

**ECONOMIC EVOLUTIONS AND THEIR RESILIENCE:  
A Model**

Manfred Breitenecker and Hans-Richard Grümm  
*International Institute for Applied Systems Analysis, Austria*

RR-81-5  
April 1981

**INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS**  
**Laxenburg, Austria**

**International Standard Book Number 3-7045-0005-4**

---

*Research Reports*, which record research conducted at IIASA, are independently reviewed before publication. However, the views and opinions they express are not necessarily those of the Institute or the National Member Organizations that support it.

---

Copyright © 1981  
International Institute for Applied Systems Analysis

All rights reserved. No part of this publication may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopy, recording, or any information storage or retrieval system, without permission in writing from the publisher.

## FOREWORD

In the past few years interest in the *phase portrait* of systems -- that is, in a qualitative description of the global structure of systems and their long-term behavior -- has grown, both at the International Institute for Applied Systems Analysis (IIASA) and elsewhere. Accordingly, the Energy Systems Program has developed a line of compact, highly aggregated models of an "abstract economy" emphasizing the energy sector; the model presented in this report is the heir of this line. The authors, Manfred Breitenecker and Hans-Richard Grümme, are interested not so much in the behavior of individual trajectories (i.e., "simulation runs") as in the global structure and its variation under parameter changes. Of course, they do not claim that this model can predict the future; they believe, however, in its usefulness in pointing at trends and in helping one to conceptualize problems, to understand structures, and, perhaps, to formulate questions to put to bigger, more realistic economic models. Obviously, this paper is just one experimental step toward a more satisfying state of the art, where a more immediate meaning will be captured by such a model.

WOLF HÄFELE  
*Leader*  
Energy Systems Program



## CONTENTS

	SUMMARY	1
1	INTRODUCTION	2
2	RESILIENCE	3
3	HISTORICAL DEVELOPMENT	4
4	THE MODEL'S STRUCTURE	5
5	ANALYSIS OF THE MODEL EQUATIONS	11
6	THE BASE CASE	17
7	TWO-COUNTRY INTERACTION	22
8	OUTLOOK AND CONCLUSION	25
	REFERENCES	29
	APPENDIX A Some Concepts of Dynamical Systems	30
	APPENDIX B Dynamics Outside the Slack-Free Region	30
	APPENDIX C A Criterion for the Number of Fixed Points	33
	REFERENCES TO APPENDIXES	33



## ECONOMIC EVOLUTIONS AND THEIR RESILIENCE: A MODEL

Manfred Breitenecker and Hans-Richard Grümh  
*International Institute for Applied Systems Analysis, Austria*

### SUMMARY

*This report designs a highly aggregated macroeconomic model that can be formulated in terms of a system of ordinary differential equations (i.e., a "dynamical system"). The report consists of two parts supplementing each other in a sort of symbiosis. One part is the abstract structure of the equations – that is, the individual dependence of the time variations of the state variables (which span the state space) on the variables themselves (which in this model are  $E$ ,  $K$ , and  $L$ ). The other part is the parameter space, each point of which is a set of parameter values that have a well-defined economic meaning and thereby endow the system with economic content.*

*A particular economy is then defined by a particular point in parameter space, together with a particular point in state space (describing the "status quo") from which it evolves deterministically in time along its trajectory.*

*The model is analyzed carefully with the help of methods from differential topology. The following questions are answered:*

- *Are there points of stationary growth in state space? If so, where are they located?*
- *What is the qualitative behavior of such a point? Is it attractive (stable) or not?*
- *Which regions of state space are slack-free – that is, describe a "desirable" economy?*
- *What is the influence of a change in the system parameters on the global behavior of a trajectory or, more generally, on the phase portrait as a whole (i.e., the set of all trajectories, roughly speaking)?*
- *Where are the regions in parameter space within which the system shows similar global behavior? In particular, where are the economic niches (regions) for which the system is globally stable?*
- *What effects do delivering and receiving investment goods (e.g., granting or receiving foreign aid) have on the qualitative behavior of the economy? To what extent can a transition out of or into a more suitable economic niche be induced by foreign aid? Similarly, what influence does the price of imported primary energy, which must be paid primarily through investment goods, have on the economy?*

*As the parameter space is high-dimensional, some essential parameters have been merged into what we call scenario variables. To a certain extent, these variables reflect a particular scenario: highly or less effective use of energy, conventional or new technology in energy production, high or low emphasis on the consumption sector. The economic niches have also been determined within this scenario space.*

*We considered as a particular application a coupling of two economies with different qualitative behaviors (the one being within, the other outside, a stable economic niche) via foreign aid and under the influence of the price of imported energy. This led to determination of an upper limit for the price of energy.*

*This work is experimental; we do not intend to present a model that is in any sense final. Rather, we would like to examine more thoroughly what structural stability means, particularly with respect to long-term problems (those with a time horizon of, say, 50 years). More work and additional contributions are obviously needed.*

## 1 INTRODUCTION

Both within the Energy Systems Program at the International Institute for Applied Systems Analysis (IIASA) and elsewhere, efforts have been under way to understand and therefore to conceptualize possible evolutions of energy and other systems over a long period of time — say, 50 years. Such a time horizon is longer than that which can be treated meaningfully by the normal techno-economic models available. In the case of input–output modeling, for instance, the evolution of the input–output coefficients over time must be known if the technique is to be used purposefully. The same applies to elasticities and other input parameters in the case of econometric models. The methodology of modeling a 50-year evolution for the purpose of conceptualization is therefore new, difficult, and specific. In *Energy in a Finite World: A Global Systems Analysis* (1981), the Energy Systems Program Group at IIASA outlines in detail one approach to this problem.

Understanding such evolutions in minute detail is not always the major problem; most often, the concern is stability, and, more precisely, the stability of underlying structures. In the context of the energy problem, which is in the forefront here, a good example is the price of oil and its impact, not only on a particular economy (either importer or exporter), but also on world trade as a whole — that is, on the overall structure of economic interactions. Will there be collapses or evolutions that inherently lead to distortions? Significantly, such a question is of a holistic nature. This approach focuses on the structure of evolutions in time (and possibly in space) as a whole, not on the summations of yearly increments. The issue is thus one of structural stability.

Capital costs of new energy technologies were a special concern of IIASA's Energy Group. Since such costs tend to be high, one may wonder whether energy still works for the economy or whether the economy works for energy. While this has recently become less of a concern, it was once an important point that led to the evaluation of certain new energy technologies against the background of the rest of the economy. Furthermore, the conception of the model reported here was determined by considering oil prices as well as the extent of foreign aid and of similar transfers of wealth.

Nontrivial structural problems arise only with nonlinear models. In this case one may consider the phase space of the variables in question. There are usually singular points, saddle points, sources, or sinks, which imply the existence of basins. These basins

are separate and are therefore divided from each other by separatrices (see Appendix A) consisting of one or of infinitely many trajectories. The evolution of the trajectories may be generally desirable with regard to one basin but undesirable in others. To demonstrate such features, Häfele (1975) conceived a simple model with population and per capita energy demand as the only variables. It then appeared desirable to consider a more powerful and, one might hope, more meaningful model permitting us, among other possibilities, to discuss questions of capital costs for new energy technologies.

This interest of IIASA's Energy Group coincided with another line of interest at IIASA: Holling and his team were studying the dynamics of ecological systems. One motivation for their study was to develop pest management strategies – an effective spraying policy, for example – in an ecosystem. In the framework of this research, Holling (1973, ed. 1978) used the term *resilience* to describe a system's capability to continue its evolution in the same basin when impacts on the system occur from outside. An IIASA workshop (Grümm ed. 1975) brought together ecologists, economists, and climatologists under Koopman's chairmanship, and Grümm (1976) generalized and formalized the notion of resilience. While the precise mathematical definition of the term is still subject to debate, the concept is obviously helpful and enlightening. Indeed, when considering the problems of the next half century, we are less concerned about quantitative evolutions – which nevertheless shape the overall structure – than about the possibility that the system might collapse and the trajectory continue in a different basin.

The model presented here should be seen in light of these considerations. It was designed, not to provide a final answer to the problem of structural stability while simultaneously dealing with all economic details, but rather to make sense economically and technically. We have proposed a sequence of steps to reach the final goal. So that this methodological development is most fruitful, we hope that others will help to improve the state of the art. Broadening our understanding of resilience and including models other than the ecological one would also be desirable. The modeling effort reported here is clearly experimental.

## 2 RESILIENCE

At present there are several slightly different concepts of resilience, all of which stem from Holling's work (1973). We shall briefly describe the concept preferred by IIASA's Energy Systems Program. It is strongly tied to the theory of differentiable dynamical systems, i.e., the global theory of differentiable equations (see Appendix A); the mathematical definitions of resilience (given in Grümm 1976) are expressed in terms of that theory.

Resilience – conceptually described – is the ability of systems to withstand exogenous, uncontrollable disturbances affecting the values of state variables and parameters without qualitatively changing their behavior. As this is originally a property of the system existing in reality, resilience is reflected in the mathematical model describing that system. As we tend to identify the system with the model, we also speak of resilience as a property of the latter. Models used in this context describe the evolution of the system as motion in a state space each of whose points uniquely identifies a state of the system. We assume that the motion is given by a causal (as opposed to a stochastic or nonautonomous) differential equation on state space, thus bringing the results of dynamical systems theory to bear. For any given set of parameter values, the state space contains one

or several attractors that describe steady modes of the system's behavior. The basins of these attractors are separated by basin boundaries. All parameter values corresponding to the same structure of state space form one parameter regime (economic niche). Resilience may be described as follows: if, due to the change in state variables/parameters, the state/set of parameter values has not left its previous basin/parameter regime, the system has absorbed the disturbance; if it has jumped over a boundary, qualitative – often catastrophic – changes will occur.

As basins and parameter regimes are usually open sets, disturbances below a certain level will always be absorbed, thus resolving at the same time our uncertainty about fine details of the dynamics, as well as about the exact values of state variables and parameters. Small differences between the exact values and the values taken for the mathematical model will not qualitatively affect the model's output.\* This form of the "structural stability" argument is, of course, well known from catastrophe theory.

Numerical measures for resilience can be introduced next, many of them incorporating the notion of "distance to some boundary." Full details are contained in *Definitions of Resilience* (Grüm m 1976).

In our model, resilience with respect to changes in state variables is formally trivial: there is at most one basin. However, the constraint that small disturbances or errors in the state variables should not lead to drastically different future behavior translates into the postulate – as shall be seen – that the system has an attractor, which corresponds to an economically meaningful state of the model economy. As there are different parameter regimes (niches), resilience with respect to parameter changes is a meaningful question. This leads to the problem of determining the boundaries of these regimes in parameter space; if the parameters of an "actual" economy come close to a boundary, one should begin to be concerned.

### 3 HISTORICAL DEVELOPMENT

In this section we shall summarize the evolution of highly aggregated economic models at IIASA in the past three years. We shall give only brief descriptions of four models (A to D), as full details are contained in *Economy Phase Portraits* (Grüm m and Schrattenholzer 1976).

Model A (Häfele 1975) showed for the first time in an economic model at IIASA how a saddle point, by generating a separatrix, can cause two basins with different long-term behavior to occur. The model attempts to describe phenomenologically different effects: the influence of the standard of living on the birth rate and on the level of safety expenditures, as well as the rise of energy consumption with an increase in the gross national product (GNP). The model has population and per capita energy consumption as state variables; its two basins describe two trends: toward "low population living in luxury" on the one hand and toward "growing population at a constant living standard" on the other.

Model B (Avenhaus, Grüm m, and Häfele 1975) is closer to established economic formulation. The entire economy is split into two parts, with an energy-producing sector distinguished from the rest. The other assumptions of Model A are incorporated, although

---

\*We can view these differences as "exogenous, incontrollable perturbations"!

their mathematical expression is necessarily different. The state variables are population, GNP per capita, and energy production; total GNP is split into consumption, depreciation, and net investment. A ceiling is assumed for per capita GNP. The two attractors and basins of the model correspond to two possibilities for producing this limiting GNP, one with a high energy production and a low investment in the nonenergy sector and the other with the reverse situation.

Model C is the result of combining Model B with ideas presented in Häfele and Bürk (1976). Labor is introduced in addition to energy and nonenergy capital stock as a production factor. A new feature is that the dynamics of this model are given by an infinitesimal optimization postulate: at each point in state space, we should proceed in the direction that optimizes the rise of GNP. Various phase portraits for this model are given by Grümme and Schrattenholzer (1976).

Total GNP has thus far been given by a Cobb–Douglas ansatz for the production function. In Model D the investment sector is described by a linear input–output ansatz. The fraction of total GNP available for investments (denoted by  $1 - \alpha$ ) plays a central role and is determined dynamically. Depending on the parameter values, the system has two to four basins, but its attractors are not isolated fixed points: along entire curves the system is at equilibrium.

