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On Structural Stability of Dynamic Inequalities

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Abstract. We show that in a generic case the problem of structural stability of generic dynamic inequality with bounded admissible velocities on two dimensional sphere is equivalent to the problem of structural stability of such an inequality on the plane, when near the infinity the inequality either completely controllable or have no admissible velocities at all. In particular, that implies the structural stability of such simplest dynamic inequalities on two-dimensional sphere.

Introduction

The notion of rough dynamical system was introduced in paper [1] by A.A.Andronov and L.S.Pontryagin for the case of differentiable vector fields on a two-dimensional disk, which have no tangency with the boundary. Such a field is *rough* if the family of phase curves of sufficiently C^1-close field could be carried out to such family of the initial field by homeomorphism of the disk, which is close to the identity. The necessary and sufficient conditions for such a field to be rough were also found in this paper. Later on such conditions were found for differentiable vector fields on a sphere and on any compact orientable surface [16], [17]. In addition, this notion was defined for some objects of another nature, for example, for smooth maps [4], for net of asymptotic lines on surfaces [13], for smooth control systems [7], [8], for dynamic inequalities [12] and gets the name *structural stability*.

For control system (and dynamic inequality) the concept of structural stability is the same as for vector fields, just under the trajectory of a point one needs to consider the union of the positive and negative orbits of this point. In this case, the analogue of singular point of vector field is a *local transitivity zone* of the system. This zone is the union of all points, where the system has local transitivity property, which is defined below. The role of closed trajectory plays *nonlocal transitivity zone*, which coincides with intersection of the positive and negative orbits of each of its points. A generic smooth control system on a sphere (or a closed orientable surface) is structurally stable [7], [8]. Note that for systems this stability includes also the stability of both local controllability properties and the nonlocal ones [6]. Some property of objects we call *generic* if it takes place for any object from some open everywhere dense subset in the space of objects endowed by an appropriate topology. Here we deal with smooth of sufficiently smooth fine Whitney topology.

The problem to analyze the orbits of smooth dynamical inequalities with locally bounded derivatives was posed by A.D.Myshkis in [15]. Such an inequality is defined by a smooth real function on the tangent bundle to the manifold, in each fiber of which the set of admissible velocities (in which the value of this function is non-positive) is bounded. Such an inequality could have regions with no any admissible velocities at all and the ones, in which the motion do not satisfy the conditions of existence and uniqueness theorem of integral curves. For example, there could appear sliding motion that can not be eliminated by a small perturbation of the inequality under consideration [9]. This makes the problem of structural stability for dynamic inequalities on surfaces much more complicated than for control systems. Note, that for three dimensional control systems or dynamic inequalities the property of structural stability is not generic.

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The stability of the local controllability properties for a generic dynamic inequalities on surfaces was proved in [9]. Structural stability for generic simplest dynamic inequalities on the plane

$$(\dot{x}-a(x,y))^2+(\dot{y}-b(x,y))^2 \le f(x,y),$$

where a,b and f are smooth function on $\mathbb{R}^2_{x,y}$, was proved in [12] for the case when near the infinity we have either

$$a^2+b^2<f,$$

that is the inequalities are completely controllable, or $f<0$, and so the inequalities have no admissible velocities at all. Here the vector field (a,b) stays for the drift in the flat "sea", and the function f characterizes own capacities to move of the object under control (the admissible velocities of a simplest dynamic inequality on any Riemannian manifold are determined by the sum of the drift vector field and the velocities modulo not exceeding $\sqrt{f}$). This result also provides the structural stability of characteristics net of a generic second-order linear partial equation of mixed type on a plane with a finite region of hyperbolicity, if the net is orientable (see [10] [12]).

Reduction theorem and structural stability on sphere

Here we formulate the main results.

