \int_0^{\omega} F'(z, a) \cdot \delta z(a) da$$

To be truly parsimonious, we need therefore only present and preserve the derivative  $F'$ , it being a trivial matter to recover the differential by taking the inner product. This, in fact, is how linkages are usually presented in physics or economics, at least when the driving change occurs in a single or vector variable. The differential, though, gives a clearer view of the linkage between the output variable and the function that alters it. It emphasizes that the change in  $y$  depends not only on the properties of  $F'$ , but in this functional case also on the shape and character of the driving perturbation  $\delta z$  in the age or time pattern. This is why I have used the differential form here.

The differential has one major disadvantage though. It is a first-order approximation, and thus is valid only to the extent that the functional in question remains linear over the schedule or function being perturbed. The differential is perfectly serviceable for many numerical applications and it shows the structure of the linkage clearly and correctly; but it is not always suitable for high-precision arithmetic.

There is a closely connected form, the functional line integral, that is *exact* for large perturbations. I shall not describe it in detail, but it works roughly as follows.

Suppose that instead of calculating the differential response to a full perturbation  $\delta z$ , we first allow only a scaled-down perturbation,  $(1/10)\delta z$ , and calculate the response to this. Now we update all parameters and functions, so that the initial function  $z_0$  becomes  $z_0 + (1/10)\delta z$ , and calculate the response to the next  $(1/10)\delta z$ , updating again. We repeat this until we have arrived at the tenth and last  $(1/10)\delta z$ . As output we have ten smaller differentials in  $y$ , which we can add together to form a total differential  $\delta y$ . This new differential will be a more accurate measure of the total change, since we have continually followed changes in the function and parameters as the function is perturbed. If we make the step size  $\Delta s$  smaller,  $1/100$  instead of  $1/10$ , then  $1/1000$  instead of  $1/100$ , the sum of the resulting small differentials will (under certain conditions) tend to a limit, which we can call, by analogy with standard calculus, an integral. This limit measures the *exact* change in  $y$ . For the case  $y = F(z)$ , with  $F'(z)$  known, with initial input function  $z_0$  and final function  $z_1$ , and the difference  $z_1 - z_0 = h$ , we can, following this procedure, write the exact change in  $F$  as

$$y(z_1) - y(z_0) = F(z_1) - F(z_0) = \int_0^1 \int_0^\omega F'(z(a, s))h(a)da ds \quad (82)$$

where

$$z(a, s) = z_0(a) + sh(a)$$

At the cost of a more complicated expression – involving a double integral – the response to large changes in argument function is now exact.

As an illustration, consider a not necessarily small change in the fertility schedule  $m_0$ , with a new schedule  $m_1$  and a difference  $m_1 - m_0$ . Then, using eqns. (23) and (82), the difference in intrinsic growth rates is

$$r(m_1) - r(m_0) = \int_0^1 \int_0^\omega \frac{\omega e^{-r(m(s))a} p(a)(m_1(a) - m_0(a)) da ds}{A_m(m(s), r(m(s)))} \quad (83)$$

where

$$m(s) = m_0 + s(m_1 - m_0)$$

This result, not an approximation, could be used to update the intrinsic growth rate in a precise numerical calculation, given any arbitrary change in the fertility schedule.

In sum, for efficient storage of information all we need is the functional derivative. For insight into the linkage mechanism itself the differential is clearest. And for high-precision numerical work or for large changes in the input schedules we require the more complicated line integral. All three forms are related and, for small changes, are equivalent.

## 9 CONCLUSION

In this paper I have attempted to show that several classes of problems in mathematical demography fall into a common format – that of estimating or analyzing the

linkage between certain aggregate variables of interest and changes in the age schedules or time sequences that describe individual demographic behavior. I have also attempted to provide a method that obtains closed-form expressions for such linkages and illustrated it with several examples: the linkage of stable parameters to general changes in fertility and mortality schedules and to an age shift in fertility; the transient response of the age composition and of vital rates to short-run changes in demographic behavior; and the analysis of errors in demographic estimation techniques. While much is known qualitatively about these linkages, in some instances with considerable analysis of special cases, the results in this paper apply more generally than before, and most of them are believed to be new.

There are several uses for linkage analysis. Not only do closed-form expressions for demographic linkages allow the analyst to compute changes (e.g., a change in the growth rate caused by a new contraceptive), but they also afford him considerable qualitative insight into the mechanisms at work. In some cases these closed-form expressions are also useful numerically; they offer a direct computational method for updating parameters without the repeated numerical solution of implicit equations. In other cases these expressions lead to general statements, or theorems, on the response to change.

Throughout this paper I have presented the various linkage mechanisms in the form of functional differentials. These, while they show the linkage clearly and are perfectly serviceable for many numerical purposes, are still, of course, first-order approximations to the true response, and hold best for marginal changes. Should we require the exact response to large changes in age or time schedules, though, the line-integral form is available, albeit at extra computational and notational expense.

For the most part, in this exploratory paper I have chosen problems for analysis about which much is already known; this allowed us to compare our results with previous experience in these problems. Several other problems could have been as easily looked at. For example, given an appropriate economic–demographic model it is possible to analyze the economic consequences of arbitrary changes in mortality risks (see Arthur 1981). It would also be possible to look at the spatial consequences of changing migration patterns, or the economic consequences of changing labor-participation patterns. And, given appropriate biological theory, it might be possible to investigate why the mortality and reproductive age patterns of a given species should provide for evolutionary success.

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