As the following description makes clear, many of the “building blocks” of the present model are contained in these four models. The overall structure of our model owes much to the work of Häfele and Bürk (1976).

Two further, unpublished “mini-models” by Bürk and Grümme give a phenomenological treatment of the price rise for a scarce resource and thereby justify the “logistic transition between two different technologies” assumed in the present model.

#### 4 THE MODEL’S STRUCTURE

We selected energy, capital, and labor force, denoted by  $E$ ,  $K$ , and  $L$ , respectively, as basic variables spanning the state space of our model. Their precise economic relevance is demonstrated in Fig. 1.  $E$ ,  $K$ , and  $L$  denote the respective stocks of

- Total installed power (we also refer to this as the total invested energy-related capital stock)
- Total invested non-energy-related capital stock
- Total available stock of skilled labor

Illustrations of these quantities follow. (In the phenomenological spirit of the model, we do not give economically exact definitions.)

$E$  includes, for example, power stations, with their integrated equipment, such as turbines, generators, dams (in the case of hydropower), electric grids, and so forth; oil refineries; pipelines; and tankers. As we assume a constant load factor,  $E$  may also be interpreted as the total energy output (or input into the economy) per time unit. In this report we use 1 year as the time unit and 0.75 as the load factor. Hence 1 W of installed power yields 0.75 Wyr of energy per year.

$K$  denotes all factories in operation (assuming no spare capacity), with their machinery and equipment to produce

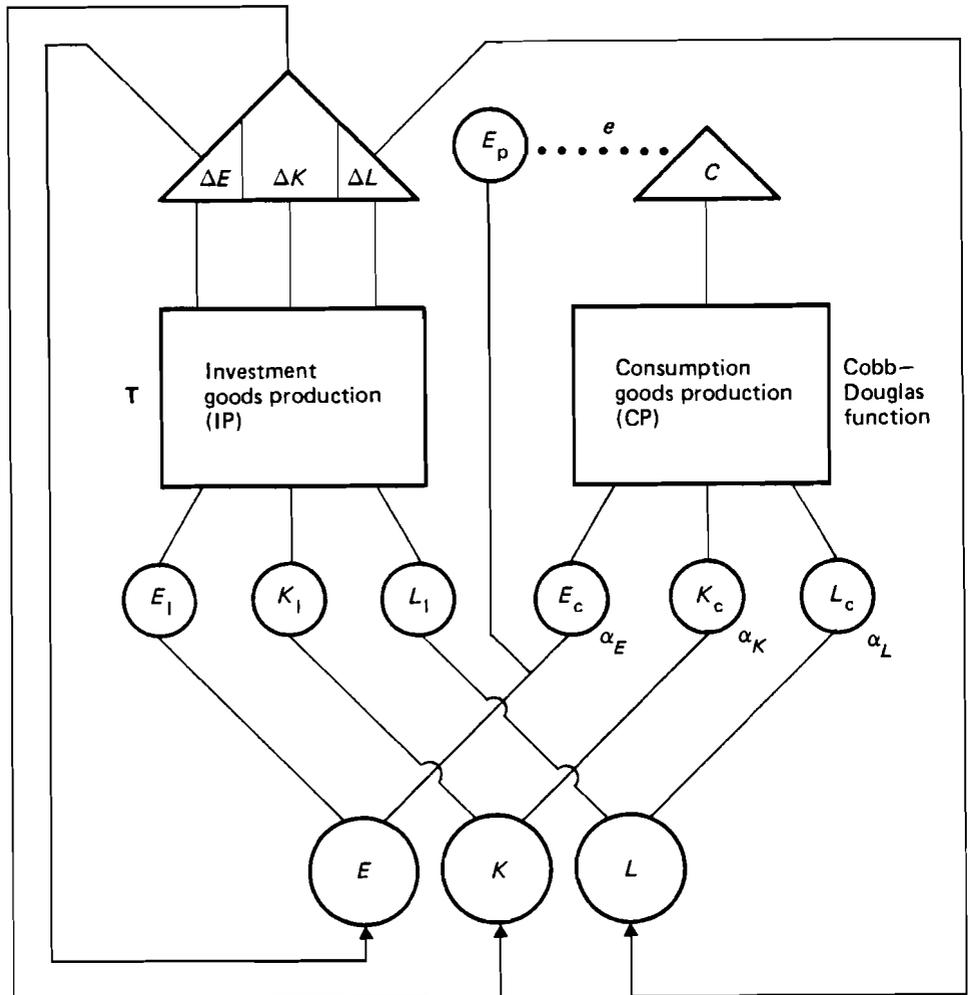


FIGURE 1 Structure of the model.

- Energy-related capital goods (the stock denoted by  $E$ ), such as turbines; generators; and cement for existing dams and power stations
- Non-energy-related capital goods, such as new machinery (which may eventually produce new turbines) or cement for new factories, schools, universities, or other means of “skill production” (but not cement for new dams, power stations, or private homes)
- Consumption goods, such as private cars, private homes (i.e., cement to build private homes), and everything else that does not produce anything in turn

$K$  also denotes existing schools, universities, and other means of “skill production.” The unit for  $K$  is \$1.

$L$  represents the available stock of labor (assuming no idle workers) weighted with skill: effectiveness, know-how, sophistication of such tools as pocket calculators, and so

forth.\* With a load factor of 0.25, 1 worker of unit skill corresponds to 2000 effective man-hours per year or 0.25 effective man-years per year. By introducing skill, we are able to increase  $L$  without increasing the number of workers and to omit a term  $\sim e^{mt}$  (describing technological progress) from our production functions.

We assume two lines of production within our system. The first is investment goods production (IP), which produces (per year)

- Energy-related capital goods  $\Delta E$
- Non-energy-related capital goods  $\Delta K$
- Skilled labor force  $\Delta L$

The second is consumption goods production (CP), which produces consumption goods  $C$  (per year).  $C$  does not include private energy consumption. Energy is required to utilize and to maintain consumption goods, especially such middle- and long-term durable goods as cars and homes. We call the part of  $E$  allocated for this purpose  $E_p$  (private energy consumption). Because even homes are of finite durability (say, 50 years), the appropriate portion is in fact “consumed” each year. Hence in a first approximation  $E_p$  is assumed to be proportional to  $C$ :

$$E_p = eC \quad (1)$$

This assumption is backed by statistics.

Combining what has been said about the stocks and their allocation, we have

$$\begin{aligned} E &= E_I + E_c + E_p \\ K &= K_I + K_c \\ L &= L_I + L_c \end{aligned} \quad (2)$$

We express the stocks allocated to CP in terms of the total quantities, using the coefficients  $\alpha_E, \alpha_K, \alpha_L$ :

$$E_c = \alpha_E E \quad K_c = \alpha_K K \quad L_c = \alpha_L L \quad (3)$$

The  $\alpha$ s clearly describe the emphasis on CP within the economy.

The inputs into IP are  $E_I, K_I$ , and  $L_I$ ; the outputs,  $\Delta E, \Delta K$ , and  $\Delta L$ . Part ( $d_x X$ ) of the outputs has to compensate for the respective depreciations (in the case of skilled labor, retirements of laborers); the other part ( $\dot{X}$ ) is the annual net increase in the respective stocks:

$$\Delta X = \dot{X} + d_x X \quad X = E, K, L \quad (4)$$

where the  $d_x$  denote the respective depreciation rates.

---

\*Again, we do not define *skill* quantitatively; we assume that it describes both “subjective” (e.g., better training) and “objective” (e.g., better technology) increases in productivity. We could use a similar concept of *effective capital*.

Common sense tells us that stocks and outputs have to be nonnegative, i.e.,

$$\begin{aligned} E \geq 0 \quad K \geq 0 \quad L \geq 0 \\ \Delta E = \dot{E} + d_E E \geq 0 \quad \Delta K = \dot{K} + d_K K \geq 0 \quad \Delta L = \dot{L} + d_L L \geq 0 \end{aligned} \quad (5)$$

Thus our state space is  $R_+^3$ .

We now assume that the IP is of the linear input–output type with minimum production functions (see Dorfman, Samuelson, and Solow 1958; Samuelson 1947, 1976; Baumol 1977; and Hicks 1969)

$$\Delta X = \min_{Y=E,K,L} (Y_{I,X}/a_{YX}) \quad X = E,K,L \quad (6)$$

More explicitly,

$$\begin{aligned} \Delta E &= \min \left( \frac{E_{I,E}}{a_{EE}}, \frac{K_{I,E}}{a_{KE}}, \frac{L_{I,E}}{a_{LE}} \right) = \dot{E} + d_E E \geq 0 \\ \Delta K &= \min \left( \frac{E_{I,K}}{a_{EK}}, \frac{K_{I,K}}{a_{KK}}, \frac{L_{I,K}}{a_{LK}} \right) = \dot{K} + d_K K \geq 0 \\ \Delta L &= \min \left( \frac{E_{I,L}}{a_{EL}}, \frac{K_{I,L}}{a_{KL}}, \frac{L_{I,L}}{a_{LL}} \right) = \dot{L} + d_L L \geq 0 \end{aligned} \quad (6)$$

$a_{YX}$  denotes the minimal amount of stock  $Y_I$  (the portion of the total stock  $Y$  allocated to IP) required to produce one unit of output of stock of type  $X$ ; correspondingly,  $Y_{I,X}$  is the total amount of  $Y_I$  necessary to produce  $\Delta X$ . Another way of looking at eq. (6') is

$$a_{YX} \Delta X \leq Y_{I,X} \text{ for all } X, Y = E,K,L \quad (7)$$

Adding the left- and right-hand sides for  $X = E,K,L$  while keeping  $Y$  fixed, we obtain

$$\sum_{X=E,K,L} a_{YX} \Delta X \leq \sum_{X=E,K,L} Y_{I,X} = Y_I \quad Y = E,K,L \quad (8)$$

Hence  $(Y_{I,E}, Y_{I,K}, Y_{I,L})$  is a particular allocation of  $Y_I$  to the three production lines within the IP.

We first assume that our IP runs optimally – that is, without slacks. Thus all ratios  $Y_{I,X}/a_{YX}$ ,  $X$  fixed,  $Y = E,K,L$  over which the minimum has to be taken in eq. (6') are the same, and the inequality in eq. (8) becomes an equality. In part of state space, this requirement of optimality is inconsistent with eq. (5).

In matrix notation this reads

$$T \Delta X = X_I \quad (9)$$

where

$$\Delta X = \begin{pmatrix} \Delta E \\ \Delta K \\ \Delta L \end{pmatrix} \text{ and } X_1 = \begin{pmatrix} E_1 \\ K_1 \\ L_1 \end{pmatrix} \quad (10)$$

$$T \doteq \begin{pmatrix} a_{EE} & a_{EK} & a_{EL} \\ a_{KE} & a_{KK} & a_{KL} \\ a_{LE} & a_{LK} & a_{LL} \end{pmatrix}$$

is called the technological matrix because it reflects the technological situation within our model economy.

The examples that follow illustrate the significance of the matrix coefficients. (We estimate the values of the coefficients for our base case in Section 6.)

- $a_{EE}$  is the number of watts allocated to IP in order to increase the installed power  $E$  by 1 W per year
- $a_{EK}$  is the number of watts allocated to IP in order to increase the non-energy-related capital stock  $K$  by \$1 per year
- $a_{EL}$  is the number of watts allocated to IP in order to increase the number of skilled workers  $L$  by 1 person per year
- $a_{KE}$  is the amount of capital invested in IP required to increase  $E$  by 1 W per year
- $a_{KK}$  is the amount of capital invested in IP required to increase  $K$  by \$1 per year
- $a_{KL}$  is the amount of capital invested in IP required to increase  $L$  by 1 person per year
- $a_{LE}$  is the number of skilled workers employed in IP required to increase  $E$  by 1 W per year
- $a_{LK}$  is the number of skilled workers employed in IP required to increase  $K$  by \$1 per year
- $a_{LL}$  is the number of skilled workers employed in IP required to increase  $L$  by 1 person per year

The choice of eq. (6) as the production function for the IP implies nonsubstitutability among the production factors. This property seems to be realistic; the lines of production within heavy industry, for instance, are rather inflexible and allow only minor deviations from an optimal path.

