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VARIANT

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June 1974

WP-74-16

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THE SOLUTION OF A TWO-PERSON POKER VARIANT

by

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June 1974

ABSTRACT

This note presents the solution of a two-person poker variant considered by Friedman [1]. The solution is derived using a general algorithm proposed by the author to solve two-person zero sum games with 'almost' perfect information [2].

1. Description of the Game

The following poker game is a slight generalization of Friedman's "Simple Bluffing Situation with Possible Reraise" [1]. Player 1 has a low card up and one card down, Player 11 has a high card up and one card down. If both players have either a high or a low card down, then Player 11 wins; otherwise, the player with the high card down wins. There are n units in the pot. Player 1 may either drop or raise 1 unit. Then Player 11 may either drop, call or reraise $(m-1)$ units. Finally, Player 1 may either drop or call (in Friedman's example $n=1, m=4$).

Let p_0 and q_0 be the respective probabilities that Player 1 and Player 11 have a high card down. Of course, each player knows whether his own card down is high or low.

2. Computation of the Value of the Game

The solution will be derived using a general algorithm proposed by the author to solve two person zero sum games with 'almost' perfect information [2]. For convenience we shall use the same notation as in [2].

Let the letters D, C, R stand for drop, call or raise respectively and let $m_1 \in \{D, R\}, m_2 \in \{D, C, R\}, m_3 \in \{D, C\}$. Let $V^{m_1 m_2 m_3}$ be the value of the m_1 - m_2 - m_3 restricted game; that is, the game in which the players' choice sets are restricted to the unique elements m_1, m_2, m_3 respectively. Then we have (theorem 1 in [2])

$$V^{m_1 m_2} = \underset{\Omega_I}{\text{Cav}} \{ \underset{m_3}{\text{Max}} V^{m_1 m_2 m_3} \} ,$$

$$V^{m_1} = \underset{\Omega_{II}}{\text{Vex}} \{ \underset{m_2}{\text{Min}} V^{m_1 m_2} \} ,$$

and finally: $V = \underset{\Omega_I}{\text{Cav}} \{ \underset{m_1}{\text{Max}} V^{m_1} \} .$

We shall make the computation stage by stage and represent the functions on the unit square ($0 \leq p \leq 1, 0 \leq q \leq 1$). It will turn out that all functions will be "rectangle wise" linear (of the form $\alpha pq + \beta p + \lambda q + \delta$ on rectangles) so that only the values at the extremal points of the rectangles need be computed.

For the computation of the optimal strategies, it will also be helpful to keep track of how the Cav and Vex are constructed. This will be done by labeling the corresponding vertices of the rectangles. (For instance, for $q \in [0, (m+n+1)/(2m+n)]$, V^{RR} is a convex combination of V^{RRD} at $p = 0$ and V^{RRC} at $p = 1$).

$$\begin{array}{c}
 (L,H)^* \quad V^{RRD} \quad (H,H) \\
 \begin{array}{|c|c|}
 \hline
 -1 & -1 \\
 \hline
 -1 & -1 \\
 \hline
 \end{array} \\
 (L,L) \quad (H,L)
 \end{array}$$

$$\begin{array}{c}
 (L,H) \quad V^{RRC} \quad (H,H) \\
 \begin{array}{|c|c|}
 \hline
 -m & -m \\
 \hline
 -m & n+m \\
 \hline
 \end{array} \\
 (L,L) \quad (H,L)
 \end{array}$$

$$\begin{array}{c}
 (L,H) \quad V^{RR} \quad (H,H) \\
 \begin{array}{|c|c|}
 \hline
 -1 & -1 \\
 \hline
 -1 & -1 \\
 \hline
 \end{array} \\
 \begin{array}{l}
 D \\
 \frac{m+n+1}{2m+n} \\
 C
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 (L,H) \quad V^{RD} \quad (H,H) \\
 \begin{array}{|c|c|}
 \hline
 n & n \\
 \hline
 n & n \\
 \hline
 \end{array} \\
 (L,L) \quad (H,L)
 \end{array}$$

$$\begin{array}{c}
 (L,H) \quad V^{RC} \quad (H,H) \\
 \begin{array}{|c|c|}
 \hline
 -1 & -1 \\
 \hline
 -1 & 1+n \\
 \hline
 \end{array} \\
 (L,L) \quad (H,L)
 \end{array}$$

$$\begin{array}{c}
 (L,H) \quad V^R_R \quad (H,H) \\
 \begin{array}{|c|c|}
 \hline
 -1 & -1 \\
 \hline
 -1 & -1 \\
 \hline
 \end{array} \\
 \begin{array}{l}
 R \\
 \frac{m+n+1}{2m+n} \\
 C \\
 \frac{n+1}{n+2}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 (L,H) \quad V^D \quad (H,H) \\
 \begin{array}{|c|c|}
 \hline
 0 & 0 \\
 \hline
 0 & 0 \\
 \hline
 \end{array} \\
 (L,L) \quad (H,L)
 \end{array}$$

$$\begin{array}{c}
 (L,H) \quad V \quad (H,H) \\
 \begin{array}{|c|c|}
 \hline
 0 & 0 \\
 \hline
 0 & 0 \\
 \hline
 \end{array} \\
 \begin{array}{l}
 D \\
 \frac{n(m+n+1)}{(n+1)(2m+n)} \\
 D \\
 F \\
 n \\
 R \\
 n
 \end{array}
 \end{array}$$

