ANALYTICAL AND NUMERICAL EXPERIMENTATION WITH A LOW-ORDER POPULATION-ENERGY MODEL

A. A. Harms

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

ABSTRACT

An experimental analysis is undertaken to formulate and examine a tractable national population-energy model. On the basis of plausibility arguments a mathematical representation is established that serves to describe population and per-capita energy consumption in a coupled dynamic relationship. The conceptual basis of the model is tested numerically with the help of historical data on distinct national population-energy patterns. The structural form of the model and its limitations are further examined and extensions are suggested.

Analytical and Numerical Experimentation With A Low-Order Population-Energy Model

1. INTRODUCTION.

The formulation of analytical models in IIASA's Energy Systems Program attempting to characterize patterns of the population and energy relationship has been the objective of several studies (Häfele 1976; Häfele and Bürk 1976; Grümm and Schrattenholzer 1976; Breitenecker and Grümm 1981). These investigations try to both simulate and represent specific relationships involving national parameters--such as population, energy consumption, labor input, gross domestic product, etc.

The present investigation endeavors to continue that line of analysis but sets out from an entirely different starting point. By way of analytical manipulation based on a well-accepted methodology, it develops a phase-space representation that formally corresponds to previous IIASA studies but also incorporates empirical data.

In the following, we first proceed to formulate a plausibility representation for the dynamics of the population-energy interaction and then employ empirical data to determine specific numerical-graphical representations. Finally, we suggest some extensions to this model and comment on some details of the implicit structural characteristics of the model.

2. ANALYTICAL-PLAUSIBILITY FORMULATION.

We consider a nation--or any other suitable region--that is characterized by a time-dependent population $P(t)$ and a time-dependent per-capita energy consumption $q(t)$. We accept the premise that these two functions are coupled in a manner which is case-specific and may vary with time. The form and strength of the population-energy coupling is taken to be determined by the specific mathematical form and the magnitude (as well as sign) of appropriate coefficients.

In order to establish the mathematical model of interest here, we use the following plausibility argument applicable to a finite, though arbitrary, time interval Δt . We hypothesize that the fractional change in the per-capita energy consumption over time interval Δt possesses two basically different components: one that is essentially constant and another that is linearly dependent on the population. That is, the proportionality relationship using superposition is written as

$$\frac{\Delta q}{q(t)} \propto [k + P(t)] \quad \text{over} \quad \Delta t , \qquad (1)$$

with k being a constant.

Equation (1) can be better understood by way of Fig. 1 which illustrates two distinct linear variations of $\Delta q/q$ with P over the time interval Δt . The line with a positive slope may, for example, be associated with the determination and resourcefulness of a growing population to provide an increasing per-capita energy accessibility; the negative slope, however, implies a diminishing per-capita energy consumption with an increasing population suggesting, for example, that the total energy resource is essentially constant and must be distributed among more people. Differences between the trends are understood to stem from dissimilarities in the technological development and resource situation of the countries considered.

By translating this qualitative argument into mathematical terms, the proportionality relationship of Equ.(1) can be

