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REACHABLE SETS APPROACH TO MULTI-OBJECTIVE PROBLEMS AND ITS POSSIBLE APPLICATIONS TO WATER RESOURCES MANAGEMENT IN THE SKANE REGION

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ABSTRACT

Decision making in a lot of resources supply and resources allocation problems is related to sophisticated multiobjective analysis. The concept of a man-computer simulation system was suggested as a tool for decision making in problems of this kind, especially in the case of water resources (Moiseev et al. 1980). Within the framework of such a system the analyst and the expert employ a full range of operational research methods (simulation, optimization, multiobjective, informal and game-theoretical ones) to address multiobjective problems by means of the hierarchical system of mathematical models of the system under study. Various forms of mathematical models can be studied by means of simulation experiments. To establish control variables (to formulate scenarios) in a simulation study the expert may use optimization techniques applied to models simpler than the simulation ones. It is reasonable to study the problem of criteria formulation in optimization problems (the objectives convolution problem) by means of multiobjective techniques and simple (screening) models. The multiobjective study is the most important part of investigation based on the simulation system, because it is the multiobjective investigation that gives a general understanding of the system under study.

This paper treats a new approach to multiobjective problems investigation. This approach is called the Generalized Reachable Sets (GRS) approach and belongs to generating multiobjective methods (Cohon 1978). It employs an explicit representation of a set of all reachable objective values. In contrast to different generating multiobjective methods, the mathematical background of the GRS approach is the linear inequalities techniques.

This approach is used now at the Computing Center of the USSR Academy of Sciences in various tasks.

The structure of the paper is as follows: first the mathematical background of the approach is outlined, and then possible applications of the approach to the Skane water resources management are discussed.

REACHABLE SETS APPROACH TO MULTIOBJECTIVE PROBLEMS AND ITS POSSIBLE APPLICATIONS TO WATER RESOURCES MANAGEMENT IN THE SKÅNE REGION

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INTRODUCTION

There exist two main approaches to solving multiobjective problems (Bell et al. 1977, Cohon 1978, Hwang et al. 1980): preference-oriented methods and generating methods. Preference oriented methods are based on constructing a formal process which leads the decision maker to the solution of the problem. The basic idea of this approach is the quantification of the preference of the decision maker (on a priori grounds or in man-computer interaction). The manner of the quantification distinguishes one preference-oriented technique from another (Wierzbicki 1979b).

The generating methods (Cohon 1978) are based on presentation of the set of all noninferior (nondominated, effective) points (Pareto set) in objective space to the decision maker. In this case the decision maker is being informed on the possibilities of the system under study. The nonformal process of striking a compromise among the competing objectives is left to the decision maker. The methods of this group have explicit advantages if the decision maker has no consistent preference or if his concept is a convenient abstraction only since the decision is a product of compromise between a group of decision makers, each of them having his own goals.

The mathematical presentation of the system under study provided with the mathematical formalization of objectives contains the implicit description of the noninferior set. The generating techniques are distinguished by the manner of explicit representation of the noninferior set. Four groups of generating techniques are described in Cohon 1978: weighting methods, constraint methods, multiobjective simplex methods (Zeleny 1974) and noninferior set estimation methods. The alternative approach discussed herein consists in constructing (or approximating) a set of all reachable (attainable) values of objectives by means of a finite number of hyperplanes. The set of all reachable values of objectives is a particular case of so-called Generalized Reachable Set (GRS) which is a generalization of the concept of reachable set in control theory (Lee et al. 1967). This is why our approach to multiobjective problems is called GRS approach.

The development of GRS methods began at the Computing Center of the USSR Academy of Sciences in the later sixties, first results were obtained in the early seventies (Lotov 1972, 1973a), other results being presented in (Lotov 1973b, 75a, 75b, 78, 79, 80, 81, Lotov et al. 1980, Bushenkov et al. 1980, Ognivtsev 1977).

THE MATHEMATICAL STATEMENT OF THE GRS APPROACH

We shall investigate mathematical models presented in the form

$$y \in G_y \subset Y \quad , \tag{1.1}$$

where y is the vector of the variables of the model, Y is a space of vectors y, G_y is a set of feasible vectors y. We do not specify the nature of the space Y at this moment. In some cases it will be the finite dimensional Euclidean space, in others it will be a functional space. We shall assume the set G_y to be not empty. Usually the vector y satisfying (1.1) is not unique.

Let the mapping $F: Y \to E^m$ be given, where E^m is m-dimensional Euclidean space. If we treat the vector

$$f = Fx \tag{1.2}$$

as an objective vector (or vector of performance criteria), the mapping defines the consequences of each decision or alternative y.

Definition. The Generalized Reachable Set (GRS) for the model (1.1) with the mapping (1.2) is the set G_f defined as follows:

$$G_f = \{f \in E^m : f = Fy, y \in G_y\} \quad . \tag{1.3}$$

If vector f is the objective vector the GRS coincides with the set of all reachable objective values. The GRS approach to multiobjective problems consists in constructing G_f in an explicit form

$$G_f = \{f \in E^m : Df \leqq d\} \quad . \tag{1.4}$$

If the set G_y is convex and the mapping F is linear, the set G_y is convex as well, and may be, at least approximately, represented in the form (1.4). This case will be analyzed in this paper.

The set G_y presented to the decision maker gives him the information on the set of noninferior values of objectives since the noninferior set P is a part of the boundary of G_f (see Figure 1). The basic mode of display mechanism in generating multiobjective methods consists in providing the decision maker with various two-dimensional projections and cross-sections (slices) of the noninferior set. The idea to provide the decision maker with projections and slices was introduced in (Meisel 1973, Lotov 1973a). If the GRS is constructed in the form (1.4) it takes only a few seconds to provide the decision maker with projections and slices on display of the computer upon request. So it is possible to present about hundred two-dimensional pictures to the decision maker in man-computer dialogue investigation of the GRS. By our experience this number of projections and slices is sufficient for a proper understanding of the structure of a convex set in objective space with five to ten dimensions.