* (Player 1's hidden card, Player 11's hidden card)

3. Computation of the Optimal Behavioral Strategies

3.1 Player 1's first optimal move

For $p_o \in [0, (n+1)/(n+2)]$ and $q_o \in [0, n(m+n+1)/(n+1)(2m+n)]$

Step 1

It is easily seen that

$$V(p_o, q_o) = \alpha_D V^D(0, q_o) + \alpha_R V^R((n+1)/(n+2), q_o) ,$$

with $p_o = \alpha_D 0 + \alpha_R (n+1)/(n+2) .$

Thus, $p_D = 0, p_R = (n+1)/(n+2), \alpha_R = p_o(n+2)/(n+1), \alpha_D = 1 - \alpha_R .$

So that Player 1's optimal move may be written as follows

Player 1	Prob ($m_1 L$)	Prob ($m_1 H$)
$m_1 = D$	$\alpha_D / (1 - p_o)$	0
$m_1 = R$	$\alpha_R (1 - p_R) / (1 - p_o)$	1

For $p_o \in [(n+1)/(n+2), 1]$ and $q_o \in [0, n(m+n+1)/(n+1)(2m+n)] ,$

$$V(p_o, q_o) = V^R(p_o, q_o) .$$

Hence Player 1 raises independently of his state.

For $q_o \notin [0, n(m+n+1)/(n+1)(2m+n)] ,$

$$V(p_o, q_o) = V^D(p_o, q_o) .$$

Hence Player 1 drops independently of his state.

3.2 Player 11's optimal first move

Given that Player 1 raised, Player 11 may either drop, call or reraise.

For $p_0 \in [0, (n+1)/(n+2)]$ and $q_0 \in [0, n(m+n+1)/(n+1(2m+n))]$

Step 1

Since Player 1 raised, we have $p_1 = p_R$. It is easily seen that we have two extremal Bayesian best responses for Player 11.

$k=1$

$$V^R(p_R, q_0) = \beta_D^1 V^{RD}(p_R, 0) + \beta_R^1 V^{RR}(p_R, (m+n+1)/(2m+n)) ,$$

with $q_0 = \beta_D^1 0 + \beta_R^1 (m+n+1)/(2m+n) .$

Thus $q_D = 0, q_R^1 = (m+n+1)/(2m+n), \beta_R^1 = q_0(2m+n)/(m+n+1), \beta_D^1 = 1 - \beta_R^1$

y^1	Prob ($m_2 L$)	Prob ($m_2 H$)
$m_2 = D$	$\beta_D^1 / (1 - q_0)$	0
$m_2 = R$	$\beta_R^1 (1 - q_R^1) / (1 - q_0)$	1

$k=2$

$$V^R(p_R, q_0) = \beta_C^2 V^{RC}(p_R, 0) + \beta_R^2 V^{RR}(p_R, (m+n+1)/(2m+n)),$$

with $q_0 = \beta_C^2 0 + \beta_R^2 (m+n+1)/(2m+n) .$

Thus $q_C = 0, q_R^2 = (m+n+1)/(2m+n), \beta_R^2 = q_0(2m+n)/(m+n+1), \beta_C^2 = 1 - \beta_R^2 .$

y^2	Prob ($m_2 L$)	Prob ($m_2 H$)
$m_2 = C$	$\beta_C^2 / (1 - q_0)$	0
$m_2 = R$	$\beta_R^2 (1 - q_R^2) / (1 - q_0)$	1

(notice that $q_R^1 = q_R^2$ and $\beta_R^1 = \beta_R^2$ so that the k index may be dropped).

Step 2

We now have to find the convex combination of these two Bayesian best responses which is in equilibrium with Player 1's first move (that is, which makes him indifferent between bluffing or not, if he has a low card).

The supporting hyperplane to $V(p, q_0)$ for $p \in [0, p_R]$ has for equation:

$$\gamma = [n - q_0(n+1)(2m+n)/(m+n+1)](n+2)p/(n+1) \quad .$$

The hyperplanes associated with the two Bayesian best responses are easily identified since y^1 is a Bayesian best response for $p \in [p_R, 0]$ and y^2 for $p \in [0, p_R]$. Thus

$$\gamma_1 = n - q_0(n+1)(2m+n)/(m+n+1),$$

$$\gamma_2 = [1 - q_0(2m+n)/(m+n+1)](n+2)p - 1$$

So that $\gamma = \mu_1 \gamma_1 + \mu_2 \gamma_2$ with

$$\mu_1 = 1/(n+1) [1 - q_0(2m+n)/(m+n+1)], \quad \mu_2 = 1 - \mu_1$$

Player 11's optimal strategy may be interpreted as follows:

- if he has a high hand, he reraises,
- otherwise, he reraises with probability $\beta_R(1 - q_R)/(1 - q_0)$ or, given that he does not reraise, then he will drop with probability μ_1 , and call with probability μ_2 .