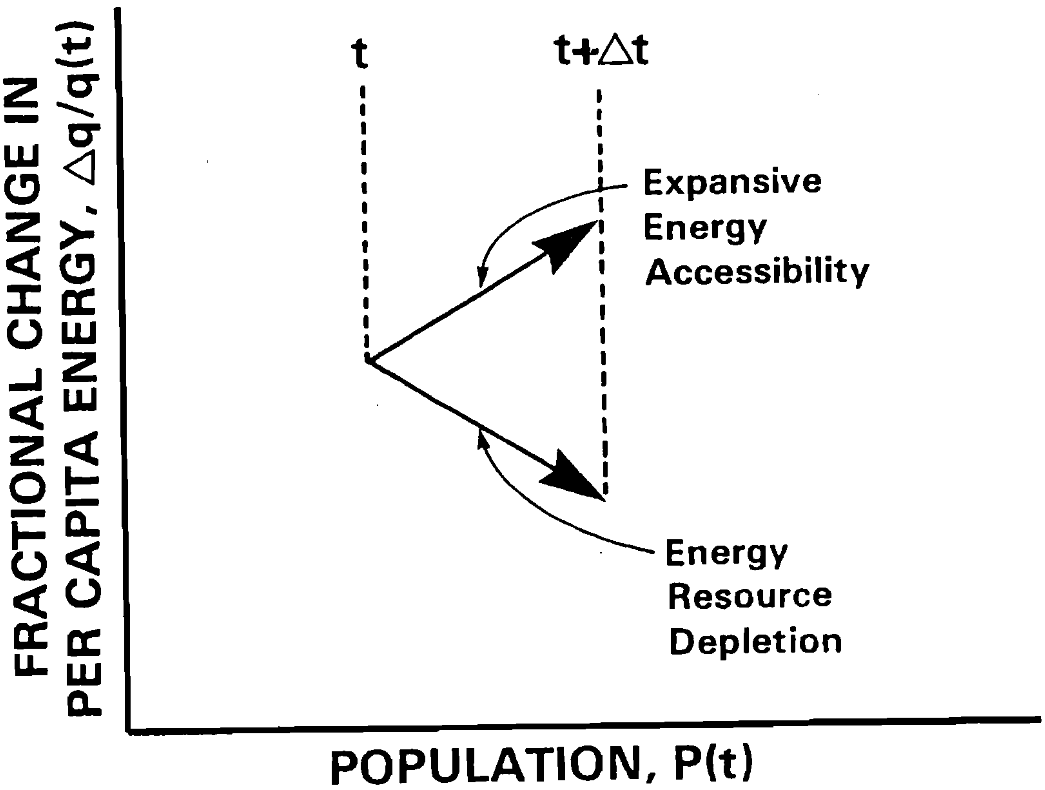

Figure 1: Schematic illustration showing hypothesized relationship of a fractional per-capita energy change as a function of population during time interval Δt. A conceivable explanation for the trend is indicated.

explicitly written as a finite difference equation

$$\frac{\Delta q/q(t)}{\Delta t} \sim \mu_1 + \mu_2 P(t) \qquad (2)$$

or

$$\frac{\Delta q}{\Delta t} \sim \mu_1 q(t) + \mu_2 q(t) P(t) \ , \qquad (3)$$

where μ_1 and μ_2 are specific constants. Taking the limit $\Delta t \to 0$ and allowing t to be arbitrary one obtains the differential equation

$$\frac{dq}{dt} = \mu_1 q(t) + \mu_2 q(t) P(t) \ . \qquad (4)$$

While this equation has a useful form, it represents only one differential relation of two functions $q(t)$ and $P(t)$. We therefore proceed to establish a comparable plausibility argument for $P(t)$ as a function of $q(t)$.

Considering the fractional change in population we postulate that it is also related to the per-capita energy consumption in a linear manner:

$$\frac{\Delta P}{P(t)} \propto [k + q(t)] \quad \text{over} \quad t \ . \qquad (5)$$

The positive slope in Fig. 2 suggests that, for example, increasing energy consumption leads to improved health care, which reduces infant mortality and extends life expectancy. On the other hand, the falling slope may imply a deliberate birth rate reduction in order to increase the standard of living by increased energy consumption. We proceed to write therefore