In contrast, the production function of the CP should allow for substitutability; using a simple ansatz, we describe it by a Cobb–Douglas function:

$$C = A E_c^\alpha K_c^\beta L_c^\gamma \quad (11)$$

$$\alpha + \beta + \gamma = 1 \quad (12)$$

Equation (12) accounts for constant returns to scale. Combining eqs. (1), (2), (3), (4), (9), and (11), we arrive at the system of ordinary differential equations that we sought and that we shall analyze by global techniques:

$$T(\dot{X} + dX) = (1 - \alpha)X - Xp \quad (13)$$

where we have used the notation

$$\mathbf{d} = \begin{pmatrix} d_E & 0 & 0 \\ 0 & d_K & 0 \\ 0 & 0 & d_L \end{pmatrix} \quad \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \alpha = \begin{pmatrix} \alpha_E & 0 & 0 \\ 0 & \alpha_K & 0 \\ 0 & 0 & \alpha_L \end{pmatrix} \quad X_p = \begin{pmatrix} E_p \\ 0 \\ 0 \end{pmatrix} \quad (13')$$

with  $E_p = eA\alpha_E^\alpha\alpha_K^\beta\alpha_L^\gamma E^\alpha K^\beta L^\gamma$ .

Equation (13) may be rewritten as

$$\dot{X} = \mathbf{T}^{-1}\{(\mathbf{1} - \alpha - \mathbf{Td})X - X_p\} \quad (14)$$

Inserting eq. (14) in eq. (5) leads to a condition defining a region in the state space where the optimality assumption is consistent with the positivity requirement, eq. (5). We call this region the slack-free region: inside it, the dynamics of our system (its evolution in time) are given by eq. (14).

Our considerations will be largely restricted to the slack-free region because such questions as the existence of equilibrium states and their stability can be discussed within it. Possible dynamics outside the slack-free region are described in Appendix B; they differ from the formal continuation of eq. (14) to the outside. As shown in the appendix, this difference does not affect the results presented in the following sections, which use eq. (14) in the entire state space.

To analyze eq. (14) we would normally calculate the critical element – in our case, the fixed point (FP). Obviously,  $X = 0$  is an FP and we can easily show that, in general, it is the only FP.

We put  $\dot{X} = 0$ ; hence, from eq. (14),

$$(\mathbf{1} - \alpha - \mathbf{Td})X = X_p \quad (15)$$

From the expansion of eq. (15), we express  $E$  and  $K$  as linear functions of  $L$  and substitute them into the first line. Since  $E_p$ , the first component of  $X_p$ , shows constant return to scale, the right-hand side of eq. (15) is linear in  $L$ ; so, also, is the left-hand side. Hence  $L \neq 0$  drops out, and we are left with a restriction on the parameters; we cannot expect this restriction to hold generally.

If we assume variable returns to scale in the Cobb–Douglas function (i.e., if instead of eq. (12) we have  $\alpha + \beta + \gamma \neq 1$ ), there will be a second FP in the  $E, K, L$  space. We then look for a fixed ray instead of an FP in state space; this ray describes an equilibrium state of stable growth.

The property of constant returns to scale of both production functions allows us to introduce new variables and to reduce three dimensions to two. Inserting

$$\epsilon := E/L \quad \kappa := K/L \quad \lambda := \log L \quad (16)$$

in eq. (13') yields, after straightforward algebra,

$$\begin{pmatrix} \epsilon \\ \kappa \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\epsilon \\ 0 & 1 & -\kappa \\ 0 & 0 & 1 \end{pmatrix} \mathbf{T}^{-1} \left\{ (\mathbf{1} - \alpha - \mathbf{Td}) \begin{pmatrix} \epsilon \\ \kappa \\ 1 \end{pmatrix} - \begin{pmatrix} B & \epsilon^\alpha & \kappa^\beta \\ 0 & & \\ 0 & & \end{pmatrix} \right\} = \begin{pmatrix} f(\epsilon, \kappa) \\ g(\epsilon, \kappa) \\ h(\epsilon, \kappa) \end{pmatrix} \quad (17)$$

where  $B = eA\alpha_E^\alpha\alpha_K^\beta\alpha_L^\gamma$ . Thus  $\dot{\lambda}$  is the growth rate of skilled labor. We observe that the right-hand side of eq. (17) depends only on  $\epsilon$  and  $\kappa$  – not on  $\lambda$ ; the problem has become two-dimensional. One first solves for  $\epsilon$  and  $\kappa$  and then substitutes the solutions into the expression for  $\dot{\lambda}$  and integrates. Details are given in Section 5. The system parameters, which must be chosen consistently, are as follows: nine coefficients of  $\mathbf{T}$ , the technological matrix,  $a_{XY}, X, Y = E, K, L$ ; three ratios,  $\alpha_E, \alpha_K, \alpha_L$ ; three depreciation rates,  $d_E, d_K, d_L$ ; and three Cobb–Douglas coefficients,  $\alpha, \beta$ , and  $B$  (where  $A$  and  $e$  are combined). These parameters make up a parameter space of eighteen dimensions.

## 5 ANALYSIS OF THE MODEL EQUATIONS

For a qualitative analysis of eq. (17), we follow the standard procedure:

- Determination of FPs and their properties
- Determination of the slack-free region according to eq. (5)
- In the case of a stable FP, determination of the central region, i.e., the region of initial values within the slack-free region such that the trajectories do not subsequently leave the slack-free region

As previously mentioned, an FP  $\dot{\epsilon} = \dot{\kappa} = 0$  corresponds to a time-invariant ray (direction) in  $E, K, L$  space. Equations (18) and (19) are valid only inside the slack-free region. This amounts to solving the (nonlinear) eigenvalue problem

$$\dot{\mathbf{X}} = \mathbf{T}^{-1} \{ (\mathbf{1} - \alpha - \mathbf{T}\mathbf{d})\mathbf{X} - X_p \} = n\mathbf{X} \quad (18)$$

in terms of eq. (14) or

$$\mathbf{T}^{-1} \left\{ (\mathbf{1} - \alpha - \mathbf{T}\mathbf{d}) \begin{pmatrix} \epsilon \\ \kappa \\ 1 \end{pmatrix} - \begin{pmatrix} B & \epsilon^\alpha & \kappa^\beta \\ 0 & & \\ 0 & & \end{pmatrix} \right\} = n \begin{pmatrix} \epsilon \\ \kappa \\ 1 \end{pmatrix} \quad (19)$$

in terms of eq. (17).

To get a feeling for the situation, we may analyze a simplified version of eq. (18) or (19). We use the argument of structural stability and extrapolate eqs. (18) and (19) outside the slack-free region. We assume all depreciations to be equal,  $d_E = d_K = d_L = d$ , consider  $X_p$  as a perturbation that we put to zero in our simplification,\* and solve the remaining linear problem.  $n$  then appears as the simultaneous growth rate of  $E, K$ , and  $L$  at the point of stable growth, i.e., at the fixed ray.

Simple algebra then leads to the respective equations

$$[(\mathbf{1} - \alpha)^{-1}\mathbf{T} - z]\mathbf{X} = 0 \quad (18')$$

or

$$[(\mathbf{1} - \alpha)^{-1}\mathbf{T} - z] \begin{pmatrix} \epsilon \\ \kappa \\ 1 \end{pmatrix} = 0 \quad (19')$$

where  $z = 1/(d + n)$ .\*\*

\*This is justified by looking at actual numerical values.

\*\* $z$  is an auxiliary quantity introduced to apply a Frobenius theorem.

As  $(1 - \alpha)^{-1} \mathbf{T}$  is a matrix with strictly positive elements, a theorem of Frobenius is applicable. This theorem implies that exactly one eigenvector with positive components exists and that its corresponding eigenvalue is positive and is the largest among the real eigenvalues. As eqs. (18') and (19') are of third degree, there are two possibilities: either three real solutions for  $z$  (and hence for  $n$ ) or one real and two complex conjugate solutions.

In the first case, we have three fixed points in the  $\epsilon$ - $\kappa$  plane with the respective

$$n_i = 1/z_i - d \quad i = 1, 2, 3 \quad (19'')$$

as growth rates. The FP corresponding to the smallest growth rate is in the positive quadrant and is a sink. The other two are a saddle point and a source, respectively, but their location is outside the positive quadrant. Qualitatively, the full phase portrait then appears as shown in Fig. 2.

In the second case, we have only one FP, with a growth rate  $n > -d$ . It is again located within the positive quadrant and may be attractive or repulsive; the corresponding phase portraits are shown in Fig. 3.

Owing to structural stability, these statements also remain true, within certain limits, for differing  $d$ s and nonvanishing  $X_p$ . For the base cases that we studied, the second alternative holds; it prevails even for large deviations from the base case data. Appendix C gives a criterion for distinguishing between the two alternatives.

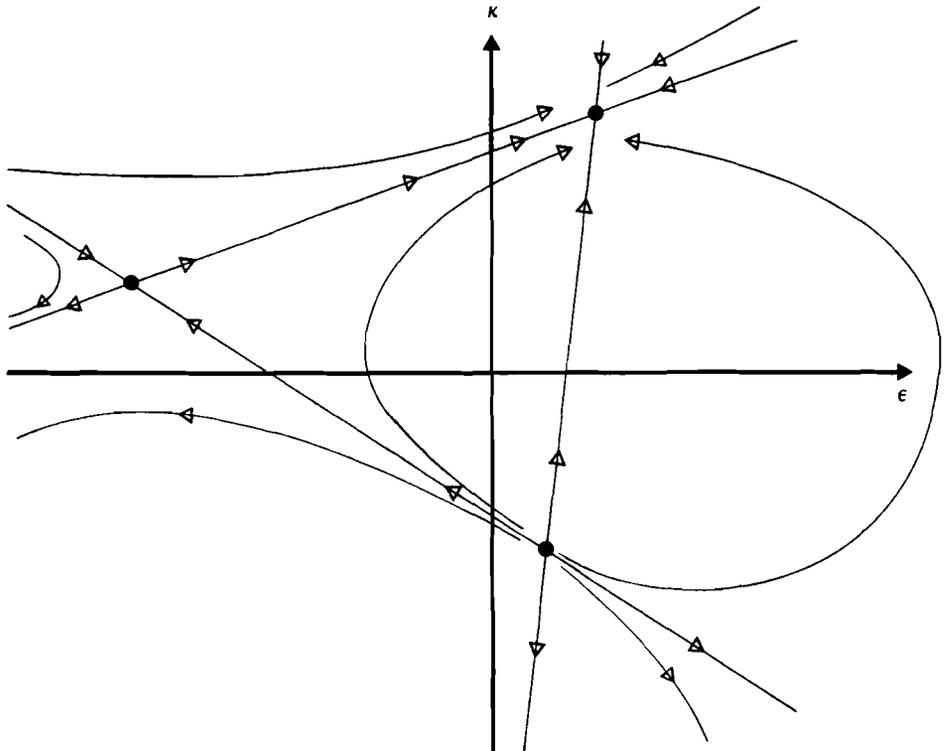


FIGURE 2 Three real fixed points.

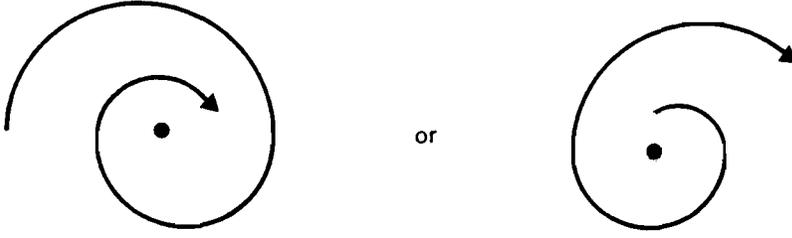


FIGURE 3 One real fixed point.

Returning to eq. (19), the fixed point condition  $\dot{\epsilon} = \dot{\kappa} = 0$  yields  $(\epsilon_0, \kappa_0)$  as a solution of

$$f(\epsilon, \kappa) = 0 \text{ and } g(\epsilon, \kappa) = 0 \quad (20)$$

Subsequent substitution into  $h$  leads to

$$n_0 = h(\epsilon_0, \kappa_0) \quad (21)$$

where the functions  $f$ ,  $g$ , and  $h$  are defined by eq. (17). This is the simultaneous growth rate of  $E$ ,  $K$ , and  $L$  at the FP.