A control system or dynamic inequality has *local transitivity property* at a point if for any neighborhood of this point there exist time $T>0$ and another neighborhood of the point such that any two points from the second neighborhood are attainable from one another for a time less than T and along the admissible trajectory lying in the first neighborhood. If the second neighbourhood exists for any time $T>0$ then the inequality has *small time local transitivity property*. For example, simple motion $\dot{x}^2+\dot{y}^2\le 1$ (without drift) on the plane has such a property at any point of the plane, but the one

$$(\dot{x}-2)^2+\dot{y}^2\le 1$$

has no any point with such property at all.

Theorem 1 *For a generic smooth dynamic inequality with bounded derivatives on two dimensional sphere there exists point P and its neighbourhood such that at any point of the neighbourhood the inequality either has local transitivity property or has no any admissible velocities at all.*

Theorem 1 implies the following reduction theorem.

Theorem 2 *In three dimensional Euclidian space the problem of structural stability of a generic smooth dynamic inequality with locally bounded derivatives on the unit sphere with the center at origin is equivalent to the problem of structural stability of dynamic inequality on the plane, which is obtained from the initial one by stereographic projection from a point P of the sphere from Theorem 1 to the plane, which goes through the origin O and is orthogonal to vector OP.*

But the stereographic projection is continuous map in Whitney topology for any bounded domain in the plane of the projection. Hence Theorem 2 inplies

Theorem 3 *The problem of structural stability of generic smooth dynamic inequalities with bounded derivatives on two dimensional sphere is equivalent to the problem of structural stability of such inequalities on the plane, which near the infinity either has local transitivity property or has no any admissible velocities at all.*

The last theorem and main result of [12] imply

Theorem 4 *On two dimensional sphere endowed by Riemannian metric a generic smooth simplest dynamic inequality is structurally stable.*

Proofs

Here the proofs of the main results are presented.

Proofs of Theorems 2, 3, 4

Firstly we prove Theorem 2 and 3.

According to Theorem 1 for a generic smooth dynamic inequality F on two dimensional sphere there exist a point P and its neighborhood such that at any point from this neighborhood the inequality either has small time transitivity property or has no any admissible velocities at all. Due to [12] the local controllability properties of generic dynamic inequality with locally bounded derivatives on two dimensional surface S are stable up to small perturbation of the inequality. Namely, the set of points with the same local transitivity properties for a generic inequality and the ones for the sufficiently close inequality (in fine smooth or sufficiently smooth Whitney topology) are transferred one into another by a homeomorphism of the surface, which is close to the identity.

Hence there exists a neighborhood V of the inequality F and the neighborhood U of the point P such that any inequality $\tilde{F} \in V$ either has small time local transitivity property or has no admissible velocities at any point in U.

Now, let the surface S be the unit sphere $x^2 + y^2 + z^2 = 1$ in three dimensional Euclidian space with standard coordinates x, y and z. Consider the stereographic projection σ of this sphere from the point P to the plane π, which passes through the origin $O = (0,0,0)$ and is orthogonal to vector OP. As the result we get a dynamic inequality $\sigma_* F$ in the plane, which is the image of the initial dynamic inequality under the projection.

Note that although the admissible velocities of the initial inequality are bounded on the whole sphere, for the resulting inequality they can grow at infinity. Nevertheless, the last inequality will be an inequality with locally bounded derivatives. In addition, at any point of the neighborhood σU of the infinity in the plane π any inequality from the image $\sigma_* V$ either has small time transitivity property or has no any admissible velocities at all.

Hence the structural stability of inequality $\tilde{F} \in V$ is equivalent to the structural stability of inequality $\sigma_* \tilde{F} \in \sigma V$, which either has small time transitivity property or has no any admissible velocities at all at any point of σU.

Thus the statement of Theorem 2 is true.

Consider now a generic dynamic inequality with bounded derivatives on two-dimensional sphere and any diffeomorphism from this sphere to unit sphere in three dimensional Euclidian space. That obviously preserves property of structural stability of the inequality. Now, using the statement of Theorem 2, we get the statement of Theorem 3.

Statement of Theorem 4 follows immediately from statement of Theorem 3 and main result of [12].

Proof of Theorem 1

Without loss of generality we consider a generic dynamic inequality F with bounded derivatives on the unit sphere S in three-dimensional Euclidean space with a standard volume element. This element naturally determines the area element in each tangent plane to the sphere.