$$\frac{\Delta P/P(t)}{\Delta t} \sim \sigma_1 + \sigma_2 q(t) \ ; \qquad (6)$$

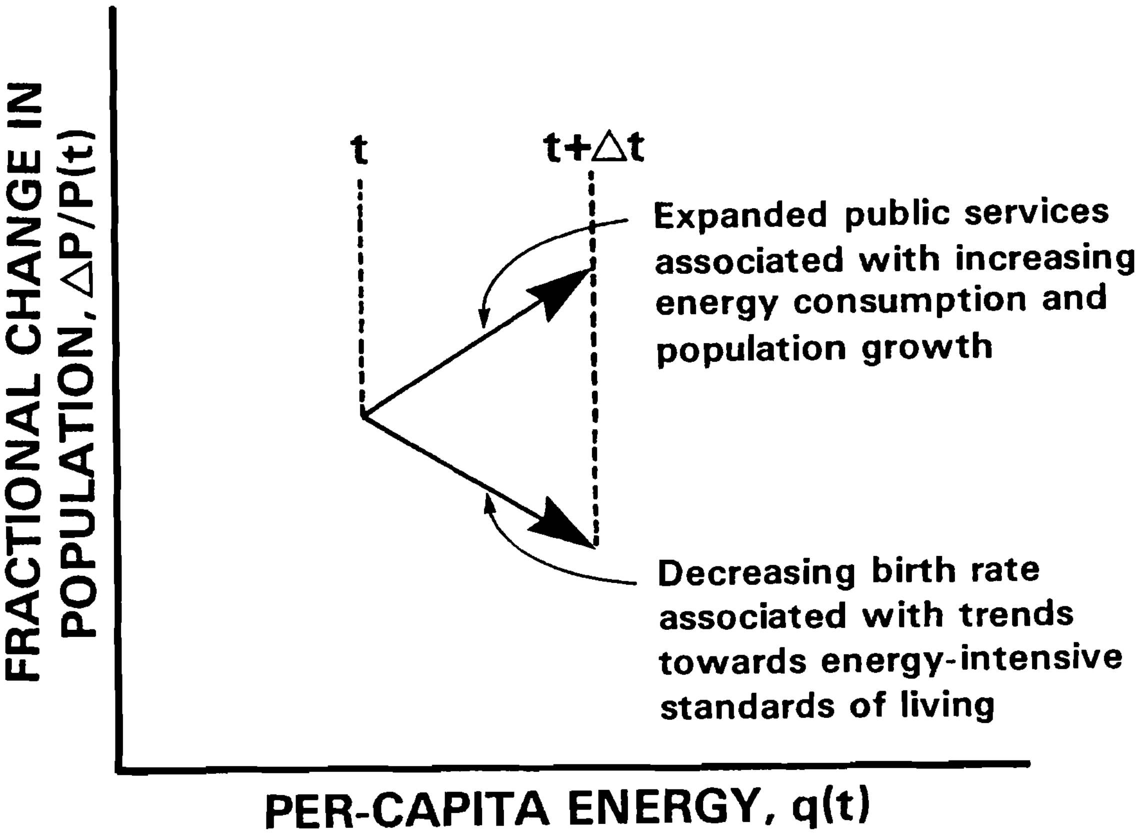

Figure 2: Graphical depiction of hypothesized relationship between a fractional population change as a function of per-capita energy consumption during time interval Δt. A plausible explanation for such linear trends is indicated.

from which, by the analogous argument used to obtain Equ.(4), we obtain

$$\frac{dP}{dt} = \sigma_1 P(t) + \sigma_2 q(t) P(t) \tag{7}$$

Equations (4) and (7) define the mathematical model that describes the energy-population pattern. They possess a symmetry which becomes more obvious if they are written in a general form for arbitrary functions $x(t)$ and $y(t)$:

$$\frac{dx}{dt} = a_1 x + a_2 xy \ , \tag{8a}$$

and

$$\frac{dy}{dt} = b_1 y + b_2 xy \ , \tag{8b}$$

with no apparent restriction on the sign or magnitude of coefficients a_1, a_2, b_1 and b_2 .

3. COMMENT ON THE DYNAMICAL EQUATIONS.

Equations (4) and (7) represent a system of coupled dynamical equations whose form are known to appear in various contexts (Kemeny and Snell 1962; Haken 1977). Of interest to us here are certain aspects associated with the coefficients σ_1, σ_2, μ_1 and μ_2 .

In view of the formulation of the problem, there is no physical reason to expect any restrictions on the signs of the coefficients; both populations and per-capita energy consumption can increase or decrease with time resulting therefore in positive and negative signs as appropriate.

If Equs.(4) and (7) constitute an acceptable representation of "reality" then the coefficients are, in a complex way, functions of the various pertinent independent parameters such as birth rates, death rates, gross national product, standard of living, cost of capital, type of economy, and many others. This would then suggest that the coefficients σ_1, σ_2, μ_1 and μ_2 might best be studied by fitting Equs.(4) and (7) to specific regional data. Further, since these determining factors change with time, these coefficients are likely to vary with time.

The space and time dependency of coefficients σ_1, σ_2, μ_1 and μ_2 suggests that careful attention be paid to the applicability of the equations. The space dependence is easily accommodated by calculating these coefficients for each region of interest. The time dependence imposes the restriction that, once the coefficients are known, the dynamical equations are useful as predictive tools only if the coefficients do not vary with time; otherwise, only a descriptive property can be associated with the mathematical characterization.