For purely technical reasons it is advantageous first to solve

$$g(\epsilon, \kappa) = 0 \text{ and } h(\epsilon, \kappa) = n \quad (22)$$

to obtain  $\epsilon$  and  $\kappa$  as functions of  $n$ , and then to substitute them into  $f(\epsilon, \kappa) = 0$  and to solve for  $n$ . Thus, after some manipulation,

$$\begin{aligned} (d_E + n)a_{KE}\epsilon + [(d_K + n)a_{KK} - (1 - \alpha_K)]\kappa + (d_L + n)a_{KL} &= 0 \\ (d_E + n)a_{LE}\epsilon + (d_K + n)a_{LK}\kappa + (d_L + n)a_{LL} - (1 - \alpha_L) &= 0 \\ [(d_E + n)a_{EE} - (1 - \alpha_E)]\epsilon + (d_K + n)a_{EK}\kappa + B\epsilon^\alpha\kappa^\beta + (d_L + n)a_{EL} &= 0 \end{aligned} \quad (23)$$

These equations must be solved for  $\epsilon$ ,  $\kappa$ , and  $n$ , respectively. With the abbreviations

$$\begin{aligned} D_1 &:= a_{KE}a_{LK} - a_{KK}a_{LE} & D_2 &:= a_{KK}a_{LL} - a_{KL}a_{LK} \\ D_3 &:= a_{KE}a_{LL} - a_{KL}a_{LE} & D(n) &:= (d_E + n)[a_{LE}(1 - \alpha_K) + (d_K + n)D_1] \end{aligned} \quad (24)$$

we obtain

$$\begin{aligned} \epsilon(n) = \frac{1}{D(n)} & [(1 - \alpha_K)(1 - \alpha_L) - (1 - \alpha_K)(d_L + n)a_{LL} - (1 - \alpha_L)(d_K + n)a_{KK} \\ & + (d_K + n)(d_L + n)D_2] \end{aligned}$$

$$\kappa(n) = \frac{d_E + n}{D(n)} [(1 - \alpha_L)a_{KE} - (d_L + n)D_3] = \frac{(1 - \alpha_L)a_{KE} - (d_L + n)D_3}{(1 - \alpha_K)a_{LE} + (d_K + n)D_1} \quad (25)$$

Substitution of eq. (25) into the third part of eq. (23) yields a transcendental equation for  $n$ .

Three remarks are in order. First, as previously mentioned, the simplified system of equations (all  $ds$  equal,  $X_p = 0$ ) always has a solution with  $\epsilon \geq 0, \kappa \geq 0$ . On the other hand, the general set of equations  $G$  need not have a solution at all; there need not exist a domain for  $n$  where both  $\epsilon(n)$  and  $\kappa(n)$  are nonnegative (they obviously should be, as eq. (5) demonstrates).

Second,  $n$  always appears in connection with depreciation rates; hence, changing the three depreciation rates by the same amount and simultaneously changing  $n$  by the same amount (but with the opposite sign) leaves  $\epsilon_0$  and  $\kappa_0$  unchanged.

Third, as functions  $\epsilon(n)$  and  $\kappa(n)$  depend on the 18 system parameters, the previously mentioned positivity condition restricts the allowed parameter values to certain regions or economic niches within the parameter space.

To analyze the properties of the FP  $(\epsilon_0, \kappa_0)$  (assuming the conditions of eq. (5) to be fulfilled), we follow the standard procedure: linearization of eq. (17) yields, after straightforward calculations,

$$\begin{aligned} \begin{pmatrix} \epsilon' \\ \kappa' \\ \lambda' \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ n_0 \end{pmatrix} \\ &+ \left\{ \begin{pmatrix} 1 & 0 & -\epsilon_0 \\ 0 & 1 & -\kappa_0 \\ 0 & 0 & 1 \end{pmatrix} \Gamma^{-1} \left[ 1 - \alpha - \mathbf{T}d - \begin{pmatrix} \alpha B \epsilon_0^{\alpha-1} & \kappa_0^\beta & \beta B \epsilon_0^\alpha & \kappa_0^{\beta-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right] - n_0 \mathbf{1} \right\} \begin{pmatrix} \epsilon' \\ \kappa' \\ 0 \end{pmatrix} \\ &+ \dots = \begin{pmatrix} 0 \\ 0 \\ n_0 \end{pmatrix} + \mathbf{S} \begin{pmatrix} \epsilon' \\ \kappa' \\ 0 \end{pmatrix} + \dots \end{aligned} \quad (26)$$

where we introduced

$$\epsilon' = \epsilon - \epsilon_0 \quad \kappa' = \kappa - \kappa_0 \quad \lambda' = \lambda - \lambda_0 \quad (26')$$

and where

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$

denotes the matrix between the braces.\* Considering the first two parts of eq. (22), we have, with

$$\mathbf{S}_1 := \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

\*Explicit expressions for matrix elements of  $\mathbf{S}$  tend to become cumbersome; as they must be evaluated numerically in any case, they are omitted here.

$$\begin{pmatrix} \epsilon' \\ \kappa' \end{pmatrix}^* = \mathbf{S}_1 \begin{pmatrix} \epsilon' \\ \kappa' \end{pmatrix} + \dots \quad (27)$$

According to the general theory, the behavior of the solutions near the FP is determined by the eigenvalues  $\mu_{1,2}$  of  $\mathbf{S}_1$ , which are given by

$$\mu_{1,2} = 1/2 \{ \text{tr } \mathbf{S}_1 \pm [(\text{tr } \mathbf{S}_1)^2 - 4 \det \mathbf{S}_1]^{1/2} \}^* \quad (28)$$

The FP  $(\epsilon_0, \kappa_0)$  is stable if the real parts of  $\mu_{1,2}$  are negative

$$\text{Re } \mu_{1,2} < 0 \quad (29)$$

It is called real stable if  $\text{Im } \mu_{1,2} = 0$  and complex stable in any other case. For both real stable and complex stable FPs, the trajectories  $(\epsilon(t), \kappa(t))$  approach the FP in the future. If eq. (29) does not hold, the FP is unstable; with the exception only of special cases, the trajectories leave any neighborhood of the FP in the future, even if there are finite time periods during which the FP is approached.

Analyzing eq. (28) in more detail, we may have

$$(\text{tr } \mathbf{S}_1)^2 \geq 4 \det \mathbf{S}_1 \quad (30)$$

or

$$(\text{tr } \mathbf{S}_1)^2 < 4 \det \mathbf{S}_1 \quad (30')$$

In the case of eq. (30'), the  $\mu_{1,2}$  will be real. We now have to distinguish

$$\det \mathbf{S}_1 > 0 \quad (31)$$

from

$$\det \mathbf{S}_1 < 0 \quad (31')$$

for eq. (31). Both  $\mu$ s have the same sign as  $\text{tr } \mathbf{S}_1$ . Hence

$$\det \mathbf{S}_1 > 0 \text{ and } \text{tr } \mathbf{S}_1 < 0 \quad (32)$$

is the stable case;

$$\det \mathbf{S}_1 > 0 \text{ and } \text{tr } \mathbf{S}_1 > 0 \quad (32')$$

is the unstable case.

For eq. (31') the  $\mu$ s have the opposite sign and the FP is unstable. In the case of eq. (30'), the eigenvalues will be conjugate complex, with

$$\text{Re } \mu_1 = \text{Re } \mu_2 = \frac{1}{2} \text{tr } \mathbf{S}_1 \quad (33)$$

Hence  $\text{tr } \mathbf{S}_1 > 0$  means instability, and  $\text{tr } \mathbf{S}_1 < 0$ , stability, of the FP.

---

\*Here  $\text{tr } \mathbf{S}_1 = \mathbf{S}_{11} + \mathbf{S}_{22} = \mu_1 + \mu_2$  (11') denotes the trace of  $\mathbf{S}$ , and  $\det \mathbf{S}_1 = \mathbf{S}_{11}\mathbf{S}_{22} - \mathbf{S}_{12}\mathbf{S}_{21} = \mu_1 \cdot \mu_2$  (11'') denotes the determinant of  $\mathbf{S}_1$ , respectively.

Combining these results, we can say that eqs. (30), (31), and (32) yield a real stable FP, whereas eqs. (30') and (32) yield a complex stable FP. All other situations are unstable.

The economic relevance of the stability or instability of an FP is significant. The ratios  $\epsilon = E/L$  and  $\kappa = K/L$  attain certain constant values  $\epsilon_0 = E_0/L_0$  and  $\kappa_0 = K_0/L_0$ , respectively, at the FP. This means that  $E$ ,  $K$ , and  $L$  have the same time evolution, given by  $n_0$  of eq. (21), at the point  $(E_0, K_0, L_0)$ .

If the FP  $(\epsilon_0, \kappa_0)$  is unstable, trajectories will move away from it in time, and the system will move into unrealistic regions, i.e., arbitrarily large or small values of  $E/L$  and  $K/L$ . Although our model is unrealistic for such values of the state variables, we can interpret this behavior as a prediction of catastrophic, and certainly undesirable, behavior of our model economy. If, on the other hand, the FP is stable, the system will tend to stable values of  $E/L$  and  $K/L$  and will achieve stable growth (or decrease, if  $n_0$  happens to be negative). Although at this point we can draw this conclusion only for evolution within the feasible region, so that eq. (18) is valid, Appendix B shows it also holds in a neighborhood of the slack-free region. This is certainly a more desirable economic situation.

In the case of stability, we can distinguish within the slack-free region a central region consisting of points the entire evolution of which will remain in the slack-free region (e.g.,  $A$  in Fig. 4). In contrast, points such as  $B$  in Fig. 4 will for some time leave the slack-free region, although they, too, will come back and tend toward the FP. Thus their evolution will be governed for some time by the dynamics discussed in Appendix B, which imply large-scale variations of the economy. We may regard the central region as the set of "best initial conditions" because a smooth evolution toward stable growth is assured there.

We note that the central region is different from the slack-free region only if the FP is a stable focus (i.e., has two complex eigenvalues with negative real parts). For an unstable FP, the concept of a central region is meaningless, as all trajectories except that coinciding with the FP will leave the slack-free region.

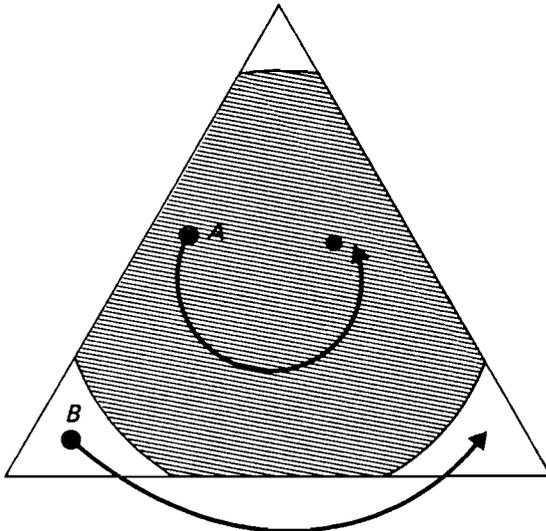


FIGURE 4 The slack-free region.

Note, too, that eq. (18) becomes undefined if  $\mathbf{T}$  is not invertible, i.e., if  $\det \mathbf{T} = 0$ . For such a technology matrix,  $E_I, K_I, L_I$  would be restricted to a plane. We assume that this is not the case, from a genericity argument. Care must be taken, however, if  $|\det \mathbf{T}|$  becomes too small or if, by an adiabatic variation of the technology matrix (see Section 6), we should cross the hypersurface in parameter space, where  $\det \mathbf{T} = 0$ .

## 6 THE BASE CASE

We chose a base set of parameters for actual calculations. As the elements of the technology matrix  $\mathbf{T}$  describe the technological standard of the model economy, the numbers could be expected to differ significantly according to their correspondence to the situation of a developed country (DC) or to that of a less developed country (LDC). In the case of a DC, most could be taken directly from data available at IIASA, while some had to be deduced from statistical material. In particular, the last column of  $\mathbf{T}$ , referring to skill production, required comparison of the relative numbers of teachers and students, identification of the depreciation rate with the rate of retirements, and so forth. In the case of an LDC, orders of magnitude of the required numbers were obtained from LDC specialists; these estimates are necessarily crude.

The primary purpose of the numerical calculations was not to obtain "predictions" but rather to acquire some feeling for the position and size of the economic niches. More explicitly, Section 5 contained the first step toward a division of the parameter space into economically meaningful parts and useless regions, according to the character of the FP. The second step involves selection of definite base case values within an economically meaningful part and estimation of the size of the region around these values, such that the qualitative behavior of the base case remains unchanged. For a stable FP we call this region a favorable economic niche. Favorable and unfavorable economic niches are separated by hypersurfaces; it is interesting to consider which parameters are primarily responsible for crossing such boundaries. In other words, which parameters allow the least range of variation? Since stability or instability of the FP (i.e., favorableness or unfavorableness of the corresponding economic niche) is described by the eigenvalues  $\lambda_i$  (eq. (26)), this amounts to analyzing which parameters show the strongest influence on the  $\lambda_i$ .

The eigenvalues are, however, not the only indicators of the economy; the growth rate at the FP is also important. Even within a favorable niche, the growth rate may be negative, which means that a trajectory will move toward equilibrium, but with shrinking  $E$ ,  $K$ , and  $L$ . As previously mentioned, growth rate  $n$  always appears in connection with depreciations. The unwanted situation of a negative growth rate could therefore immediately be improved by reducing the depreciations, i.e., by producing goods of higher quality and greater durability.