A system of applied programs POTENTIAL was developed (Bushenkov et al. 1980) in order to construct the GRS in the form (1.4) and to present it to the decision maker. The algorithms of the system are based on linear inequalities theory. The general idea of the method is the following one. The graph of the mapping F denoted by Z is defined as

$$Z = \{z = \{y,f\} : f = Fy \ , \ y \in G_y\} \quad . \tag{1.5}$$

The set G_f is an orthogonal projection of the graph Z into the objective space E^m. The POTENTIAL system is based on orthogonal projection of polyhedral sets in finite dimensional spaces (convolution methods). Let the polyhedral set M which belongs to $(k+\ell)$-dimensional Euclidean space $E^{k+\ell}$, be described in the form of the solution of a finite system of linear inequalities

$$Av + Bw \leqq c \quad , \quad v \in E^k \quad , \quad w \in E^\ell \quad . \tag{1.6}$$

The matrices A and B as well as the vector c are given. We want to construct the set M_w of all points $w \in E^\ell$, for which there exists such a point $v \in E^k$ that $\{v,w\} \in E^{k+\ell}$ belongs to the set M. The set M_w being the orthogonal projection of the set M into the space E^ℓ is to be constructed in the form

$$M_w = \{w \in E^\ell : Dw \leqq d\} \quad . \tag{1.7}$$

For this the convolution techniques can be used. They consist in excluding variables of the systems of (1.6) type. The first convolution method was introduced by J.B. Fourier (1890). To provide a general understanding of convolution methods we shall discuss a simple example. Let the system (1.6) be the following:

(1) $v+w \leq 3$

(2) $-v+w \leq 5$

(3) $-w \leq -0.5$

(4) $v-2w \leq 0$

(5) $-2v-w \leq -1$

(6) $v \leq 2$,

where v and w are scalars. The set M is presented in Figure 2. To construct M_w it is necessary to divide each inequality on the absolute value of the coefficient by v (if this coefficient is not zero) and to sum all pairs of inequalities whose elements in the first column have opposite signs. For the system under study we obtain (in brackets the numbers of equations being combined are given)

(1) - (2)	$2w \leq 8$
(1) - (5)	$0.5w \leq 2.5$
(2) - (4)	$-w \leq 5$
(2) - (6)	$w \leq 7$
(3)	$-w \leq 0.5$
(4) - (5)	$-2.5w \leq 0.5$
(5) - (6)	$-0.5w \leq 1.5$.

So the set M_w is described by the inequality

$$0.5 \leq w \leq 4 \quad .$$

This idea can be applied to any system (1.6). To transform the system (1.6) into the system (1.7) it is necessary to fulfill k steps described here.

The main disadvantage of the Fourier method lies in the exponential growth of the number of inequalities. But most of the inequalities obtained are superfluous in the description of the set M_w. In our example we have got seven inequalities but only two of them are necessary to describe M_w. In the 20th century the Fourier method was modified (Motzkin et al. 1953, Chernikov 1965) and some new methods have been developed in order to remove part of the superfluous inequalities. The method (Chernikov 1965) removes all superfluous inequalities while the elements of matrix B and vector c are parameters. Additional methods used in the POTENTIAL system remove all superfluous inequalities and construct an approximation of the set M_w if necessary (Bushenkov et al. 1980).

In many cases the decision maker may be satisfied with any other set $\tilde{G}_f$ instead of the set G_f, having the same set of noninferior points that is $P(G_f) = P(\tilde{G}_f)$, where $P(G)$ is the noninferior (Pareto) boundary of the set G. For the set G_f the noninferior boundary is described as follows:

$$P(G_f) = \{f \in G_f : \{f^* \geqq f \ , \ f^* \in G_f\} \curvearrowright f^* = f\} \quad . \tag{1.8}$$

Let us define the set G_f^p

$$G_f^p = \{f \in E^m : f = f_1 + f_2 \ , \ f_1 \in G_f \ , \ f_2 \leqq 0\} \quad . \tag{1.9}$$

It is easy to show that $G_f \subset G_f^p$ and $P(G_f^p) = P(G_f)$. Let the set G_f^p denote the Generalized Pareto-Reachable Set (GPRS). Sometimes the set G_f^p is described by a smaller number of inequalities than the set G_f but contains sufficient information.

The methods for the construction of GRS and GPRS are described in the next section of this paper. Herein we shall discuss some features of the GRS approach to multiobjective problems. First of all, the GRS techniques construct the whole set of reachable objective values while the noninferior (Pareto) set is part of it. The feature is related to three advantages of the GRS approach. It is much easier to imagine a convex set (GRS) than a nonconvex Pareto set given by the points in the multidimensional space. It is easier to produce two dimensional slices for the set (1.4) than for the Pareto set given by the points. In many cases the decision maker may be interested not only in the Pareto set but also in inferior points (for example, in gaming and real situations of game type).

The second main feature of the GRS approach consists of using linear inequalities techniques instead of optimization techniques used in multiobjective methods usually. We believe the nature of optimization techniques is more related to preferance-oriented multiobjective methods. Linear inequalities techniques proved to be more effective than optimization methods in various problems containing about thirty variables, fifty linear restrictions and five to ten objectives. In the case of two objectives optimization methods (weighting methods, multiobjective simplex method and noninferior set estimation methods) are usually more effective, but when the number of objectives is getting bigger the computational work in the GRS approach is not growing exponentially as in the optimization oriented generating multiobjective methods. It seems to be very effective to combine possibilities of noninferior set estimation methods, which are now at the early stage of development (Cohon 1978), with GRS techniques to construct GRS for problems containing about hundred variables and about ten objectives.

To investigate the problems with hundreds of variables and ten or more objectives it is necessary to combine generating methods with preference-oriented methods. The combination of the GRS and reference objective methods (Wierzbicki 1979a) seems to be very effective. In the multiobjective problem with a complex model for which the construction of the set of all reachable objective values might be too cumbersome, the GRS techniques may be applied to a simplified version of the model. Provided with projection and slices of GRS the decision maker can choose the best compromising solution for a simplified version of the model. This solution could happen to be nonreachable for an initial model but it might serve as reference objectives (aspiration levels) in multiobjective studies on the basis of optimization techniques (Wierzbicki 1979a).