The above points suggest a potential domain of application of the dynamical equations, Equs.(4) and (7), under conditions that the coefficients can be accurately calculated; it refers to the identification of trajectory tendencies for the national population-energy patterns and indicates the intrinsic (geometric) structure characterizing the pattern of population and energy.

The establishment of phase-space representations, as suggested in previous IIASA studies quoted above, is particularly relevant in this regard.

4. PHASE-SPACE CHARACTERIZATION.

To obtain the phase-plane representation and the associated trajectory, we eliminate time from Equs.(4) and (7) by division

$$\frac{dP(t)}{dq(t)} = \frac{P(t)[\sigma_1 + \sigma_2 q(t)]}{q(t)[\mu_1 + \mu_2 P(t)]} , \qquad (9)$$

and identify the pertinent equilibrium point on the P-q phase plane by the coordinates

$$(P_E, q_E) = \left(-\frac{\mu_1}{\mu_2}\ ,\ -\frac{\sigma_1}{\sigma_2}\right) \quad . \tag{10}$$

The solution of Equ.(9) can be deduced and, for the case that the signs of σ_1, σ_2, μ_1 and μ_2 are arbitrary, is given by the transcendental equation

$$\{|\frac{\sigma_2}{\mu_1}|q(t)\}^{S(\sigma_1)} \exp\{S(\sigma_2)|\frac{\sigma_2}{\sigma_1}|q(t)\}$$

$$x[\{|\frac{\mu_2}{\mu_1}|P(t)\}^{S(\mu_1)} \exp\{S(\mu_2)|\frac{\mu_2}{\mu_1}|P(t)\}]^{-|\mu_1/\sigma_1|} = C \quad . \tag{11}$$

Here C is an integration (i.e., contour) constant and $S(z) = \text{sign}\ (Z) \times 1(=\pm 1)$.

The resultant phase-space surfaces, or phase-portraits, can be conveniently characterized by the sign of the ratios of the coefficients:

$$\left(\frac{\mu_1}{\sigma_1}\right)_i = \begin{cases} < 0, & \text{elliptic} \\ > 0, & \text{hyperbolic} \end{cases} \tag{12}$$

Figure 3 provides one illustration, a hyperbolic phase-space portrait in this case, for which the equilibrium points exist in the positive $P(t)$ and positive $q(t)$ quadrant. Note the characteristic trajectories in the four domains and the pattern of the separatrix. Other substantially different P-q portraits are mathematically feasible as we will show.

5. NUMERICAL EVALUATION.

To assess the adequacy of the formulation for our purposes here and to determine whether or not some useful parameterization might result, we undertook to use historical data for the determination of the coefficients as temporal