The model was implemented on a desk computer (HP8830A) and on IIASA's PDP11/70. Implementation allows

- Alternation between countries (i.e., parameter sets (DC and LDC)) and variation of scenario variables
- Computation of the FP and its eigenvalues
- Plotting of the feasible region

- Numeric integration and plotting of specific system trajectories
- Adiabatic variations of scenario variables during the numeric integration

With regard to the last point, the model implementation should be able to simulate an “economy in transition” in the sense used by Häfele *et al.* (1976) – for example, an economy changing from conventional to nuclear energy production. One must thus change some parameters continuously while running the model. We have assumed a transition from initial to final values using a logistic curve; this assumption is confirmed by the data collected by Marchetti and Nakicenovic (1979). The term *adiabatic* refers to the system’s smooth response to parameter changes if the time scale of those changes is long compared to that of the system.

Table 1 shows the parameter values of the base case representing a DC, together with the band width of allowed variation of each parameter (all other parameters remaining fixed). Numbers with a single asterisk do not denote the boundary of the niche but rather the point of transition to a negative growth rate. Numbers followed by < are still within the favorable niche, but we did not pursue the upper limit further.

The corresponding FP is located at  $\epsilon_0 = 10\,600$  Wyr/smyr,  $\kappa_0 = 16\,700$  \$yr/smyr, with a growth rate of  $n_0 = 3.3$  percent and eigenvalues  $\lambda = -0.047 \pm 0.31i$ .

The extent of the favorable niche in some parameter directions is quite wide, while in others (particularly  $a_{KK}$  and  $a_{LL}$ ) it is quite narrow.

Table 2 shows a set of parameter values that might represent the economy of some LDC. In this case the FP is not stable and we again indicate the extension of the unfavorable niche in each direction.

The corresponding FP is located at  $\epsilon_0 = 7\,500$  Wyr/smyr,  $\kappa_0 = 360$  \$yr/smyr, with a growth rate of  $n_0 = 2.2$  percent and eigenvalues  $\lambda = +0.58 \pm 0.66i$ .

Obviously, the extent of the niches in each direction changes if the parameters are not kept fixed at their base case values.

We may think of the base cases as the representation of certain scenarios:  $T$  describes a particular standard of investment goods production,  $e$  describes how effectively the energy allocated to private consumption is used, the  $\alpha$ s describe the emphasis on CP within the economy, and the  $a_{ij}$  describe a particular technology.

In order to study the effect of a scenario change, we introduced scenario variables, which enable one to study the economic niches in parameter space without needing to vary 18 parameters at the same time; moreover, most independent variations of parameters are unrealistic. Each set of values of the scenario variables, however, is assumed to represent at least a consistent model economy.

$H_1$  accounts for a transition from standard to new (e.g., nuclear or solar) energy options. We model the full transition by increasing  $a_{EE}$  by a factor of 30 and  $a_{KE}$  by a factor of 10, i.e.,

$$\begin{aligned} a_{EE} &\rightarrow a_{EE}(1 + 29 H_1) \\ a_{KE} &\rightarrow a_{KE}(1 + 9 H_1) \quad 0 \leq H_1 \leq 1^* \end{aligned} \tag{34}$$

Intermediate stages are represented by intermediate values of  $H_1$ ; this is similar for the other scenario variables.

---

\* $H_1$  may also be taken as larger than one, which corresponds to still more expensive energy options.

TABLE 1 Parameter values of the base case (DC).

*Technological matrix*

$$a_{EE} = 0.04 \text{ yr} \\ (0-0.7)$$

$$a_{EK} = 3.4 \text{ Wyr/\$} \\ (2-100 <)$$

$$a_{EL} = 300 \text{ Wyr/smyr}^{**} \\ (0-22400)$$

$$a_{KE} = 0.1 \text{ \$/yr/W} \\ (0-1.9)$$

$$a_{KK} = 2.5 \text{ yr} \\ (0-3)$$

$$a_{KL} = 1.8 \times 10^4 \text{ \$/yr/smyr} \\ (1.4 \times 10^4 - 2 \times 10^5 <)$$

$$a_{LE} = 8 \times 10^{-5} \text{ smyr yr/W} \\ (2.5 \times 10^{-5} - 2 \times 10^{-3*} - 3 \times 10^{-3})$$

$$a_{LK} = 8 \times 10^{-5} \text{ smyr yr/\$} \\ (4 \times 10^{-5} - 1.5 \times 10^{-3*})$$

$$a_{LL} = 0.7 \text{ yr} \\ (0-1)$$

*Depreciation rates*

$$d_E = 0.04$$

$$d_K = 0.04$$

$$d_L = 0.025$$

*Consumption fractions\*\*\**

$$\alpha_E = 0.35 \\ (0-0.4)$$

$$\alpha_K = 0.75 \\ (0-0.85)$$

$$\alpha_L = 0.8 \\ (0-0.9)$$

$$e = 0.9 \text{ W/\$} \\ (< 0-1.8)$$

*Cobb–Douglas constants*

$$\alpha = 0.12 \\ (< 0-0.17)$$

$$\beta = 0.12 \\ (< 0-0.17)$$

$$A = 425$$

\*Denotes point of transition to a negative growth rate.

\*\*Skilled man-year.

\*\*\* $\alpha$ s are varied simultaneously.

TABLE 2 Parameters of the base case (LDC).

<i>Technological matrix</i>		
$a_{EE} = 0.04 \text{ yr}$ (0-1 <)	$a_{EK} = 0.8 \text{ Wyr}/\$$ (0.03-7)	$a_{EL} = 40 \text{ Wyr/smyr}^*$ (0-160)
$a_{KE} = 0.02 \text{ \$/yr/W}$ (0-0.2 <)	$a_{KK} = 3 \text{ yr}$ (2.4-8 <)	$a_{KL} = 500 \text{ \$/yr/smyr}$ (0-700)
$a_{LE} = 2 \times 10^{-4} \text{ smyr yr/W}$ (0-3.3 $\times 10^{-4}$ )	$a_{LK} = 1.4 \times 10^{-3} \text{ smyr yr}/\$$ (0-2.4 $\times 10^{-3}$ )	$a_{LL} = 0.4 \text{ yr}$ (0.13-4)
<i>Depreciation rates</i>		
$d_E = 0.02$	$d_K = 0.02$	$d_L = 0.02$
<i>Consumption fractions **</i>		
$\alpha_E = 0.8$ (0-0.83)	$\alpha_K = 0.8$ (0-0.83)	$\alpha_L = 0.9$ (0-0.94)
<i>Cobb-Douglas constants</i>		
$\alpha = 0.4$ (0-0.42)	$\beta = 0.4$ (0-0.42)	$A = 16$
$e = 0.3 \text{ W}/\$$ (0-0.43)		

\* Skilled man-year.

\*\*  $\alpha$ s are varied simultaneously.

Looking at Table 1, we might expect  $H_1 = 1$  to bring the system close to the boundary of the base case niche, if not beyond. Our interpretation – always within the limits of the model – might then be that the base case economy could hardly afford to replace conventional energy options totally by new technology without paying the price somewhere else – on the consumption side, for example.  $H_2$ , the second scenario variable, describes the emphasis on CP. We assume, for the sake of simplicity, that the  $\alpha$ s are changed simultaneously.

$$\alpha_{E,K,L} \rightarrow \alpha_{E,K,L}(1 + 0.2H_2) \quad -1 \leq H_2 \leq 1 \quad (35)$$

$H_2 = -1$  represents a 20 percent reduction of CP inputs;  $H_2 = +1$ , a 20 percent increase. Not surprisingly, calculations show a high sensitivity of the growth rate  $n$  on  $H_2$ .

We also examined the effect of increasingly efficient energy use. This would decrease  $a_{EK}$  and  $e$  but would also change  $a_{KK}$ . Classical arguments about substitutability would suggest an increase in  $a_{KK}$  ( $\sigma = +1$  in the following equations); however, the highly aggregate character of our model makes this argument doubtful. In fact, adherents to the “small is beautiful” school of thought have claimed that saving energy according to that philosophy could actually result in a decrease in  $a_{KK}$  ( $\sigma = -1$ ). Thus we have incorporated both alternatives in a scenario variable  $H_3$ :

$$\begin{aligned} a_{EK} &\rightarrow a_{EK}(1 - 0.2H_3) \\ a_{KK} &\rightarrow a_{KK}(1 + \sigma 0.2H_3) \quad \sigma = \pm 1 \\ e &\rightarrow e(1 - 0.3H_3) \quad 0 \leq H_3 \leq 1 \end{aligned} \quad (36)$$

Figures 5 and 6 show the boundary in space of scenario variables  $H_1$ ,  $H_2$ , and  $H_3$ ; this boundary separates a favorable from an unfavorable niche. The direction of scenario

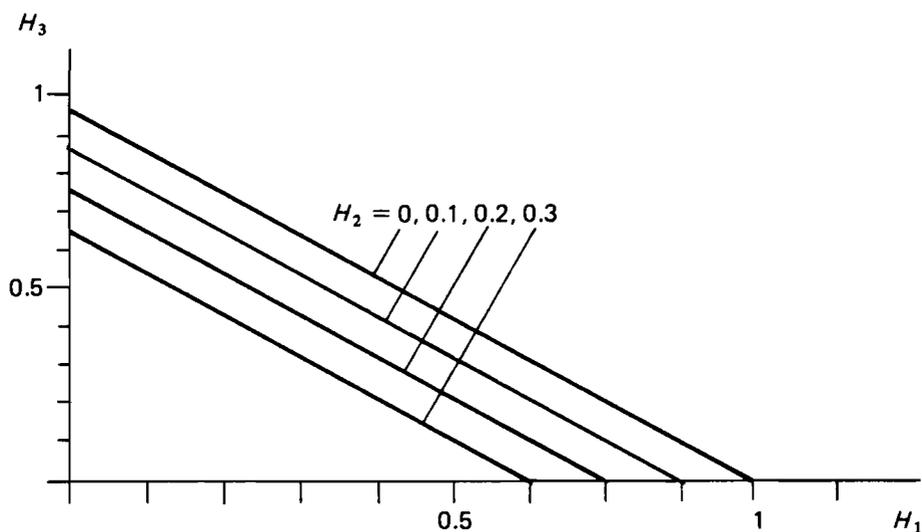


FIGURE 5 Boundaries of the stable niche (DC): “Big is beautiful” ( $\sigma = +1$ ).

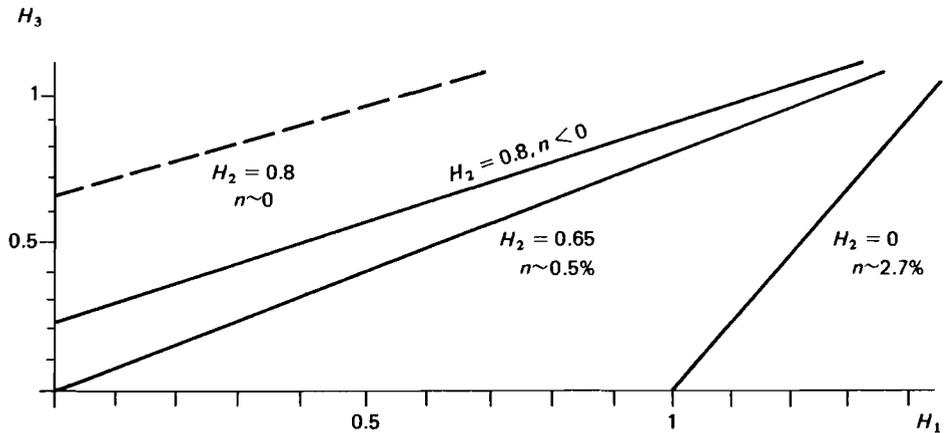


FIGURE 6 Boundaries of the stable niche (DC): “Small is beautiful” ( $\sigma = -1$ ).

variable  $H_2$  is perpendicular to the plane of the figure, and the intersections of the boundary surface with the planes  $H_2 = 0, 0.1, 0.2,$  and  $0.3$  (Fig. 5) and  $H_2 = 0, 0.65,$  and  $0.8$  (Fig. 6) are shown as solid lines. The region of stability always lies to the left of the respective lines. The growth rate along the line  $H_2 = 0.8$  in Fig. 6 is negative; therefore, the actual boundary lies within the stable region. This is indicated by the dotted line, which represents the locus of zero growth.

Figure 7 shows a typical trajectory for our DC base case. The FP is located at  $\epsilon_0 = 10\,600$  Wyr/smyr,  $\kappa_0 = 16\,700$  \$yr/smyr, with a growth rate of  $n_0 = 3.3$  percent. It will take five years to move along the trajectory from one square to the next.

## 7 TWO-COUNTRY INTERACTION

Thus far, the model has been used to describe an economy isolated from the rest of the world. Now, to be more realistic, we introduce two important links to other economies. First, most economies must import a significant part of their energy sources and must pay for it, with, for example, non-energy-related capital goods. Second, a rich economy may give away part of its  $\Delta K$  – thereby reducing its growth rate, but not so much as to leave its favorable economic niche – to support a poor economy and thus induce its transition into a favorable niche.

