

S1 Proof of Proposition 1

(i) For each $i = 1, \dots, n$, we use (21) and obtain

$$\begin{aligned}
\mathbb{P}[W_i \leq w_i \mid V_i; \alpha, \beta] &= \mathbb{P}\left[\exp\left(\frac{X_i + V_{i-1} - \alpha}{\beta}\right) - \exp\left(\frac{V_{i-1} - \alpha}{\beta}\right) \leq w_i \mid V_i; \alpha, \beta\right] \\
&= \mathbb{P}\left[\exp\left(\frac{X_i + V_{i-1} - \alpha}{\beta}\right) \leq w_i + \exp\left(\frac{V_{i-1} - \alpha}{\beta}\right) \mid V_i; \alpha, \beta\right] \\
&= \mathbb{P}\left[X_i \leq \beta \log\left(w_i + \exp\left(\frac{V_{i-1} - \alpha}{\beta}\right)\right) + \alpha - V_{i-1} \mid V_i; \alpha, \beta\right] \\
&= 1 - \exp\left(\exp\left(\frac{V_{i-1} - \alpha}{\beta}\right) - \exp\left(\frac{\beta \log\left(w_i + \exp\left(\frac{V_{i-1} - \alpha}{\beta}\right)\right) + \alpha - V_{i-1} - \alpha + V_{i-1}}{\beta}\right)\right) \\
&= 1 - \exp(-w_i).
\end{aligned}$$

Therefore, we have $W_i \sim \text{Exp}(1)$, which concludes the proof of the first claim.

(ii) We verify that the joint distribution for X_1 and X_2 is given by

$$\begin{aligned}
\mathbb{P}[X_1 \leq x_1, X_2 \leq x_2] &= \mathbb{P}[X_1 \leq x_1] \cdot \mathbb{P}[X_2 \leq x_2 \mid X_1 \leq x_1] \\
&= \mathbb{P}[X_1 \leq x_1] \cdot \mathbb{P}[X_2 \leq x_2 + qx_1].
\end{aligned}$$

Consequently, the joint distribution of X_1, \dots, X_n with $n > 2$ can be written as

$$\mathbb{P}[X_1 \leq x_1, \dots, X_n \leq x_n] = \prod_{j=1}^n \mathbb{P}[X_j \leq x_j + V_{j-1}],$$

and the density function for $\mathbf{X} = (X_1, \dots, X_n)$ is given by

$$f_{\mathbf{X}}(x_1, \dots, x_n \mid v_{i-1}; \alpha, \beta) = \prod_{i=1}^n f_{X_i}(x_i \mid v_{i-1}; \alpha, \beta). \quad (23)$$

In order to show that the variables W_1, W_2, \dots, W_n are independent, we need to show that

$$f_{W_1, \dots, W_n}(w_1, \dots, w_n \mid v_{i-1}; \alpha, \beta) = \prod_{i=1}^n f_{W_i}(w_i \mid v_{i-1}; \alpha, \beta). \quad (24)$$

Using the change of variables

$$W_i = \exp\left[\frac{X_i + V_{i-1} - \alpha}{\beta}\right] - \exp\left[\frac{V_{i-1} - \alpha}{\beta}\right],$$

and solving to $X_i + V_{i-1}$, we obtain

$$X_i + V_{i-1} = \beta \log\left(W_i + \exp\left(\frac{V_{i-1} - \alpha}{\beta}\right)\right) + \alpha.$$

The joint density of the W_1, W_2, \dots, W_n , is given by

$$f_{W_1, \dots, W_n}(w_1, \dots, w_n \mid v_{i-1}, \alpha, \beta) = f(\varphi_1(x_1 + v_0, \dots, x_n + v_{n-1}), \dots, \varphi_n(x_1 + v_0, \dots, x_n + v_{n-1}) \mid \alpha, \beta, q) \times J,$$

where

$$\varphi(x_1 + v_0, \dots, x_n + v_{n-1}) = \beta \log \left(w_1 + \exp \left(\frac{v_0 - \alpha}{\beta} \right) \right) + \alpha$$

and J is the determinant of the $(n \times n)$ -matrix given by

$$\begin{aligned} J &= \det \left(\frac{\partial \varphi_i(x_1, \dots, x_n)}{\partial w_j} \right)_{i,j} \\ &= \det \begin{bmatrix} \frac{\beta}{w_1 + \exp \left(\frac{v_0 - \alpha}{\beta} \right)} & 0 & \dots & 0 \\ 0 & \frac{\beta}{w_2 + \exp \left(\frac{v_1 - \alpha}{\beta} \right)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\beta}{w_n + \exp \left(\frac{v_{n-1} - \alpha}{\beta} \right)} \end{bmatrix} \\ &= \prod_{i=1}^n \frac{\beta}{w_i + \exp \left(\frac{v_{i-1} - \alpha}{\beta} \right)} \end{aligned} \quad (25)$$

Thus, according to (25), we have:

$$\begin{aligned} f_{W_1, \dots, W_n}(w_1, \dots, w_n \mid v_{i-1}; \alpha, \beta) &= \left\{ \prod_{j=1}^n \frac{1}{\beta} \exp \left[\frac{x_j + v_{j-1} - \alpha}{\beta} - \overbrace{\exp \left[\frac{x_j + v_{j-1} - \alpha}{\beta} \right]}^{-w_j} + \exp \left[\frac{v_{j-1} - \alpha}{\beta} \right] \right] \right\} \\ &\quad \times \prod_{i=1}^n \frac{\beta}{w_i + \exp \left[\frac{v_{i-1} - \alpha}{\beta} \right]}. \end{aligned}$$

Replacing $x_i + v_{i-1}$ in the last equality, we obtain:

$$\begin{aligned}
& f_{W_1, \dots, W_n}(w_1, \dots, w_n \mid v_{i-1}; \alpha, \beta) \\
&= \frac{1}{\beta^n} \left\{ \prod_{j=1}^n \exp \left(\frac{\beta \log(w_j + \exp((v_{j-1} - \alpha)/\beta)) + \alpha - \alpha}{\beta} - w_j \right) \right\} \\
&\quad \times \beta^n \prod_{i=1}^n \frac{1}{w_i + \exp \left(\frac{v_{n-1} - \alpha}{\beta} \right)} \\
&= \prod_{j=1}^n \left(w_j + \exp \left(\frac{v_{j-1} - \alpha}{\beta} \right) \right) \times \exp(-w_j) \times \prod_{i=1}^n \frac{1}{w_i + \exp \left(\frac{v_{i-1} - \alpha}{\beta} \right)} \\
&= \prod_{j=1}^n \exp(-w_j)
\end{aligned}$$

The last equality is independent of v_i , and hence W_1, W_2, \dots, W_n are independent and identically distributed, which concludes the proof of the second claim.