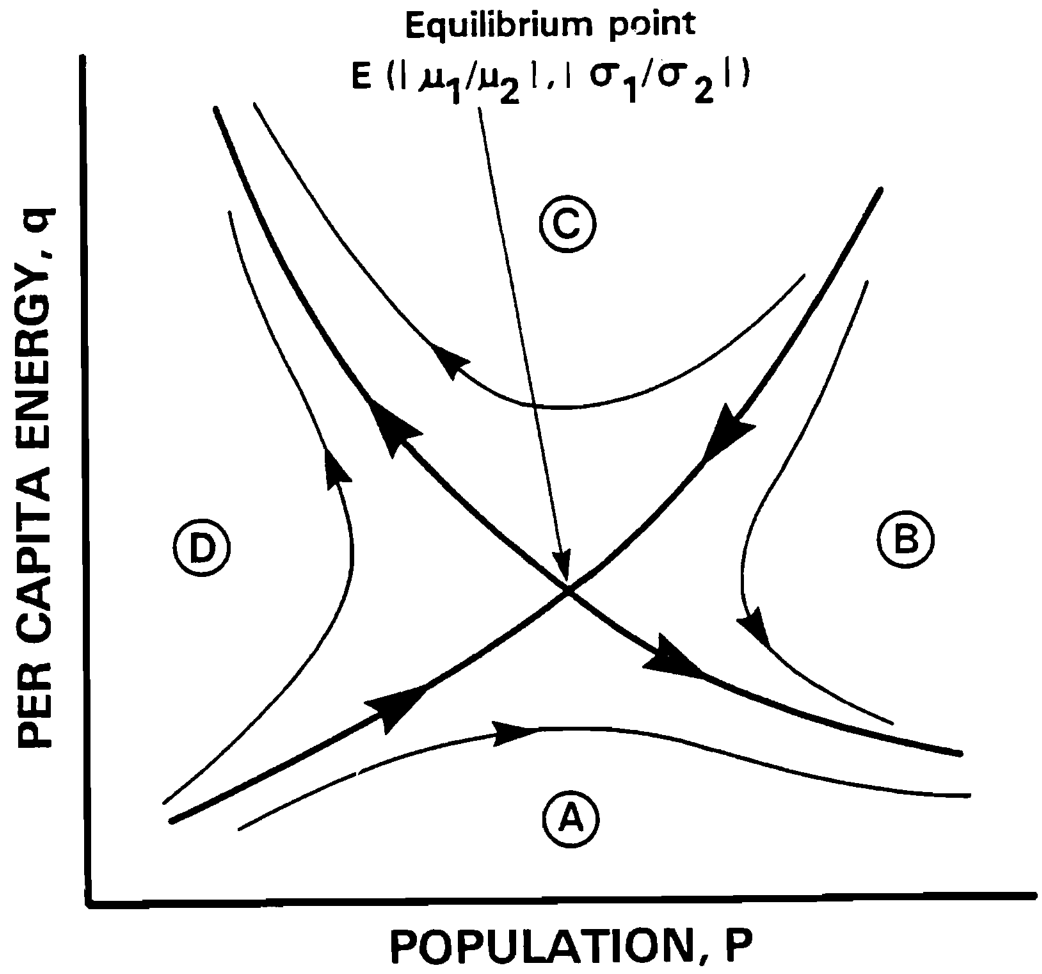

Figure 3: Example of trajectories and domains of the P-q phase portrait; other portraits are possible.

averages for countries which seemed to represent sufficiently diverse population-energy characteristics: Canada, Czechoslovakia, India, Nigeria, Sweden, USA, and USSR. The population data, $P(t)$, and the per-capita energy consumption data in units of tons-equivalent-of-coal, $q(t)$, for the period 1957-66 (World Energy Supplies 1976) were selected; since this period seemed generally free from large-scale perturbation, the calculation of such temporal-average coefficients σ_1, σ_2, μ_1 and μ_2 appeared appropriate.

However, the apparent noise in the $q(t)$ data complicated the determination of the coefficients α_1, α_2, μ_1 and μ_2, as well as identification of the phase-space portrait. To eliminate this noise effect we smoothed the raw historic $P(t)$ and $q(t)$ data by fitting least squares polynomials of the form

$$X(t) = a_0 + a_1 t + a_2 t^2 , \tag{13}$$

and used the resultant values to obtain least-squares fits for σ_1, σ_2, μ_1 and μ_2 to Equs.(4) and (7), which were written in the linear form

$$\frac{1}{P(t)} \frac{dP(t)}{dt} = \sigma_1 + \sigma_2 q(t) \tag{14a}$$

and

$$\frac{1}{q(t)} \frac{dq(t)}{dt} = \mu_1 + \mu_2 P(t) \quad . \tag{14b}$$

Figure 4 displays a typical result showing both the raw data and the smoothed data.

This procedure yielded excellent fits with the correlation coefficients well in excess of 0.8 with typical values of 0.99; Table 1 lists coefficients, correlations, equilibria points, and the geometric phase-space feature.