THE CONSTRUCTION OF THE GRS FOR FINITE-DIMENSIONAL MODELS

First, we shall discuss the problem of constructing the GRS for linear static models. Let the space Y be the n-dimensional Euclidean space E^n, and let G_y be

$$G_y = \{y \in E^n : Ay \leqq b\} \quad , \tag{2.1}$$

where A and b are the given matrix and vector. The mapping (1.2) in described in this case by the matrix F, having m rows and n columns. The graph of the mapping is a polyhedral set, described by the following system of equations and inequalities

$$Z = \{\{y,f\} : f = Fy , Ay \leqq b\} \quad . \tag{2.2}$$

Since the GRS is an orthogonal projection of the polyhedral set Z, we have the possibility to construct it in this case using the POTENTIAL system.

Now let us discuss the GPRS construction. Once the set G_f is given in an explicit form (1.4) it is sufficient to find an orthogonal projection of the set

$$W = \{\{f,f^1\} : Df^1 \leqq d , f-f^1 \leqq 0\}$$

into the space

$$f \in E^m \quad .$$

It is reasonable to construct the GPRS without intermediate construction of the GRS. This is possible in the case of the block structured model (2.1)

$$y^j \in G_{y^j} \quad , \qquad j = 1,\dots,J \quad ,$$

$$F^j = F^j y^j \quad , \qquad j = 1,\dots,J \quad ,$$

$$\sum_{j=1}^{J} A_j f^j \geqq b \quad ,$$

where $y^j \in E^{n_j}$, $f^j \in E^{m_j}$, $j = 1,\dots,J$. We shall denote the vectors y^j as block variables and the vectors f^j as intermediate objectives. Let the objective vector f be dependent only on the intermediate objectives

$$f = \sum_{j=1}^{J} F_j f^j \quad .$$

If we denote

$$G_{f^j} = \{f^j \in E^{m_j} : f^j = F^j y^j , y^j \in G_{y^j}\} \quad ,$$

then the set G_f may be represented as

$$G_f = \{f \in E^m : f = \sum_{j=1}^{J} F_j f^j \ , \ \sum_{j=1}^{J} A_j f^j \geqq b \ , \ f^j \in G_{f^j} \ , \ j=1,\dots,J\} \ .$$

The following theorem makes it possible to construct the set G_f^p without constructing the set G_f.

Theorem. Let for any point $f^j \in G_{f^j}$ exist a point $f_*^j \in P(G_{f^j})$ satisfying $f_*^j \geqq f^j$. If $F_j \geqq 0$, $A_j \geqq 0$, $j = 1,\dots,J$, then for any set

$$\tilde{G}_f = \{f : f = \sum_{j=1}^{J} F_j f^j \ , \ \sum_{j=1}^{J} A_j f^j \geqq b \ , \ f^j \in \tilde{G}_{f^j} \ , \ j=1,\dots,J\}$$

satisfying

$$P(\tilde{G}_{f^j}) = P(G_{f^j}) \quad , \qquad j = 1,\dots,J \quad ,$$

we have

$$P(\tilde{G}_f) = P(G_f) \quad .$$

First we shall prove a lemma.

Lemma. Let any sets $G_j \subset E^{m_j}$, $j = 1,\dots,J$, have the following property: for any $f^j \in G_j$ there exists a point $f_*^j \in P(G_j)$ satisfying $f_*^j \geqq f^j$. If $F_j \geqq 0$, $A_j \geqq 0$, $j = 1,\dots,J$, then for any point $f_* \in P(G)$, where

$$G = \{f : f = \sum_{j=1}^{J} F_j f^j \ , \ \sum_{j=1}^{J} A_j f^j \geqq b, \ f^j \in G_j \ , \ j=1,\dots,J\} \quad ,$$

there exist $f_*^j \in P(G_j)$, $j = 1,\dots,J$ satisfying

$$\sum_{j=1}^{J} A_j f_*^j \geqq b \quad \text{and} \quad f_* = \sum_{j=1}^{J} F_j f_*^j \quad .$$

Proof. For any $f_* \in P(G) \subset G$ there exist $f_{**}^j \in G_j$, $j = 1,\dots,J$, satisfying the restriction $\sum_{j=1}^{J} A_j f_{**}^j \geqq b$ and the equality

$$F_* = \sum_{j=1}^{J} F_j f_{**}^j \quad .$$

For any $f^j_{**} \in G_j$ there exists $f^j_* \in P(G_j)$ satisfying $f^j_* \geqq f^j_{**}$, $j = 1,\dots,J$. Let $f_{**} = \sum_{j=1}^{J} F_j f^j_*$. Since $A_j \geqq 0$, $j = 1,\dots,J$, then $\sum_{j=1}^{J} A_j f^j_* \geqq \sum_{j=1}^{J} A_j f^j_{**} \geqq b$. Since $F_j \geqq 0$, $j = 1,\dots,J$, then $f_{**} = \sum_{j=1}^{J} F_j f^j_* \geqq \sum_{j=1}^{J} F_j f^j_{**} = f_*$. Since $f_* \in G$, $f_{**} \geqq f_*$ and $f_* \in P(G)$ then $f_{**} = f_*$ and $f_* = \sum_{j=1}^{J} F_j f^j_*$. Thus the lemma has been proved.

Proof of the theorem. It follows from lemma that for any $f_* \in P(G_f)$ there exist f^j_* $P(G_{f^j})$, $j = 1,\dots,J$, satisfying $\sum_{j=1}^{J} A_j f^j_* \geqq b$ and $f_* = \sum_{j=1}^{J} F_j f^j_*$. But $P(\tilde{G}_{f^j}) \subset G_{f^j}$. Therefore $f_* \in \tilde{G}_f$. For $f_* \in \tilde{G}_f$ we can find $f_{**} \in P(\tilde{G}_f)$ that satisfies $f_{**} \geqq f_*$. Since $f_{**} \in P(\tilde{G}_f)$ then by the lemma there exist $f^j_{**} \in P(\tilde{G}_{f^j})$, $j = 1,\dots,J$, satisfying $\sum_{j=1}^{J} A_j f^j_{**} \geqq b$ and $f_{**} = \sum_{j=1}^{J} F_j f^j_{**}$. But $P(G_{f^j}) \subset \tilde{G}_{f^j}$. Therefore $f_{**} \in G_f$. So we have $f_{**} \geqq f_*$, $f_{**} \in G_f$, $f_* \in P(G_f)$. Therefore $f_{**} = f_*$ and $f_* \in P(\tilde{G}_f)$. We repeat the similar arguments to prove that for any $f_* \in P(\tilde{G}_f)$ we have $f_* \in P(G_f)$. Thus the theorem has been proved.