If for the inequality F with bounded derivatives there exists point $P \in S$ without admissible velocities then due to continuity of F and closeness of $F \leq 0$ there is neighborhood of the point, at any point of which there are no admissible velocities also. So in such a case the statement of Theorem 1 takes place.

If the inequality F with bounded derivatives has admissible velocity at any point of S then in a generic case the set of admissible velocities has nonempty interior in tangent plane at any point of the sphere (that follows from the results of [9]). In such a case consider the vector field v provided by the geometric centers of mass of the set $F \leq 0$ in each tangent plane. The following statement is utile.

Proposition 1 *For a generic inequality on the sphere S with bounded derivatives, which has admissible velocities at any point of this sphere, the field v is well defined, continuous and its value at any point belongs to the interior of convex hull of admissible velocities at the point.*

Now with the help of this proposition we complete the proof of the theorem, and then prove the proposition itself. Due to the proposition the field v is continuous, and so it has to have singular point since the genus of the sphere is 2. But due to the last statement of the proposition the respective zero velocity belongs to the interior of the convex hull of admissible velocities at the respective point. Hence in a generic case this point and all point nearby has small time transitivity property [9], [18], [19]. Thus the statement of Theorem 1 again takes place.

Thus Theorem 1 is proved modulo the proposition.

Let us prove the proposition.

The statement of the proposition is local one. So without loss of generality we consider dynamic inequality $F(x,y) \leq 0, x = (x_1, x_2), y = (y_1, y_2) = \dot{x}$, with locally bounded derivatives on the plane and take the area element in the velocity plane in standard form ($= dy_1 \times dy_2$).

Consider now F as a two-parameter family of functions of y. For a generic family F with locally bounded derivatives In particular the domain $F(x,.) \leq 0$ could have only standard bifurcation under the change of parameter x [2], [4], [14].

Outside the bifurcations this intersection has smooth boundary, which changes smoothly under the change of x, and so the field v is also smooth. Hence one needs only to account the impact of the bifurcations of this intersection on the field v. But in generic case these bifurcations preserve the continuity of the field due to genericity of the restriction of projection $(x,y) \mapsto x$ to the surface $F=0$ [14] and local normal forms of the bifurcations [2], [4] (in [5] some singularities of the area of the set $F \leq 0$ as the function of x are described). Hence the field v is continuous

Finally one needs to show that at any point x the value $v(x)$ belongs to the interior of convex hull of $F(x,.) \leq 0$. But considered case the set $F(x,.) \leq 0$ for a generic inequality has nonempty interior in tangent plane at any point. Hence the geometric center mass of such set has to belong to the boundary of its convex hull.

Hence the statement of Proposition 1 is true.

Conclusion

Thus we reduce the problem of structural stability of generic dynamic inequality with locally bounded derivatives on two-dimensional sphere to the same problem for such an inequality on the plane, which at any point near infinity either has small time local transitivity property or has no admissible velocities at all. By this result we get structural stability of generic simplest dynamic inequalities on two-dimensional sphere (endowed by Riemannian metric).

Note that without genericity assumption the statement of Proposition 1 is wrong. For example, on the real plane $\mathbb{R}_{x_1,x_2}$ for the inequality $F(x_1,x_2,\dot{x}_1,\dot{x}_2) = [(\dot{x}_1-1)^2+\dot{x}_2^2-1]f(x_1)+[(\dot{x}_1+1)^2+\dot{x}_2^2-1]f(-x_1)$ with a smooth function f being positive by $x_1>0$ and zero for $x_1 \leq 0$ the vector field v has value $(-1,0)$ by $x<0$ and $(1,0)$ by $x>0$. Thus the field is not continuous at any point of the plane $x_1=0$.

While the property of structural stability is not generic for three dimensional control systems (or dynamic inequalities) the problem of stability of local transitivity properties makes sense and is open. Significant progress in analysis of the properties of local controllability for three-dimensional systems was done in [11].

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