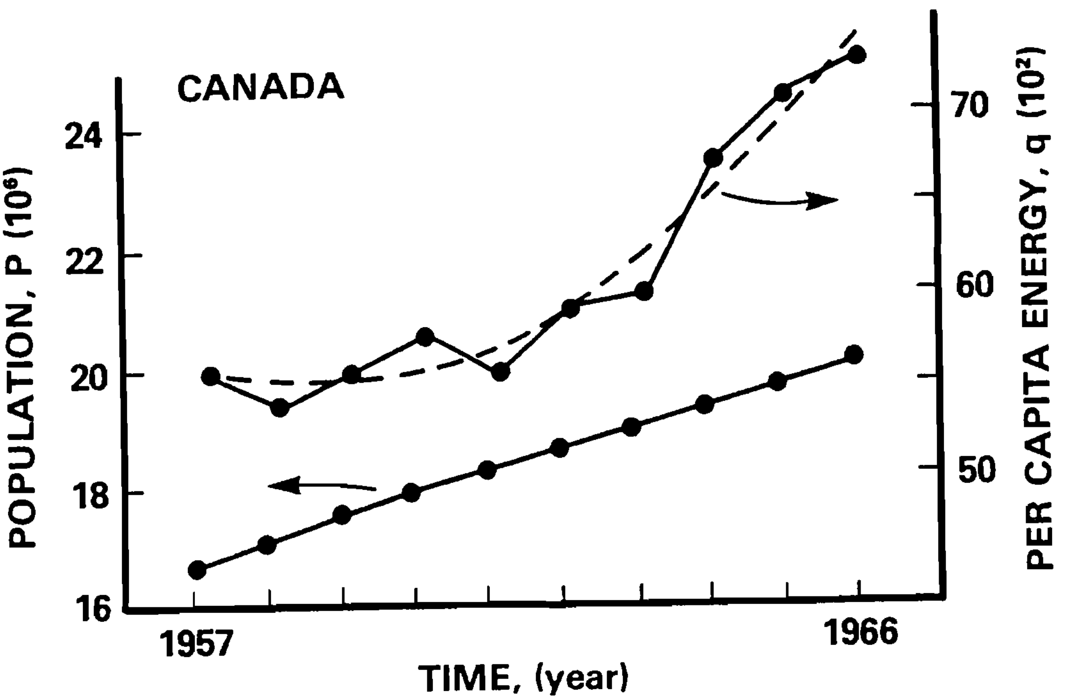

Figure 4: Population and energy data for Canada (1957-1966). The solid line refers to raw data (—) and the dashed line identifies the smoothed results (---); the population raw and smoothed data are essentially identical.

Country	σ_1	σ_2	μ_1	μ_2	Sign		Correlation		Type of Trajectory	Equilibrium Point	
					σ_1/σ_2	μ_1/μ_2	σ's	μ's		$P_E(10^6)$	$q_E(10^3)$
Canada	40.4	-0.0033	- 445	25.9	-	-	-0.90	0.99	ellipse	17.2	12.2
Czechoslovakia	10.6	-0.0007	1068	-74.4	-	-	-0.99	-0.99	hyperbola	14.4	15.1
India	20.7	0.0066	320	- 0.62	+	-	0.97	-0.99	hyperbola	516.0	- 3.14
Nigeria	12.4	0.25	- 929	21.9	+	-	0.82	0.95	ellipse	42.4	- 0.050
Sweden	- 3.12	0.0024	- 453	67.9	-	-	0.99	0.91	hyperbola	6.7	1.30
U.S.A.	41.4	-0.0031	- 318	1.84	-	-	-0.94	0.99	ellipse	173.0	13.4
U.S.S.R.	36.0	-0.0065	- 235	1.27	-	-	-0.97	0.99	ellipse	185.0	5.54

Table 1

Evaluated data based on the smoothed historical data 1957-1966

6. ILLUSTRATION OF RESULTS.

The phase-space portraits for the seven countries and for the 10-year period 1957-66 are shown in Figs. 5 to 11. Note again that if the parameters σ_1, σ_2, μ_1 and μ_2 do not change with time the specific phase-portrait is stationary and the evolution of the population and per-capita energy pattern is thus fully prescribed. These patterns are characterized by elliptic and hyperbolic geometrics about the equilibrium point suggesting either cyclic or asymptotic trajectory tendencies. Note also the considerable distortion of these two classes of surfaces, especially in the two cases--India and Nigeria--where in reality equilibria points would be unattainable.

Interestingly, as displayed in Table 2, the type of trajectory and location of the equilibrium point can be associated with a particular type of resource economy. While the results exhibit interesting and distinct features of the different national energy-population patterns, they highlight in particular the temporal average state of the 10-year period 1957-66. An assessment of how the P-q surfaces vary with time is clearly of interest and is examined below.