For $p_0 \in [0, (n+1)/(n+2)]$ and $q_0 \in [0, n(n+m+1)/(n+1)(2m+n)]$

Step 1

Since Player 1 raised independently of his state, $p_1 = p_0$.

There is only one Bayesian best response for Player 1; it is y^1 as described on page 6, thus it is Player 1's optimal first move.

3.3 Player 1's optimal move

The procedure used in 3.2 may be repeated. We shall only give the final result.

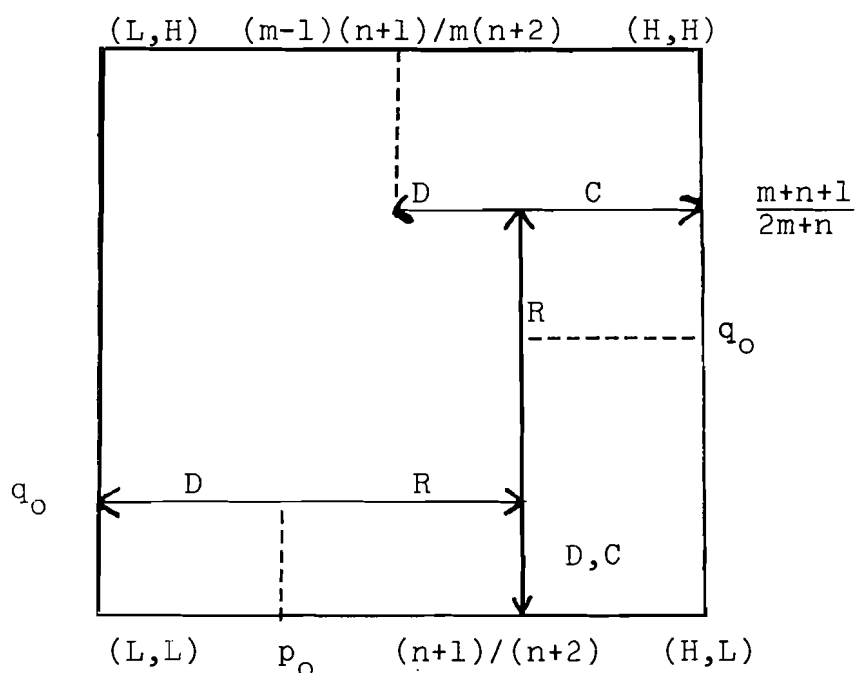
For $p_0 \in [0, (n+1)/(n+2)]$

Player 1	Prob ($m_3 L$)	Prob ($m_3 H$)
$m_3 = D$	1	$(m-1)/(m+n+1)$
$m_3 = C$	0	$(n+2)/(m+n+1)$

For $p_0 \in [(n+1)/(n+2), 0]$

Player 1	Prob ($m_3 L$)	Prob ($m_3 H$)
$m_3 = D$	1	$1 - (n+1)/(m+n+1)p_0$
$m_3 = C$	0	$(n+1)/(m+n+1)p_0$

While the optimal strategies may appear complicated, the description of the "story" of the game in terms of the graph of conditional probabilities is quite simple. Here is such a story for $p_0 \in [0, \frac{n+1}{n+2}]$, $q_0 \in [0, n(n+m+1)/(n+1)(2m+n)]$.



Starting with probability distributions $p_0 = \frac{1}{2}$ and $q_0 = \frac{1}{2}$ an observer to the game could derive the following conditional probabilities:

- Player 1 drops, he has a low card,
- Player 1 raises of n units, the probability that he has a high card jumps from p_0 to $\frac{n+1}{n+2}$;
- Player 2 calls or drops, he has a low card,
- Player 2 raises of m units, the probability that he has high cards jumps from q_0 to $\frac{m+n+1}{2m+n}$;
- Player 1 calls, he has a high card;
- Player 1 drops, the probability that he has a high card falls from $\frac{n+1}{n+2}$ to $\frac{(m-1)(n+1)}{m(n+2)}$.

This sequence of conditional probabilities and the knowledge of (μ_1, μ_2) fully describe the optimal behavioral strategies .

Ordinarily the conditional probabilities would be sufficient, except that here they do not completely specify Player 11's strategy

4. Some Comments on Computational Feasibility

The use of this algorithm for real poker is severely limited by the fact that so far no numerical procedure is available for the Cav and Vex operators in more than two dimensions. Concavifications have to be carried out by hand using "visual judgments". On the other hand, the number of reraises and their amounts may be quite arbitrary with no further complications.

REFERENCES

- [1] Friedman, L., "Optimal Bluffing Strategies in Poker"
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- [2] Ponssard, J.P., "Zero Sum Games with 'Almost' Perfect
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