We therefore modify the dynamics of eq. (16) in the following way:

- Reduce  $\Delta K$  by a term proportional to  $E$ ; the factor of proportionality is denoted by  $g$ . The numerical value of  $g$  is obtained heuristically:

$$g \doteq (\text{fraction of energy imported}) \times (\text{conversion factor bbl/Wyr}) \times (\text{oil price } \$/\text{bbl}) \times (1 - \text{fraction recycled}) = 0.3 \times 5 \times 10^{-3} \times 12 \times (1 - 0.7) \approx 5 \times 10^{-3} \text{ } \$/\text{Wyr}.$$

The last factor is included because petrodollars reinvested do not correspond to capital goods extracted from the system.

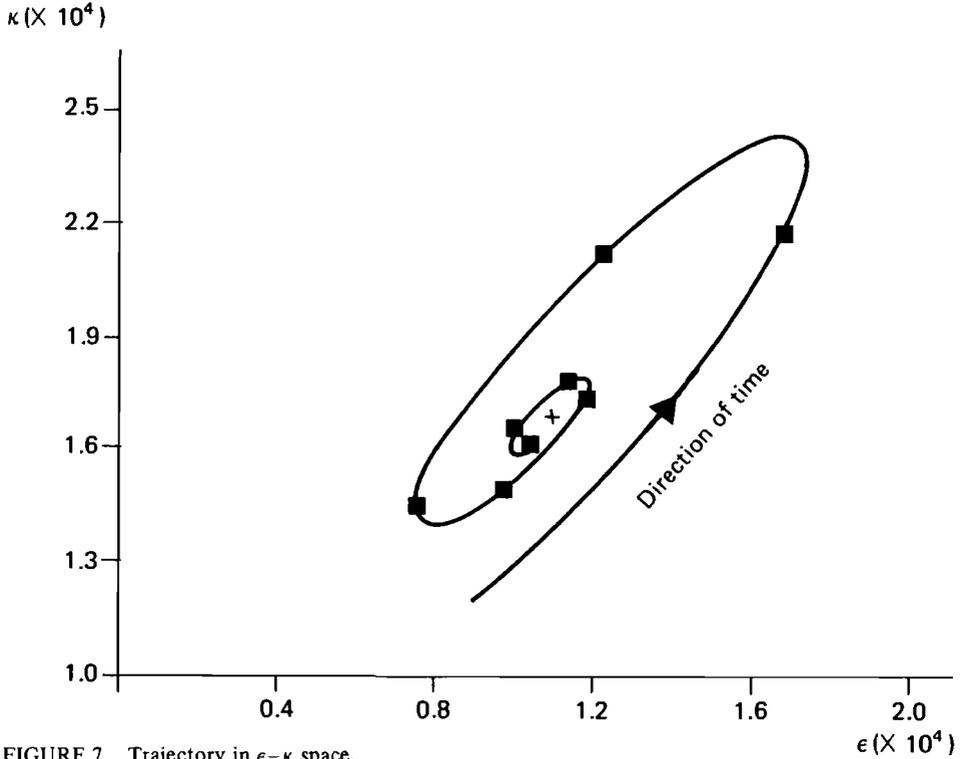


FIGURE 7 Trajectory in  $\epsilon$ - $\kappa$  space.

- Extract a fraction  $r\Delta K$  from (or, for negative  $r$ , add a fraction  $r\Delta K$  to) the total output of non-energy-related capital goods. This corresponds to foreign aid given away (or received).

In the spirit of the model, the phenomena of oil import and foreign aid are dealt with only schematically, by restricting discussion to the transfer of capital goods.  $r$  should not be confused with the well-known “0.7 percent of GNP” because foreign aid is measured on the scale of  $\Delta K$  only.

In mathematical terms, we have to replace  $\Delta K$  by  $(1 - r)\Delta K - gE$ ; this is done simply by replacing eq. (14) by

$$\dot{X} = rT^{-1}\{(1 - \alpha)X - X_p\} - d_g X \quad (37)$$

where

$$r := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-r & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad d_g := \begin{pmatrix} d_E & 0 & 0 \\ g & d_K & 0 \\ 0 & 0 & d_L \end{pmatrix} \quad (38)$$

For technical purposes,  $r$  may be incorporated into  $T$  by

$$T \rightarrow \begin{pmatrix} a_{EE} & a_{EK}/1-r & a_{EL} \\ a_{KE} & a_{KK}/1-r & a_{KL} \\ a_{LE} & a_{LK}/1-r & a_{LL} \end{pmatrix}$$

Starting from the two base cases for a DC and an LDC economy, respectively, we may consider the resilience against variations of  $g$  and  $r$  (the base cases themselves correspond to  $g = r = 0$ ). Given the price of energy (in terms of the oil price of 1976, 12 \$/bbl), how much foreign aid can the DC economy afford without leaving its favorable economic niche, decreasing its growth rate below zero, or both? How much foreign aid must be granted to an LDC economy so that both a transition from its unfavorable to a favorable economic niche and a positive growth rate are induced?

Figures 8 and 9 illustrate our findings. In Fig. 8, the solid line divides the stable region (on the left) from the unstable region (on the right); the broken lines are loci of constant

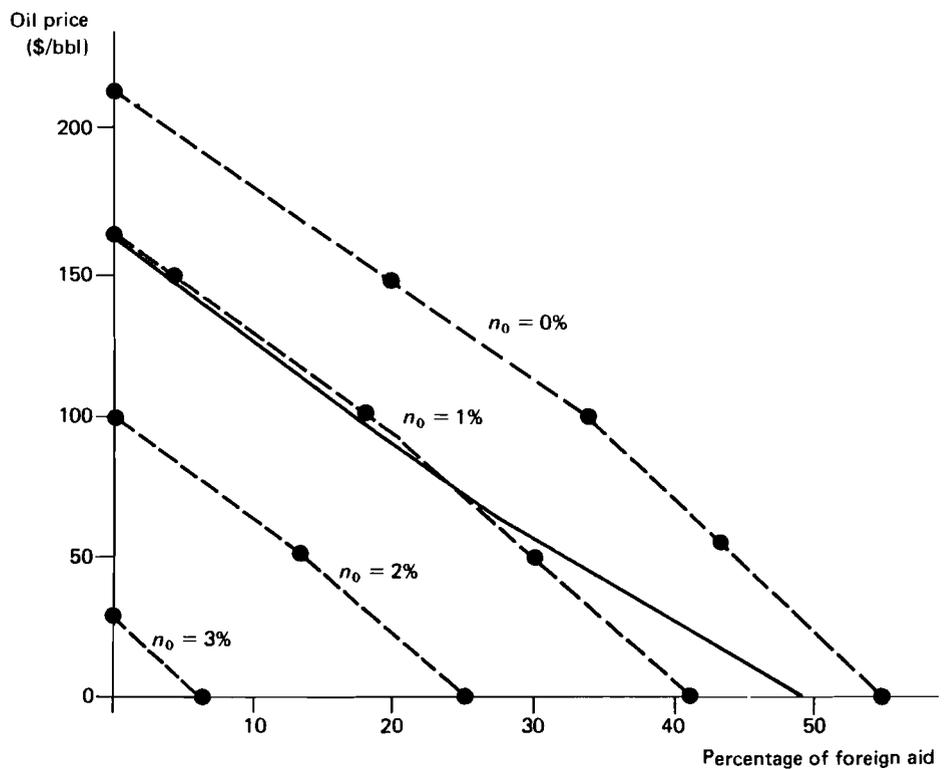


FIGURE 8 Economic aid: niche boundary (DC).

growth rate. In Fig. 9, the stable region is to the right of the boundary. In both cases, foreign aid is measured relative to  $\Delta K$  of the respective country.

We may then combine the two economies by superimposing the two figures. Care must be taken to rescale foreign aid to one of the two economies. Hence, the ratio  $\eta = \Delta K_{\text{LDC}} / \Delta K_{\text{DC}}$  must be given some value.\* The result is displayed in Fig. 10(a) with

\*The units on the abscissae in Figs. 8 and 9 are different;  $\eta$ , which relates capital goods production in the DC to that in the LDC, accounts for this difference.

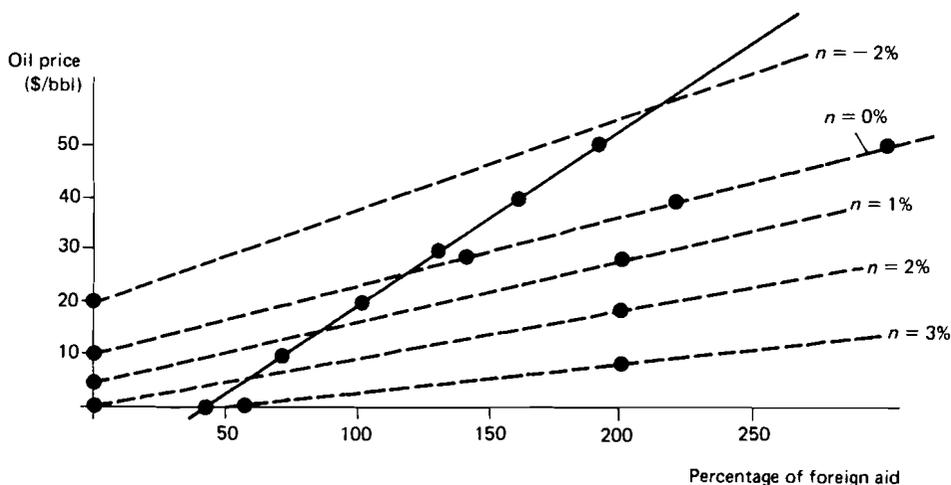


FIGURE 9 Economic aid: niche boundary (LDC).

$\eta = 0.1$  and in Fig. 10(b) with  $\eta = 0.2$ . Fig. 11 (of which Figs. 10a and 10b are actually slices) shows the situation with various values of  $\eta$ . As  $\eta$  is essentially a scale factor, diagrams with different values of  $\eta$  do not differ qualitatively.

## 8 OUTLOOK AND CONCLUSION

Two directions for further study come to mind immediately:

- Introducing an oil country as a full economy rather than merely as a sink for investment goods as in Section 7. Thus two-country (DC + oil country) or three-country (DC + LDC + oil country) interaction could be investigated. The existence of oil price thresholds (lower, upper, or both) for the stability of the oil country would be an interesting consideration.
- Relaxing our requirement of homogeneous production functions (neither economy nor diseconomy of scale). For this to be done meaningfully,  $E$ ,  $K$ , and  $L$  must be reinterpreted as quantities referring to a “typical population” (say, 50 million inhabitants) or, better, to a “typical area” (say, 500 000 km<sup>2</sup>; just as we introduced skill, we could weigh differently land of different productivity or other factors.\* We could then multiply the linear production function for the investment sector with a “congestion function” or “agglomeration function,” depending on the level of economic activity, which would be measured by a suitable linear combination of  $E$ ,  $K$ , and  $L$ .

An interesting agglomeration function has the shape shown in Fig. 12. This congestion function expresses the unfavorable effects of levels that are either too high or too low. A system with this function would behave like a single-species ecological system with

\*An FP in  $E$ ,  $K$ , and  $L$  would imply that we predict the same equilibrium energy production for the US as for Liechtenstein, as long as they have the same parameters!