On the basis of this theorem we may construct G^p_f using $G^p_{f^j}$. Usually the sets $G^p_{f^j}$ are described much simpler than the sets G_{f^j}. Thus, constructing the set G^p_f is easier than the set G_f.

For example, this theorem may be used to study various systems consisting of blocks and of upper level equations (as it happens in economic applications)

$$f = \sum_{j=1}^{J} z^j \quad , \quad z^j \geqq z^j_0 \quad , \quad \sum_{j=1}^{J} B_j y^j \leqq b \quad ,$$

where z^j is the output vector for the jth block, y^j is the resource vector for the jth block.

The upper level equations describe the objective vector f, restrictions imposed on the outputs of each block, and the common restrictions on the resources. Assuming that no resource is produced in the system, we get $B_j \geqq 0$, $j = 1,\ldots,J$.

Let us introduce the intermediate object vectors

$$f^j = \{-y^j, z^j\} \quad \text{and denote} \quad F_j = [0 \,\vdots\, I] \quad .$$

The model has got the appropriate structure to use the theorem.

Next we shall discuss methods for the construction of GRS in the case of dynamical models. For the dynamical multi-step model the general description (2.1) has a special structure. First, the operation time period of the system is split into a finite number of steps by time moments $t = 0,\ldots,T$. The values of the variables relate to certain moments of time. Second, all variables are split into two classes: controls and states. We shall denote the control vector at the time moment t as $u_t \in E^{n_u}$, $t = 0,\ldots,T-1$. When the control vectors u_t, $t = 0,\ldots,T-1$ are given it is possible to calculate the state vectors $x_t \in E^{n_x}$, $t = 0,\ldots,T$, beginning from the given initial state x_0 on the basis of the equation

$$x_{t+1} = A_t x_t + B_t u_t + a_t \quad , \qquad t = 0,\ldots,T-1 \quad , \qquad (2.3)$$

where A_t and B_t are given matrices, a_t are given vectors, $t = 0,\ldots,T-1$. The description of the system also includes restrictions on the state variables and controls

$$D_t^{(1)} x_t + D_t^{(2)} u_t \leqq d_t \quad , \qquad t = 0,\ldots,T-1 \qquad (2.4)$$

where $D_t^{(1)}$ and $D_t^{(2)}$ are given matrices, d_t are given vectors. The initial state vector x_0 belongs to the polyhedral set Γ_0

$$x_0 \in \Gamma_0 \quad . \qquad (2.5)$$

In some cases the set Γ_0 may have only one initial point. Generally speaking, one may represent the system (2.3) - (2.5) in the form (2.1), and reduce the problem of the GRS construction to the case discussed above. In practice the system obtained is too unwieldy, and it is preferable to construct the GRS using the usual reachable sets of the system (2.3) - (2.5).

First, let us discuss the case of the objective vector having the structure

$$f \equiv x_T \quad . \tag{2.6}$$

In this case the set G_f coincides with the usual reachable set Γ_t, which is defined as the set of all reachable state vectors of the system at the time moment T. So the problem of the GRS construction in this case may be solved by constructing a reachable set.

To construct the set Γ_T we shall use the method, which consists of successive construction of the sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_t, \ldots, \Gamma_T$, beginning with the set Γ_0. We shall show how we can obtain the set Γ_{t+1} on the basis of the set Γ_t represented as

$$\Gamma_t = \{x : C_t x \leqq c_t\} \quad , \tag{2.7}$$

where C_t and c_t are matrix and vector calculated on the previous step of the method. The equations (2.3), (2.4) in the time moment t and (2.7) describe a polyhedral set Y in the space $E^{2n_x+n_u}$. The set Γ_{t+1} is an orthogonal projection of the set Y in the space of state variables x_{t+1}. Thus, the problem is reduced to the construction of the orthogonal projection of the finite dimensional polyhedral set and can be solved by POTENTIAL. This approach to the reachable sets construction was proposed in (Lotov 1972, 1975b).

Now, let the objective vector f be

$$f = Fx_\Gamma \quad , \tag{2.8}$$

where F is a given matrix. In this case we first have to construct the set Γ_T and then to construct the GRS, using the GRS construction methods for static models. Of course, it is possible to reduce the case of integral objectives as well as the case of dependence of the objective vector upon state vectors over two or more moments of time to the problems discussed above.

THE GRS CONSTRUCTION FOR DIFFERENTIAL SYSTEMS WITH CONVEX STATE CONSTRAINTS

In this section we shall discuss the problems of the GRS construction for the following model

$$\dot{x} = A(t)x + B(t)u + a(t) \quad , \quad t \in [0,T] \quad , \tag{3.1}$$

$$\{x(t),u(t)\} \in Y(t) \subset E^{n_x+n_u} \quad , \quad t \in [0,T] \tag{3.2}$$

$$x(0) \in \Gamma(0) \subset E^{n_x} \tag{3.3}$$

where $x(t) \in E^{n_x}$, $u(t) \in E^{n_u}$, the matrices A(t) and B(t), as well as the vector a(t) are given, the sets Y(t) and $\Gamma(0)$ are given, and they are convex.

First we shall explore the construction of the usual reachable set $\Gamma(T)$ defined as a set of all points $x \in E^{n_x}$, which may be attained by the system (3.1) - (3.3) by the time moment T, control function u(t) being a limited measurable function on [0,T]. This problem was previously discussed in (Lotov 1975b, 1978, 1979).

