RESILIENCY, STABILITY AND THE DOMAIN OF ATTRACTION

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I. Introduction

A common thread running through the analysis of most large systems is the problem of how "elastic" is the system under study, i.e. if the system is perturbed from some equilibrium state (or set), how large a perturbation can it withstand before either breaking down or entering a new operating regime? Obviously, when the problem is stated in such a vague, intuitive fashion no quantitatively useful answers may be obtained. To make analytic progress, some mathematical "flesh and blood" must be added to the imprecise skeletal verbal system description. However, even the above crude problem statement is qualitatively useful since it makes perfectly clear the sound system-analytic principle of "stability before optimality." It is a happy mathematical coincidence that, for some special classes of systems (fortunately useful), it can be shown that an optimal control law also generates a stable system. However, in general these two notions have to be treated as separate questions with stability being the first order of business.

The idea of determining the "elasticity", or "resiliency", of a mathematical system is not a new one, essentially having its origins in the stability theory of differential equations, where an item of prime concern is the so-called "domain of attraction" of a critical point. Here, of course, the question is to determine those regions in phase space which belong to given attractors (or repellors). In general, this is a very difficult problem with no complete solution. Unfortunately, the resiliency notion has not, as yet, been systematically pursued by analysts of physical systems. A few isolated first steps have been taken, notably in ecology [1-2], but for the most part effort has been concentrated on determining either the domain of attraction under the free (uncontrolled) motion of the system, or controls which optimize some integral criteria of the system's controlled motion. In this note, we intend to investigate some of the relationships between feedback controls and system resiliency.

In the classical theory of differential equations, it is a well known fact that the linear system of n equations

$$\dot{x} = Ax \quad , \qquad x(0) = c \quad ,$$

will have $x \longrightarrow 0$ as $t \longrightarrow \infty$ for all c if, and only if, A is a stability matrix, i.e. the characteristic values of A lie in the left half-plane. Thus, the domain of attraction of the origin is, in this case, the whole space R^n. The Poincare-Lyapunov theorem partially extends this result to certain types of nonlinear systems. Namely if

$$\dot{x} = Ax + g(x) \quad , \qquad x(0) = c \qquad (*)$$

where i) A is a stability matrix, ii) g is a vector function such that $\frac{||g(x)||}{||x||} \rightarrow 0$ as $||x|| \rightarrow 0$, and iii) $||c||$ is sufficiently small, then all solutions of (*) also approach zero as $t \rightarrow \infty$. In other words, in a sufficiently small neighborhood of the origin, the linear part of (*) determines its stability characteristics. Note that this is a much weaker result than that for linear equations since now the domain of attraction of the origin has been reduced from all of R^n to a "sufficiently small" neighborhood of 0 and what "sufficiently small" means is determined by the precise structure of g.

As already noted, it is often of interest to determine the boundary of the domain of attraction in order to gain insight into the resiliency question. From a passive, purely observer-oriented point of view, the determination of this boundary is an important question and the classical theory of ordinary differential equations contains many results in this direction. But from a more fundamental, "activist" viewpoint, the question while still interesting (academically), is rather sterile. The reason is that even if one had magical techniques for precisely determining the domain of attraction, if there are no means available for influencing the behavior of the system from the outside, then one is forced to accept the free motion of the system and its associated domain of attraction. In other words, if the initial perturbations <u>and</u> the system dynamics are beyond control, then it is small consolation to know the domain of attraction. Admittedly, this situation changes if the initial disturbances are "influenceable". On the other hand, if there exists a capability to interact with the system, then the following problem arises: within the constraints of allowable interaction, determine a control input such that the domain of attraction has certain properties, e.g. is as large as possible, includes a certain region, etc. If it were known, for example, (as it is for linear systems), that by suitable feedback it was possible to arrange to have the domain of attraction be any prescribed set, then it would not be necessary to design resiliency into a system, since it could always be achieved afterwards by suitable feedback control. Naturally, such a result would allow the system designer (or controller) great flexibility in dealing with other aspects of the system, safe in the knowledge that he can always recapture an arbitrary degree of resiliency.

Our objective in this note is to establish a result along the foregoing lines for the class of nonlinear systems (*) under specific conditions as to how one is allowed to interact with the system. Applications to some ecological models will then be given, along with some possible connections to related work on minimal control fields [3-4].