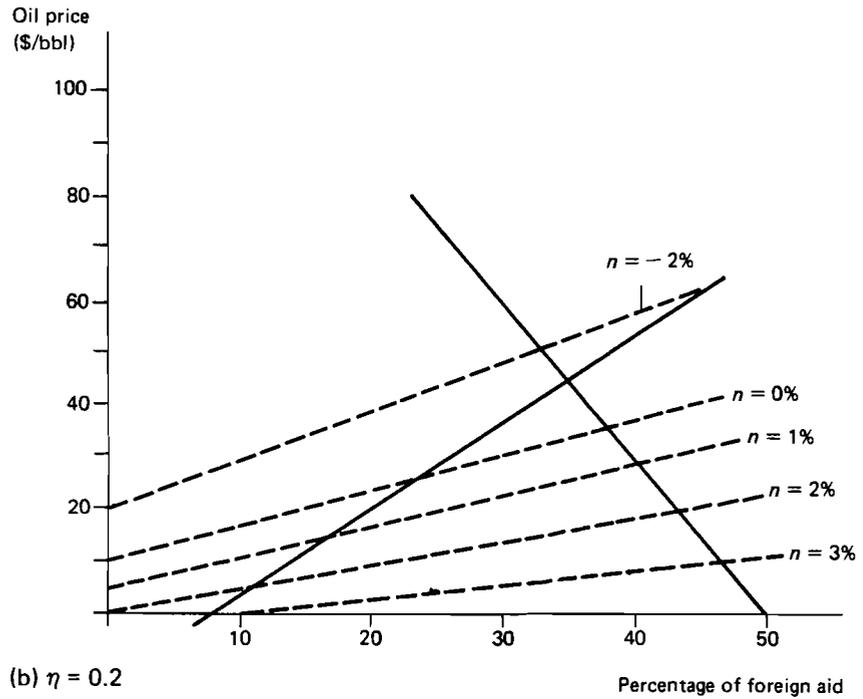
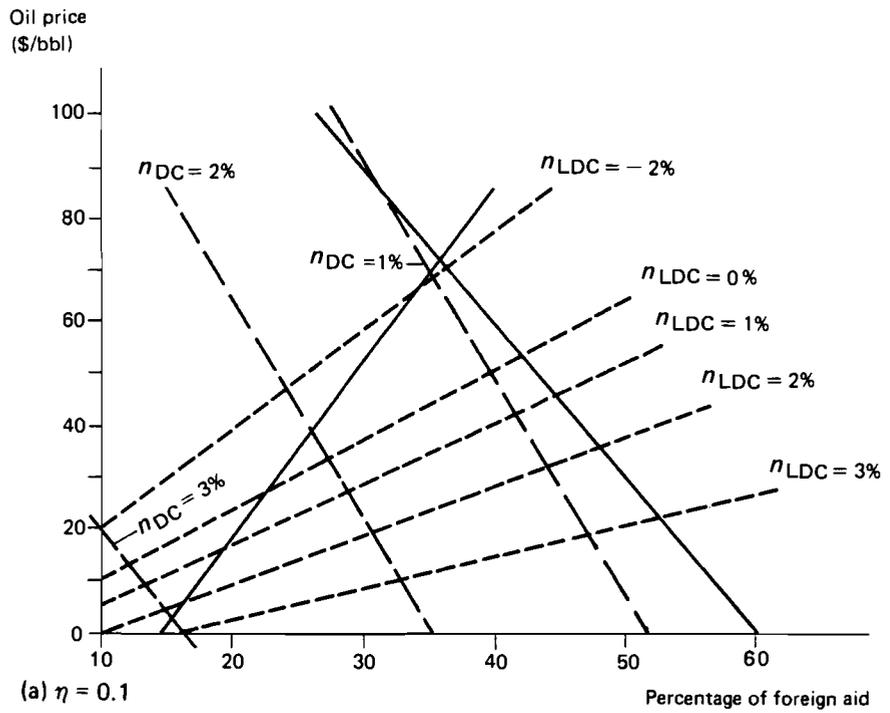


FIGURE 10 Maximum oil price.

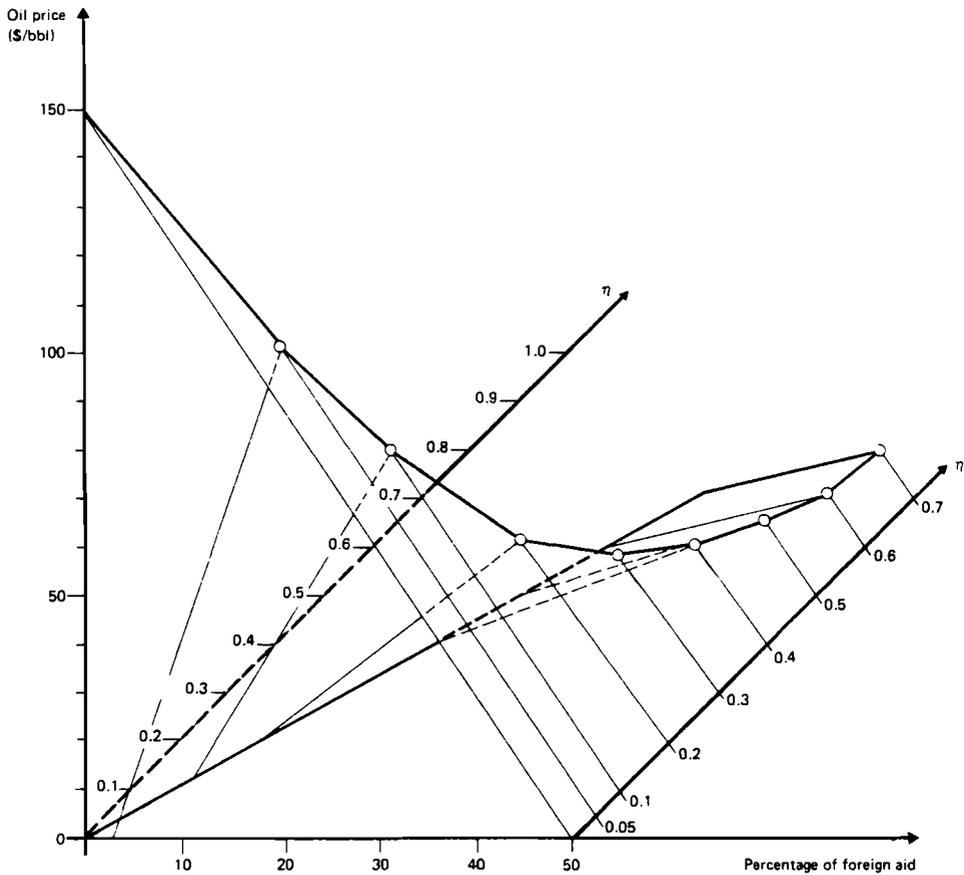


FIGURE 11 Maximum oil price variation with  $\eta$ .

the reproduction curve illustrated in Fig. 13 (Holling 1973), which shows a threshold level below which growth is negative and the system tends to zero and an equilibrium at a higher economic level; the trends are indicated by arrows. The behavior of the system transverse to the stable ray, i.e., in the  $\epsilon$ - $\chi$  space, remains unaffected; the system thus has two FPs in  $E$ ,  $K$ , and  $L$  outside of the origin, one of which can generate a separatrix.

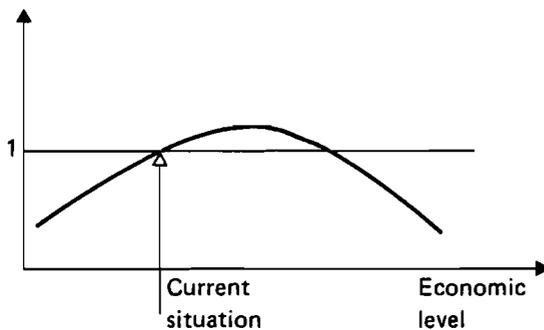


FIGURE 12 Congestion function.

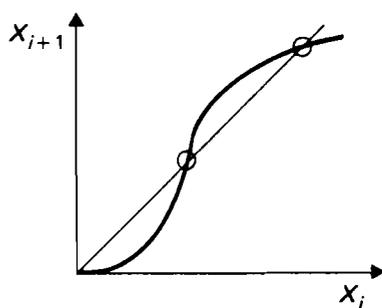


FIGURE 13 Reproduction curve.

The two-country interaction that we introduced must be considered as a first step. To help an LDC simply by pumping investment goods into its economy is certainly not sufficient: the transition to a favorable economic niche would be only a temporary one, and a cancellation (or perhaps merely a reduction) of foreign aid would result in an immediate reversal. The system parameters are only virtually, not intrinsically, changed by the exogenous support (as long as it is supplied). It is therefore necessary to introduce a coupling between the foreign aid and the parameters that corresponds to an actual improvement in the economy's infrastructure. As the economy improves, foreign aid may be reduced gradually until the transition from an LDC to a DC is completed. The mathematical way to express these mechanisms is unclear.

In summarizing our results, we must stress both what our approach can do and what it cannot do. Any detailed economic prediction – with quantitative results that inspire confidence – of course requires a much larger model. While it would be ridiculous to claim that an economy can be described accurately by just three state variables, we do feel justified in making three observations.

First, to the extent that the structure of our model (i.e., the choice of the state variables and the form and interrelation of the production functions) has something to do with an actual economy, we can deduce the existence of a slack-free region within the state space. As it can be said definitely that unpleasant situations will arise at the boundary (e.g., one or several outputs will tend to zero), proximity to the boundary of the slack-free region should be avoided.

Similarly, we have determined boundaries in parameter space (or, equivalently, in a space of scenario variables) across which the model's behavior changes drastically, showing instability, negative growth, and so forth. Because our knowledge is incomplete, parameter values close to those boundaries should be avoided as well (see Section 3).

Second, the model allows us to determine an FP (point of equal growth of  $E$ ,  $K$ , and  $L$ ) that shows one of the previously mentioned possible qualitative behaviors. These possible behaviors depend on the model economy's infrastructure, expressed in terms of a set of certain characteristic system parameters. The parameter space appears to be subdivided into cells, which we have called economic niches. By definition, all points within one niche correspond to economies with the same topological behavior. We have called niches with a stable associated FP favorable niches; their significance lies in the existence of a central region around the FP (within the slack-free region, of course) from which the slack-free region cannot be escaped. On the contrary, an economy starting outside the central region inevitably approaches the boundary of the slack-free region

within a finite time. If such a trajectory were well off the boundary, a person “living” on it might not initially recognize a problem with the economy but, even with positive growth rates of  $E$ ,  $K$ , and  $L$ , would observe that at least one of the outputs becomes zero. The initial conditions, i.e., the “right” amount of the available stocks, are essential for a favorable economic evolution.

Third, the two-country interactions studied in Section 7 model the structure to be expected for foreign aid given by an abstract DC to an abstract LDC, both of which are subject to the same oil price. An oil price sufficiently high inhibits any reasonable level of foreign aid; either the DC gives away too much or the LDC receives too little. It is a pleasant surprise that the limiting oil price comes out at the right order of magnitude – neither too close to the present level nor too high. A limiting price of, say, 10 000 \$/bbl would make this feature irrelevant.

These qualitative results suggest interesting questions that we hope will be addressed through a full-size economic model.

## ACKNOWLEDGMENTS

In preparing this paper, the authors benefited greatly from the collaboration of C. Riedel, from the continuous interest and guidance of W. Häfele, and from stimulating discussions with R. Bürk and H.-H. Rogner.

## REFERENCES

- Avenhaus, R., H.-R. Grümml, and W. Häfele (1975) *New Societal Equations*. Internal Paper. Laxenburg, Austria: International Institute for Applied Systems Analysis.
- Beaumol, W.J. (1977) *Economic Theory and Operations Analysis*. Englewood Cliffs, New Jersey: Prentice-Hall, Inc.
- Dorfman, R., P.A. Samuelson, and R.M. Solow (1958) *Linear Programming and Economic Analysis*. New York: McGraw-Hill Book Co.
- Energy Systems Program Group of the International Institute for Applied Systems Analysis (1981) *Energy in a Finite World: A Global Systems Analysis*. W. Häfele, Program Leader. Cambridge, Massachusetts: Ballinger.
- Grümml, H.-R., ed. (1975) *Analysis and Computation of Equilibria and Regions of Stability with Applications in Chemistry, Climatology, Ecology, and Economics: Record of a Workshop*. CP-75-8. Laxenburg, Austria: International Institute for Applied Systems Analysis.
- Grümml, H.-R. (1976) *Definitions of Resilience*. RR-76-5. Laxenburg, Austria: International Institute for Applied Systems Analysis.
- Grümml, H.-R., and L. Schrattenholzer (1976) *Economy Phase Portraits*. RM-76-61. Laxenburg, Austria: International Institute for Applied Systems Analysis.
- Häfele, W. (1975) *Objective Functions*. WP-75-25. Laxenburg, Austria: International Institute for Applied Systems Analysis. German version: *Zielfunktionen, Beiträge zur Kerntechnik, Bericht der Gesellschaft für Kernforschung*: 25–49. KFK 2200, Karlsruhe, FRG.
- Häfele, W., and R. Bürk (1976) *An Attempt of Long-Range Macroeconomic Modelling in View of Structural and Technological Change*. RM-76-32. Laxenburg, Austria: International Institute for Applied Systems Analysis.
- Häfele, W., *et al.* (1976) *Second Status Report of the IIASA Project on Energy Systems*. RR-76-1. Laxenburg, Austria: International Institute for Applied Systems Analysis.
- Hicks, J. (1969) *Capital and Growth*. Oxford: Oxford University Press.

- Holling, C.S. (1973) Resilience and Stability of Ecological Systems. RR-73-3. Laxenburg, Austria: International Institute for Applied Systems Analysis.
- Holling, C.S., ed. (1978) Adaptive Environmental Assessment and Management. Chichester, UK: John Wiley and Sons.
- Marchetti, C., and N. Nakicenovic (1979) The Dynamics of Energy Systems and the Logistic Substitution Model. RR-79-13. Laxenburg, Austria: International Institute for Applied Systems Analysis.
- Samuelson, P.A. (1947, 1976) Foundations of Economic Analysis. Cambridge, Massachusetts: Harvard University Press and New York: Atheneum.