The system (3.1) - (3.3) will be approximated by its multistep analogues. We shall split the time period [0,T] into N equal parts by time moments $t_i = i\tau$, i = 0,...,N. The differential equation (3.1) will be approximated by one of the multistep equations from the following class

$$\frac{1}{\tau}(x_{i+1} - x_i) = \alpha A_i^{(1)} x_{i+1} + (1-\alpha) A_i^{(2)} x_i + B_i u_i + a_i \ , \ i=0,\dots,N-1 \ , \tag{3.4}$$

where $x_i \in E^{n_x}$ is the state vector, $u_i \in E^{n_u}$ is the control vector over the time moment t_i, $A_i^{(1)} = A(t_i^{(1)})$, $A_i^{(2)} = A(t_i^{(2)})$,

$B_i = B(t_i^{(3)})$, $a_i = a(t_i^{(4)})$, $\alpha \in [0,1]$, $t_i^{(k)} \in [t_i, t_{i+1}]$, $k = 1,2,3,4$.

The following restrictions will be imposed on the vectors x_i, u_i, $i = 1,\ldots,N-1$:

$$\{\beta x_{i+1} + (1-\beta)x_i, u_i\} \in Y_i \subset E^{n_x+n_u} \quad , \tag{3.5}$$

where Y^i is a polyhedral set approximating $Y(t_i^{(5)}$, $t_i^{(5)} \in [t_i, t_{i+1}]$, $\beta \in [0,1]$. The vector x_0 belongs to the polyhedral set Γ_0 approximating $\Gamma(0)$:

$$x_0 \in \Gamma_0 \quad . \tag{3.6}$$

Once the parameters α, β, $t_i^{(k)}$, $k = 1,\ldots,5$, $i = 0,\ldots,T-1$, as well as the method for the construction Γ_0 and Y_i, $i = 0,\ldots, N-1$, are fixed, we obtain the multistep system approximating (3.1) - (3.5).

The reachable set Γ_N for the system (3.4) - (3.6) may be constructed by means of the method, discussed above. The problem is to evaluate the discrepancy between the sets Γ_N and $\Gamma(T)$. Let $\rho(Y_i, Y(t_i^{(5)}) \leq \Delta$, where $i = 0,\ldots,N-1$, and $\rho(\Gamma_0, \Gamma(0)) \leq \delta$. Let us construct the sequences of positive numbers $\{\tau_j\}$, $\{\Delta_j\}$, $\{\delta_j\}$, for which τ_j, Δ_j, $\delta_j \to 0$, and $N_j = T/\tau_j$ are integer numbers. Let us denote Γ as a set of all points $x \in E^{n_x}$, for which one can find a sequence $x_j \in \Gamma_{N_j}$, converging to x. It is possible to show that the set Γ is unique.

Theorem. Let the following conditions be satisfied:

1/ the elements of matrices A(t) and B(t) as well as the vector a(t) are continuous on [0,T];

2/ the set $Q(x,t) \in E^{n_u}$ defined as

$$Q(x,t) = \{u \in E^{n_u} : \{x,u\} \in Y(t)\} \quad ,$$

is restricted for any $x \in E^{n_x}$ and $t \in [0,T]$;

3/ there exists a constant $K > 0$, for which it holds

$$\rho(Y(\tilde{t}), \quad Y(\tilde{\tilde{t}})) \leq K|\tilde{t} - \tilde{\tilde{t}}| \quad ,$$

where $\tilde{t}$ and $\tilde{\tilde{t}}$ belong to $[0,T]$;

4/ for any $t \in [0,T]$, and for any $\Delta > 0$ there exists a polyhedral set $Y^{\Delta}(t)$ satisfying the condition

$$\rho(Y(t), \quad Y^{\Delta}(t)) \leq \Delta \quad ;$$

5/ for any $\delta > 0$ there exists a polyhedral set Γ^{δ} satisfying the condition

$$\rho(\Gamma(0), \Gamma^{\delta}) \leq \delta \quad ;$$

6/ let $\Gamma(\varepsilon,T)$ denote the reachable set for the system (3.1), (3.3) and $\{x(t),u(t)\} \in Y_{\varepsilon}(t)$, where $Y_{\varepsilon}(t)$ is the set of all points $T(t)$, the distance between each of them and the boundary of the set $Y(t)$ being more than $\varepsilon \geq 0$; there exists $\varepsilon_0 > 0$, for which $\Gamma(\varepsilon_0,T) \neq \phi$.

If the conditions 1 - 6 are satisfied, then $\Gamma(T) = \Gamma$. It should be noted that in this theorem the set $\Gamma(0)$ may not be restricted. If the set $\Gamma(0)$ is restricted, a more precise theorem may be proved.

Theorem. If the conditions of the preceding theorem are satisfied, and the set $\Gamma(0)$ is restricted, then

$$\lim_{\tau,\delta,\Delta \to 0} \rho(\Gamma_N(\tau,\delta,\Delta) \quad , \quad \Gamma(T)) = 0 \quad .$$

The proof of both theorems is too lengthy and will not be presented in this paper. See proofs of the theorems in (Lotov 1979).

MULTIOBJECTIVE ANALYSIS OF WATER RESOURCES ALLOCATION IN THE REGION OF SKANE, SWEDEN

The presence of several objectives is one of the basic aspects of water resources management in the Skane region in Sweden (Andersson et al. 1979). Decision making on water supply and allocation in Malmöhus and Kristianstad counties of Skane is related to the treatment of different goals like water supply to urban areas, industrial water supply, recreational development, and so on. This is the reason why multiobjective analysis is indispensable in any practical investigation of the water management problems in Skane.

In this section we discuss how the GRS method can be applied to a particular problem of the Skane region: the water resources allocation in the Kävlinge River System during the summer period with low precipitation. The difficulties in the water allocation problem are combined with water pollution problems arising from fertilization practices, since chemicals are partly brought to the Kävlinge River by return water. Other environmental problems are related to the water allocation as well. This problem was studied previously in (Kindler et al. 1980), the model of the Kävlinge River System was formulated and investigated by the multiobjective method developed by A. Wierzbicki (1979a).

The scheme of the Kävlinge River System is presented in Figure 3. The Kävlinge River is flowing out of the Vomb Lake. The Vomb Lake has two minor inflows. The water release to the Kävlinge River from the Vomb Lake is regulated. The Vomb Lake serves as a source of municipal water supply for the Malmö region. For this study three agricultural regions are defined which use water from the Kävlinge River System for irrigation, fertilizers being partly brought by return flow to the Kävlinge River. At control point A near the Baltic Sea the flow and concentration of the pollutant in the Kävlinge River are monitored.