2. Linear Feedback and Resiliency

As a point of departure, let us assume that the allowable external inputs to the system under study appear additively and are linear. Thus, we consider the controlled system

$$\dot{x} = Fx + Gu + h(x) \quad , \qquad x(0) = c \quad , \qquad (\Sigma)$$

where x is the u-dimensional state vector, u is an m-dimensional control vector, F and G are constant matrices of the appropriate sizes, and h is a continuous n-dimensional vector function satisfying the conditions $\frac{||h(x)||}{||x||} \to 0$ as $||x|| \to 0$ in some vector norm $||\cdot||$. Note that the matrix G specifies the allowable interactions between the control input u and the state x. Further, assume that the allowable control laws u are "linear feedback", i.e. $u(t) = -Kx(t)$, where K is an mxn constant matrix to be chosen in an appropriate manner.

The basic question we ask is the following: given F, G, h, is it possible to choose K so that the domain of attraction of the origin contains some prespecified neighborhood of the origin? In particular, can K be chosen so that the set $\Omega = \{c: ||c|| < M\}$ is contained in the domain of attraction of 0, where M is given in advance? The theorem given below asserts that, under very weak conditions on F and G, the answer to this question is yes. Thus, by suitable linear feedback, Σ may be made to have any prespecified degree of resiliency! The precise statement of the theorem is

Theorem 1. Let the pair (F,G) be completely controllable and let $M > 0$ be specified. Assume that the system Σ is as specified above. Then there exists a matrix K such that the domain of attraction of the controlled system

$$\dot{x} = (F - GK)x + h(x) \quad , \quad x(0) = c \quad , \qquad (\Sigma_c)$$

contains Ω.

Proof: The proof hinges upon the "pole-shifting" theorem of linear systems theory which asserts that, given any symmetric set Λ of complex numbers, there exists a matrix K such that the characteristic roots of F - GK are the set Λ if, and only if, (F,G) is completely controllable. Thus, in particular, we can arrange for the root with largest real part to be as far into the left half-plane as desired.

Now let Λ be a symmetric set of n complex numbers whose element with largest real part lies to the left of the real number $\sigma < 0$, where σ will be specified in a moment. Further, assume that the elements of Λ are distinct. Applying the pole-shifting theorem, we determine K so that F - GK has Λ as its set of characteristic values.

Next, make the transformation $z = Tx$ to diagonalize $(\sum_c)$. This gives the equivalent system

$$\dot{z} = Dz + \tilde{h}(z) \quad , \qquad z(0) = \tilde{c} \quad ,$$

where $D = \mathrm{diag}(\lambda_1,\ldots,\lambda_n) = T(F - GK)T^{-1}$, $\tilde{c} = Tc$, $\tilde{h}(z) = Th(T^{-1}z)$. It is easy to see that $\tilde{h}$ satisfies the same conditions as h.

Define the scalar function

$$V\,(z) = 1/2(z,z) \quad .$$

Then

$$\dot{V}\left(z(t)\right) = \sum_{i=1}^{n} |\lambda_i|^2 z_i^{\,2} + \sum_{i=1}^{n} z_i(t)\tilde{h}_i\left(z(t)\right)$$

$$\leq \sigma \sum_{i=1}^{n} z_i^{\,2}(t) + ||z(t)||\;||h\left(z(t)\right)||$$

For $t = 0$,

$$\dot{V}\left(z(0)\right) \leq \sigma \sum_{i=1}^{n} \tilde{c}_i^{\,2} + ||\tilde{c}||\;||\tilde{h}(\tilde{c})||$$

Thus, choosing $\sigma < \frac{-||\tilde{c}||\ ||\tilde{h}(\tilde{c})||}{\sum_{i=1}^{n} c_i^2}$, $\dot{V}(z) < 0$ in some neighborhood of $t = 0$.

Hence V will be decreasing in a neighborhood of $t = 0$ and we may then repeat the argument for all $t > 0$.