## APPENDIX A Some Concepts of Dynamical Systems

We assume a deterministic system described by differential equations: knowing the state of the system at a particular time, one can calculate the time derivatives of all state variables. A “geometric” point of view is inherent in this approach: we introduce *state space*, each of whose points fully specifies a state of the system at one instant in time. The state space is spanned by the *state variables*. On the state space, we have a *time-evolution law*, possibly dependent on several parameters, ranging over *parameter space*. Under it, the states of the system move along *trajectories*. In our approach, we emphasize not so much a single trajectory as the structure of all trajectories. A *fixed point* (or *equilibrium*) is a state of the system that does not change in time; it may be stable or unstable.

Under general assumptions, the state space can be divided into *basins*, each consisting of states having a common future long-term behavior. Each basin contains one *attractor* representing this common behavior; it is the region in state space toward which all trajectories originating in the basin tend. The simplest attractor is a *stable fixed point* (or *stable equilibrium*); if the attractor of a basin is a stable fixed point and if the system starts in this basin, all state variables of the system will tend toward constant values. There are many more complicated types of attractors; the list is currently incomplete.

Basins are separated from each other by *basin boundaries* or *separatrices*. States on or very close to a basin boundary have uncertain futures because small modifications of the state variables may cause them to belong to different basins and thus to exhibit completely different long-term behavior.

The *phase portrait* is a full (at least qualitatively) description of the basins and attractors of a system. In general, it depends on the parameters of the system; qualitative changes of the phase portrait caused by parameters crossing certain boundaries are called *bifurcations*. These boundaries play a role similar to that of separatrices in state space.

The mathematical theory behind these concepts can be found in Arnold (1973), Grüm (1979), and Hirsch and Smale (1974).

## APPENDIX B Dynamics Outside the Slack-Free Region

In looking at possible dynamics outside the slack-free region of the system defined in Section 4, we may rewrite eq. (14) as follows:

$$\Delta X = (\dot{X} + dX) = T^{-1}X_I \quad X_I = X(1 - \alpha) - X_p \quad (\text{B1})$$

We have chosen this form to avoid the detailed structure of  $X_I$ , which is a known function of  $X$ . As discussed, if  $\mathbf{T}^{-1}X_I \succ 0$ , i.e., if some  $(\mathbf{T}^{-1}X_I)_j < 0$ , eq. (B1) does not make economic sense. We denote the value of  $\Delta X$  obtained from eq. (B1) by  $\Delta X_{\text{vir}}$  (the “virtual” gross production). To find a realistic  $\Delta X$ , we turn again to eq. (6),

$$\Delta E = \min \left( \frac{E_{I,E}}{a_{EE}}, \frac{K_{I,E}}{a_{KE}}, \frac{L_{I,E}}{a_{LE}} \right) \quad (\text{B2})$$

with similar equations for  $\Delta K$  and  $\Delta L$ . The assumption of “allocations without slacks,” i.e., the equality of all terms in eq. (B2) and its  $\Delta K$  and  $\Delta L$  counterparts, leads to eq. (B1); thus this assumption can be fulfilled if and only if the system lies within the slack-free region. In this case, it leads to the unique dynamics contained in eq. (B1). Outside the slack-free region, we have to introduce slacks, taking into account that the terms in eq. (B2) and its counterparts will be equal. Summing the slacks occurring in the three sectors, we obtain

$$\Delta X = \mathbf{T}^{-1}X_W = \mathbf{T}^{-1}(X_I - X_S) \quad 0 \leq X_S \leq X_I \quad (\text{B3})$$

with  $X_S$  representing the stock of energy-related goods, non-energy-related capital goods, and skilled labor not allocated to production and  $X_W$  representing the stock actually “working” in the investment goods sectors.

As it stands, eq. (B3) of course does not define a unique evolution. We complement it by two requirements.

The first is the requirement of “Pareto optimality” of our allocation: no other allocation of  $E_N^I$ ,  $K_N^I$ , and  $L_N^I$  to the three sectors leads to an increase in any of the quantities  $\Delta E$ ,  $\Delta K$ , or  $\Delta L$  without decreasing at least one of them. This is an obvious extension of the “allocation without slacks” possible within the slack-free region; there, this allocation is the unique Pareto-optimal one. Outside the slack-free region, there are generally several Pareto-optimal allocations.

The second is the requirement that uneconomical processes be shut off; if  $\Delta E_{\text{vir}} < 0$ , the realistic  $\Delta E$  is set to zero. The allocation without slacks that, if possible, would give the virtual  $\Delta E$  would require the  $\Delta E$  production to run backwards; as this is not possible, we handle the situation by shutting off  $\Delta E$  production completely. This requirement follows from the first if we have only two production functions; one can argue for it using familiar arguments from linear programming.

Both requirements might still fail to define a unique evolution of our system by giving unique expressions for  $\dot{E}$ ,  $\dot{K}$ , and  $\dot{L}$ . We call any evolution of the system outside the slack-free region fulfilling eq. (B3) and these two requirements a rational evolution.

To illustrate the general situation, we assume that the system is just crossing from inside the boundary of the slack-free region corresponding to  $\Delta E = \dot{E} + d_E E = 0$ . We write for the “real” evolution outside the slack-free region

$$\begin{aligned} \dot{E} + d_E E &= 0 \\ a_{KE}(\dot{K} + d_K K) + a_{LE}(\dot{L} + d_L L) &= E_I - E_S \\ a_{KK}(\dot{K} + d_K K) + a_{LK}(\dot{L} + d_L L) &= K_I - K_S \\ a_{KL}(\dot{K} + d_K K) + a_{LL}(\dot{L} + d_L L) &= L_I - L_S \end{aligned} \quad (\text{B4})$$

together with

$$0 \leq E_S \leq E_I \quad 0 \leq K_S \leq K_I \quad 0 \leq L_S \leq L_I \quad (\text{B5})$$

and the Pareto optimality of the allocation. At least one of the slacks must obviously be zero; otherwise, we could increase  $\Delta K$ , say, without decreasing  $\Delta L$ .<sup>\*</sup> If one slack is zero, the other two are linearly related; taking the inequalities of eq. (B5) into account, we see that for all rational allocations  $X_S = (E_S, K_S, L_S)$  must lie on one of three straight segments. As eq. (B4) is an affine relation between  $(\dot{K}, \dot{L})$  and  $X_S$ , all rational allocations lead to the following net increases:

$$E = -d_E E$$

$$\begin{pmatrix} \dot{K} \\ \dot{L} \end{pmatrix} \in S \quad (\text{B6})$$

$S$  denotes the set in  $(\dot{K}, \dot{L})$  space that is the image of the previously mentioned straight segments. If  $S$  contains a "greatest point," i.e., if both  $\dot{K}$  and  $\dot{L}$  are larger than at any other point of  $S$ , there is only one rational allocation, and the time derivatives  $\dot{E}$ ,  $\dot{K}$ , and  $\dot{L}$  are uniquely determined; if not, the system will have a residual freedom. The two situations in  $(\dot{K}, \dot{L})$  space ( $S$  is the boundary of the triangle) are shown together with the virtual time derivatives in Fig. B1. Fig. B1(a) shows a unique evolution; in Fig. B1(b), any point on the upper right-hand segment corresponds to a rational evolution; nevertheless, all rational trajectories will lie inside a certain "fan."

This ambiguity is not serious: as we come close to the slack-free region,  $S$  becomes small and contracts to  $(\dot{K}, \dot{L})_{\text{vir}}$  as we reach the boundary. If the virtual evolution is stable, as analyzed in Section 5, its trajectories leaving the slack-free region will come back. As indicated in Fig. B1,  $\dot{E}_{\text{vir}} < \dot{E}$ ,  $\dot{K}_{\text{vir}} > \dot{K}$ ,  $\dot{L}_{\text{vir}} > \dot{L}$  for all rational allocations (if the trajectory crosses the  $\Delta E = 0$  boundary). Outside the slack-free region a "rational"

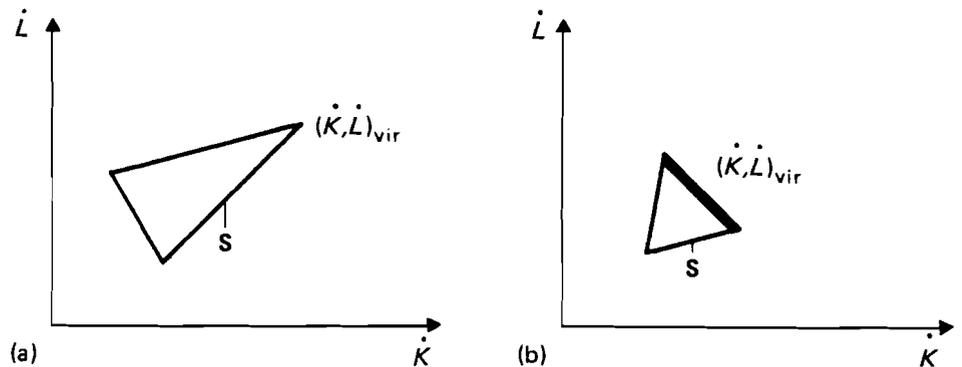


FIGURE B1 Rational evolutions.

<sup>\*</sup>Equation (B4) of course holds on the boundary of the slack-free region, too, with  $E_S = K_S = L_S = 0$ . Furthermore, the "real"  $\dot{K}$  and  $\dot{L}$  are the same as the virtual  $\dot{K}_{\text{vir}}$  and  $\dot{L}_{\text{vir}}$  defined by eq. (B1). Thus the evolutions inside and outside the slack-free region fit together continuously.

trajectory will of course not coincide with the “virtual” trajectory but, owing to these inequalities, it will lie closer to the slack-free region than will the “virtual” trajectory. Thus any rational evolution leads the system back to the slack-free region, just as the “virtual” trajectory does.

### APPENDIX C A Criterion for the Number of Fixed Points

The criterion explained here distinguishes between the two alternatives discussed in Section 5. We write eq. (18') in the form

$$\det [1 - \alpha - (d + n)\mathbf{T}] = 0 \quad (\text{C1})$$

which yields ( $x = d + n$ )

$$A + BX + CX^2 + DX^3 = 0 \quad (\text{C2})$$

Here  $A = \det (1 - \alpha)$ ,  $D = -\det \mathbf{T}$ , and  $B$  and  $C$  are defined correspondingly. Substituting

$$p := B/D - C^2/3D^2 \quad q := 2C^3/27D^3 - BC/3D^2 + A/D \quad y := X + C/3D \quad (\text{C3})$$

into the normal form of an equation of third degree,

$$y^3 + py + q = 0 \quad (\text{C4})$$

brings us back to eq. (C3). Theory now tells us that there will be one real solution of eq. (22) if

$$q^2/4 + p^3/27 > 0 \quad (\text{C5})$$

and three real solutions if

$$q^2/4 + p^3/27 < 0 \quad (\text{C6})$$

Substitutions of eq. (C3) into eq. (C5) or eq. (C6) produce a hypersurface in parameter space that separates the two alternatives. Unfortunately, this relation is too clumsy to be written down explicitly.

Equations (C5) and (C6) are only a criterion for the simplified case (equal  $d$ s and  $X_p = 0$ ); the hypersurface mentioned will be continuously deformed during the transition to the general case.

### REFERENCES TO APPENDIXES

- Arnold, V.I. (1973) *Ordinary Differential Equations*. Cambridge, Massachusetts: MIT Press.  
 Grümm, H.-R. (1979) *Introduction to Dynamical Systems*. Internal Paper. Laxenburg, Austria: International Institute for Applied Systems Analysis.  
 Hirsch, M., and S. Smale (1974) *Differential Equations, Dynamical Systems and Linear Algebra*. New York: Academic Press.

## THE AUTHORS

*Manfred Breitenecker* received his Ph.D. in physics and mathematics in 1960 and his habilitation as docent in theoretical physics in 1975 from the University of Vienna. An assistant at the Institute for Theoretical Physics at the University of Vienna, he has been with IIASA since 1976 as a consultant with the Energy Systems Program. His special interests include mathematical physics, dynamical systems, functional analysis, and teaching.

*Hans-Richard Grümm* received his M.S. in mathematical physics in 1970 from the University of Washington, his Ph.D. in the same subject in 1971 from the University of Vienna, and his J.D. in 1976 from the University of Vienna. He is an assistant at the Institute for Theoretical Physics at the University of Vienna. In 1975 he joined IIASA, where his research focuses on the qualitative theory of differentiable dynamical systems as applied to the concept of resilience.