To facilitate the application of the GRS method, the original model (Kindler et al. 1980) was slightly modified. The agricultural production was described by means of N irrigation technologies. (This form of description is traditional in economics.

Let x_{ij} be the area of the j-th region, j = 1,2,3, with i-th type of irrigation (ha), i = 1,...,N. The areas in each region are constrained by the total agricultural area of the region

$$\sum_{i=1}^{N} x_{ij} = a_j \quad , \qquad j = 1,2,3 \quad . \tag{4.1}$$

Surely, the variable x_{ij} is nonnegative

$$x_{ij} \geq 0 \quad , \qquad i = 1,\ldots,N \;\; ; \;\; j = 1,2,3 \quad . \tag{4.2}$$

The agricultural production in the j-th region is described by means of the following indices:

- y_{j1} - yield effect of the irrigation and fertilization in the j-th region (kg);
- y_{j2} - irrigation water withdrawals to this region (m^3);
- y_{j3} - amount of fertilizer (kg);
- y_{j4} - return flow (m^3);
- y_{j5} - chemicals in return flow (kg);

These indices are calculated using specified coefficients a_{kij}, where k is the number of the index, i is the number of the technology and j is the region number. The indices are calculated in the following manner:

$$y_{jk} = \sum_{i=1}^{N} a_{kij} x_{ij} \quad , \qquad j = 1,2,3 \;\; ; \;\; k = 1,\ldots,5 \; . \tag{4.3}$$

The relationships (4.1) - (4.3) describe the agricultural production in the model. The coefficients a_{kij} were specified on the basis of information presented in (Kindler et al. 1980). The values of the coefficients are the following (N = 7):

j = 1

k \ i	1	2	3	4	5	6	7	Unit
1	0	4000	5500	8000	4500	6800	9500	Kg/ha
2	0	300	300	300	550	550	550	m^3/ha
3	0	0	80	150	0	80	150	Kg/ha
4	0	60	60	60	210	110	110	m^3/ha
5	0	0	12	22.5	0	12	22.5	Kg/ha

j = 2,3

k \ i	1	2	3	4	5	6	7	Unit
1	0	4000	5500	9200	4500	6800	10800	Kg/ha
2	0	300	300	300	550	550	550	m^3/ha
3	0	0	80	180	0	80	180	Kg/ha
4 (j=2)	0	30	30	30	55	55	55	m^3/ha
4 (j=3)	0	60	60	60	110	110	110	m^3/ha
5	0	0	12	27	0	12	27	Kg/ha

The total agricultural areas in the agricultural regions are the following: a_1 = 3000 ha, a_2 = 2500 ha, a_3 = 2300 ha.

Comments. In the first technology the irrigation and the fertilization are not used. The pollution coefficients a_{5ij} are based on the assumption that about 15% of the fertilizers are brought to the river with the return flow. The coefficients for the second and for the third regions are equal (excluding the return flow). The coefficients of irrigation are specified using the assumption that the value of precipitation in this month equals 10 mm.

Let us describe now the water and pollution balances. Let q_1, q_2, q_3 and q_4 be the inflows to the system (m^3/sec). The values of the inflows are 1.8, 1.5, 0.8 and 0.7 respectively. The actual water storage volume in the Vomb Lake S is the following:

$$S = S_0 + (q_1 + q_2)T - (Z_k + Z_M)T + y_{1y} - y_{12} \quad , \qquad (4.4)$$

with T being the length of time period (2.59×10^6sec), S_0 being the initial storage volume of the lake ($3 \times 10^7 m^3$), Z_k being the release from the lake to the Kälvinge River (m^3/sec), Z_M being the water intake for Malmö (m^3/sec). Here the assumption is made that the values of inflows, releases, withdrawals and intakes are constant during the month-period under study.

The flow in the Kälvinge River at control point A denoted by v_A(m^3/sec) is the following:

$$v_A = Z_k + q_3 + q_4 + \frac{1}{T}(y_{2y} + y_{3y}) - \frac{1}{T}(y_{22} + y_{32}) \ . \qquad (4.5)$$

The pollution flow at point A denoted by w_A(kg/sec) is the following

$$w_A = \frac{1}{T}(y_{25} + y_{35}) + \psi_3 \cdot q_3 + \psi_4 q_y + w_V \quad , \qquad (4.6)$$

with ψ_3 and ψ_4 being the initial concentration of pollution in the third and the fourth inflow respectively, w_V being the pollution flow from the Vomb Lake. The value of the w_V (kg/sec) is calculated in the following manner

$$w_V = (1 - \phi)(\psi_1 \cdot q_1 + \psi_2 q_2 + \frac{1}{T} y_{15}) \quad , \qquad (4.7)$$

with ψ_1 and ψ_2 being the initial concentrations in the first and the second inflows respectively, ϕ being the coefficient of the pollution reduction in the Vomb Lake. We have $\psi_1 = \psi_2 = 10^{-3}$ kg/m^3, $\psi_3 = 2 \times 10^{-3}$ kg/m^3, $\psi_y = 1.5 \times 10^{-3}$ kg/m^3, $\phi = 0.9$.

We have the following constraint on the water and pollution balances variables. First, there are nonnegative constraints

$$Z_k \geq 0 \quad , \qquad (4.8)$$

$$z_M \geq 0 \quad , \tag{4.9}$$

$$S \geq 0 \quad . \tag{4.10}$$

Second, we have physical constraints on the water withdrawal

$$y_{12} \leq T \cdot q_1 \quad , \tag{4.11}$$

$$y_{22} + y_{32} \leq T \cdot (z_k + q_3) \quad . \tag{4.12}$$

There are constraints related to the environmental requirements

$$v_A \geq v_A^* \quad , \tag{4.13}$$

$$\frac{w_A}{v_A} \leq w_A^* \quad . \tag{4.14}$$

The constraint (4.13) requires that the flow in the Kälvinge River at point A denoted by v_A be not less than $v_A^* = 6m^3/sec$. The constraint (4.14) shows that the pollution concentration at point A must not exceed $w_A^* = 10g/m^3$. Furthermore, there exists a constraint showing that the intake for Malmö must not exceed the sufficient level $z_M^* = 2m^3/sec$:

$$z_M \leq z_M^* \quad . \tag{4.15}$$

At last, the water storage volume in the Vomb Lake is not greater than the optimal one S^* which is optimal from the environmental and recreational points of view ($S^* = 29 \times 10^6 m^3$)

$$S \leq S^* \quad . \tag{4.16}$$

INDICES

The indices of the system performance are the same as in (Kindler et al. 1980).