Remarks: (i) the condition on σ may be rephrased as a condition involving M by using the inequality $||\tilde{c}|| \leq ||T||M$, (ii) the complete controllability condition on the pair (F,G) means that the nxnm matrix

$$C = \left[G|FG|F^2G| \cdots |F^{n-1}G\right] ,$$

has rank n. This is a generic property of constant systems, i.e. the pairs (F,G) which fail to satisfy it form a null set in the space of all pairs (F,G), (iii) to insure that the real parts of all characteristic roots of F - GK lie in the half-plane Re $\lambda < \alpha$, for some fixed α, it may or may not be necessary to utilize all nm degrees of freedom available in K. If not, then other optimality factors may be considered, e.g. rapid approach to equilibrium, integral criteria, and so forth. (iv) the norm used in the above proof is the ℓ_∞ norm, i.e. $||x|| = \max_i \{|x_i|\}$.

3. An Example from Ecology

Consider the simple predator-prey problem

$$\dot{x} = x(1 - 1/2x - y) + u_1(t) ,$$

$$\dot{y} = y(-1 + x) + u_2(t) ,$$

where x and y are the levels of prey and predator, respectively, with u_1 and u_2 the two control inputs. In the absence of control, it's easy to see that this system has the two equilibrium levels (0,0) and (1,1/2), with the second being the critical point of interest. It is a stable focus.

As usual, we first shift the critical point (1,1/2) to the origin, obtaining the new system

$$\dot{x} = (x + 1)\left[-1/2x - y\right] + u_1 \quad , \qquad x(0) = \alpha_1 \quad ,$$

$$\dot{y} = (y + 1/2)x + u_2 \quad , \qquad y(0) = \alpha_2 \quad ,$$

The objective is now to choose a control vector vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = K\begin{pmatrix} x \\ y \end{pmatrix}$, such that the set of points $\left\{\max \{|\alpha_1|, |\alpha_2|\} < 1\right\}$ lie within the domain of attraction of the origin. (Note that $M = 1$ was chosen so that the domain of attraction of the origin would not include the trivial critical point corresponding to extinction of both species.

In the notation of Theorem 1,

$$F = \begin{bmatrix} -1/2 & -1 \\ 1/2 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad c = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$h = \begin{bmatrix} x(-1/2x - y) \\ x(y + 1/2) \end{bmatrix} \quad .$$

Choosing K of the form

$$K = \begin{pmatrix} k_{11} & -1 \\ 1/2 & k_{22} \end{pmatrix}$$

gives

$$F - GK = \begin{bmatrix} -1/2 - k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$$

and insures that T = I. To determine the values k_{11}, k_{22} so that the controlled system has the set $\max\{|\alpha_1|, |\alpha_2|\} < 1$ within the domain of attraction of the origin, let

$$\Omega = \{(\alpha_1,\alpha_2) : |\alpha_1| = 1 \text{ or } |\alpha_2| = 1\}$$

Then

$$||c|| = \max_{\Omega} \{|\alpha_1|, |\alpha_2|\} = 1$$

$$||h(c)|| = \max_{\Omega} \{|\alpha_1(\alpha_2 - 1/2\alpha_1)| , |\alpha_1(\alpha_2 + 1/2)|\}$$

$$= 3/2$$

$$\min_{\Omega}(\alpha_1{}^2 + \alpha_2{}^2) = 1$$

Thus, if $\sigma < -3/2$ the conditions of the theorem will be fulfilled. This implies that any choice of k_{11}, k_{22} such that

$$k_{11} > 1 \quad , \quad k_{22} > 3/2$$

will insure that the interior of Ω lies within the domain of attraction of the origin.

4. Discussion

We have demonstrated that by suitable application of linear control theory, it is possible to modify the domain of attraction of a critical point for certain nonlinear systems. This result raises several questions for future investigation:

a) how may the results be extended to a broader class of problems, in particular, to systems whose dynamics may be more complex than just ordinary differential equations, e.g. differential-delay equations, stochastic differential-integral equations, or even functional equations of a more exotic type incorporating all of these features;

b) since there may be degrees of freedom remaining in the control law after assuring the domain of attraction, what is the best way to utilize these "extra" variables. For example, they may be used in an attempt to find a stabilizing law which requires measurement of the fewest number of state variables. Some ideas along these lines have appeared in [3-4];

c) how may the foregoing ideas be made to intersect with other concepts of system stability, in particular, questions involving the stability at a set rather than a single critical point.

These and other questions will be investigated in future studies.

References

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