1. Yield effects of irrigation and fertilization in the agricultural regions

$$J_1 = y_{11} \quad , \qquad J_2 = y_{21} \quad , \qquad J_3 = y_{31} \quad .$$

2. Water deficit in Malmö

$$J_4 = z_M^* - z_M \quad .$$

3. Excess over minimal flow at point A

$$J_5 = v_A - v_A^* \quad .$$

4. The deviation from the optimal level of the Vomb Lake:

$$J_6 = b(S^* - S) \quad ,$$

with b being the coefficient connecting the level and the storage volume of the lake ($b = 6 \times 10^{-7}\ 1/m^2$)

5. The flow of pollution at point

$$J_7 = w_A \quad .$$

The GRS for the objectives listed above was constructed and will be described in a special paper. It is possible to present the GRS to decision makers in Sweden for analysis which may include formal as well as informal methods of decision making. Since the POTENTIAL system is only programmed at the Computing Center of the USSR Academy of Sciences, the GRS may be presented to the decision makers in Sweden in form of its slices and projections obtained on a priority ground. The dialogue investigation of the GRS which is a most effective mode of application of the GRS methods can be provided, after programming the POTENTIAL system at IIASA, at the Lund University of Sweden or elsewhere in Sweden.

SOME OTHER APPLICATIONS

There exist three main different directions of the GRS application:

1) aggregation of a mathematical model;
2) coordination of a system of mathematical models;
3) evaluation of potential possibilities of a system under study.

The aggregation of models by means of the GRS techniques is based on the following idea. Let us treat the vector f as vector of variables of the aggregated model. The mapping (1.2) describes the correspondence between original and aggregated variables. This correspondence is established on *a priori* grounds by the decision maker. In this case the relationship

$$f \in G_f \tag{5.1}$$

describes an aggregated model while for any $y \in G_y$ there exists a corresponding vector $f \in G_f$. The main advantage of the aggregation based on the GRS techniques consists of the fact that all the values of variables f which are feasible for the aggregated model (5.1) can be pricisely disaggregated into feasible values of the original model (1.1).

The disadvantage of the method consists in the form of the aggregated model (5.1): it may be, and usually is, not convenient for the decision maker. To avoid this, it is possible to use another aggregated model

$$f \in G(b) \quad , \tag{5.2}$$

where the model (5.2) is chosen for convenience to the decision maker while

$$G(b) \subset G_f \quad . \tag{5.3}$$

The parameters b in (5.2) are chosen to obtain the best approximation of the set G_f by the set G(b). In this case for some feasible values of the original model (1.1) the corresponding values of the aggregated model (5.2) may not exist but the property of precise disaggregation holds.

The coordination of a system of models based on GRS consists of the linearization of original models (if nonlinear) and of the construction of an aggregated description of each model by means of the GRS techniques. The aggregated description of the system of models is used by the decision maker to choose a feasible coordinated decision for the whole system in terms of aggregated variables. Since the variables of the aggregated description can be precisely disaggregated into variables of the original model (in the linear case) the decisions in terms of the original variables will be coordinated as well without additional iterations. This approach could be effective in the informal coordination of models where a big number of iterative steps of decision making seems to be unrealistic. One of the modes of coordination of models based on GRS techniques is presented in (Alexandrov et al. 1981).

The evaluation of potential possibilities of a system is based upon the representation of vector f in (1.2) as a vector of performance indices of the model (1.1) describing the system under study. The set of all reachable values of performance indices shows potential possibilities of the system. As an example of this approach we can mention the study of global biospheric models described in (Alexandrov et al. 1981).

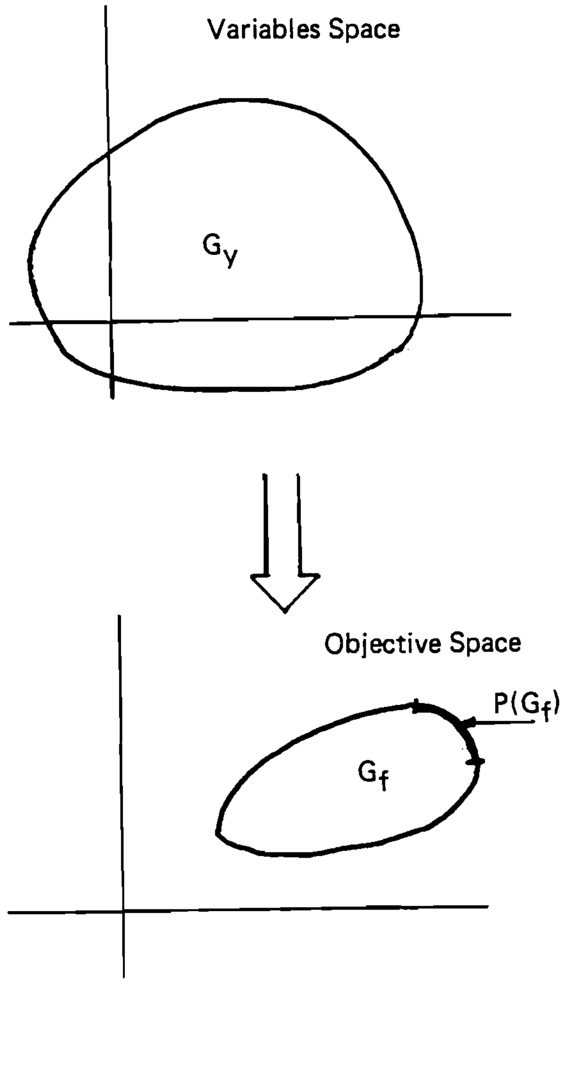


Figure 1

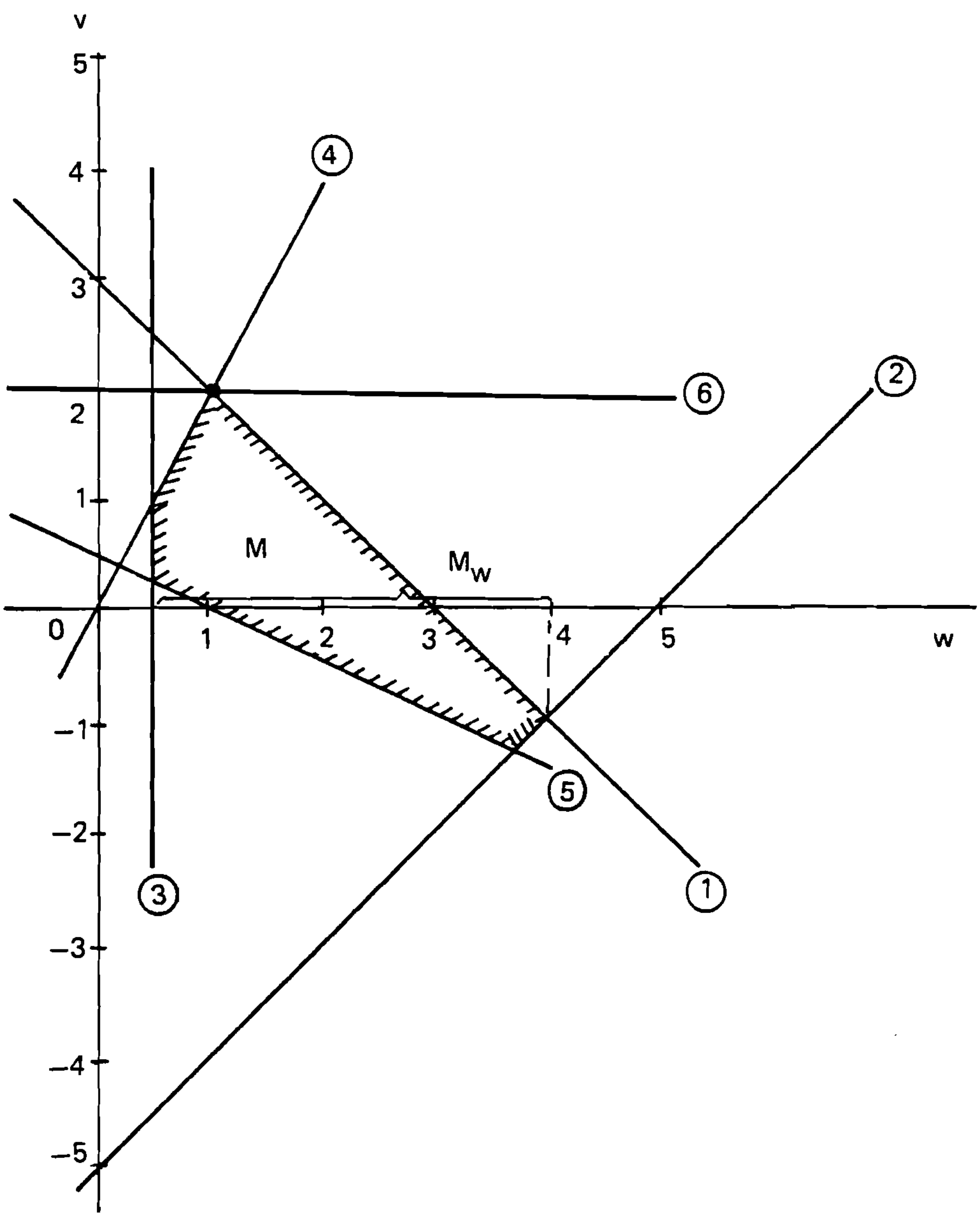
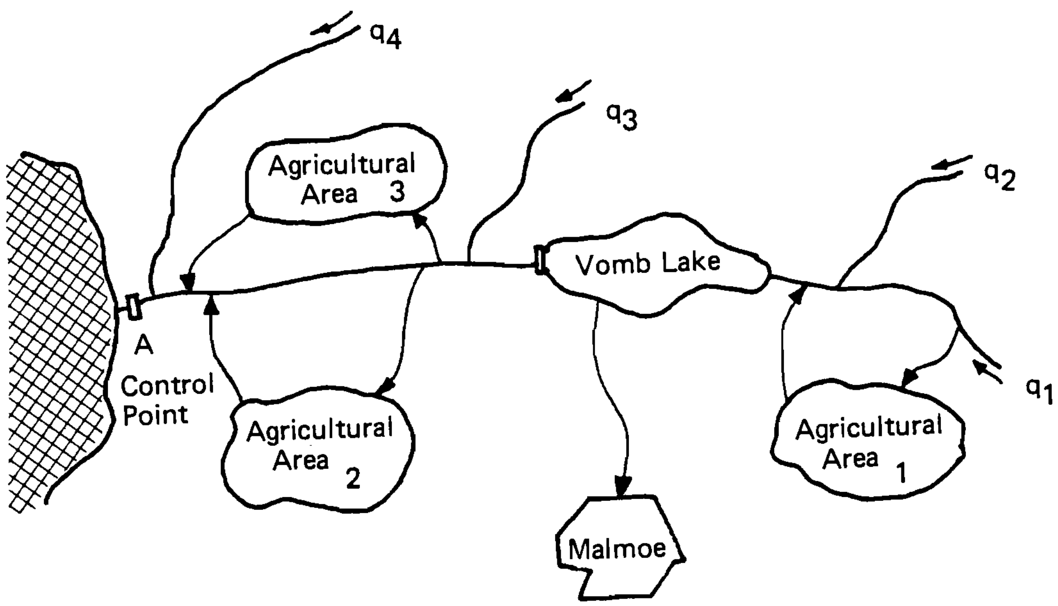


Figure 2



q_4
q_3
q_2
q_1
Agricultural Area 3
Vomb Lake
A
Control Point
Agricultural Area 2
Agricultural Area 1
Malmoe

Figure 3

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