7. TEMPORAL VARIATIONS.

The temporal variations of the population-energy surface have been assessed for the example of Canada. The relevant population-energy data (United Nations 1976 and 1979) for the 29-year period 1950-1978, are displayed in Fig. 12. As indicators of the temporal variation of the P-q surface we have chosen are the history of the equilibrium point (P_E, q_E) and the characteristic geometric shape--either hyperbolic or elliptic--for successive 10-year sliding intervals. That is, for each of the time-intervals 1950-59, 1951-60, 1952-61, 1953-62, ..., 1969-78, we calculated σ_1, σ_2, μ_1 and μ_2; from this we obtained the equilibrium point, Equ.(10), and identified the geometric shape of the corresponding (P-q) surface, according to Equ.(12).

The polynomial data smoothing procedure used previously was also applied to this part of the analysis.

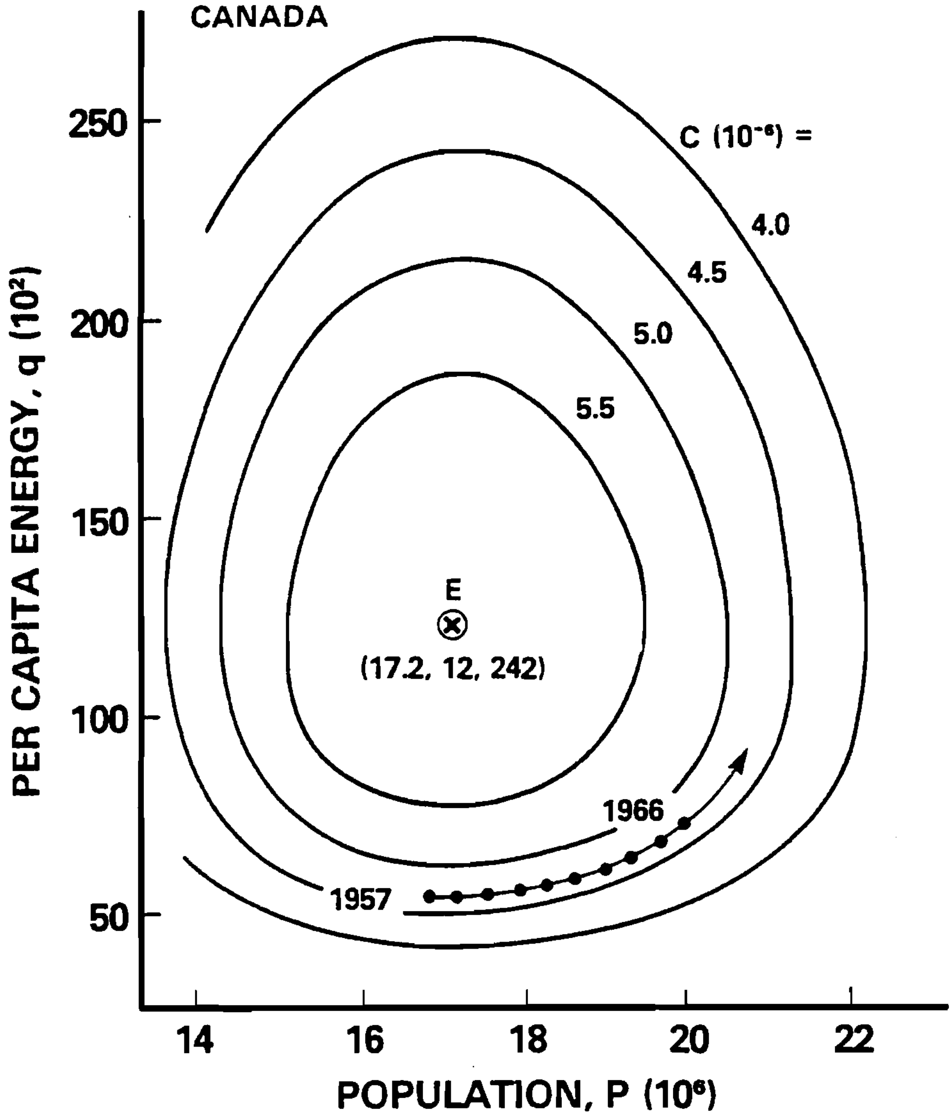

Figure 5: Population-energy phase-portrait and trajectory, Canada, 1957-1966.

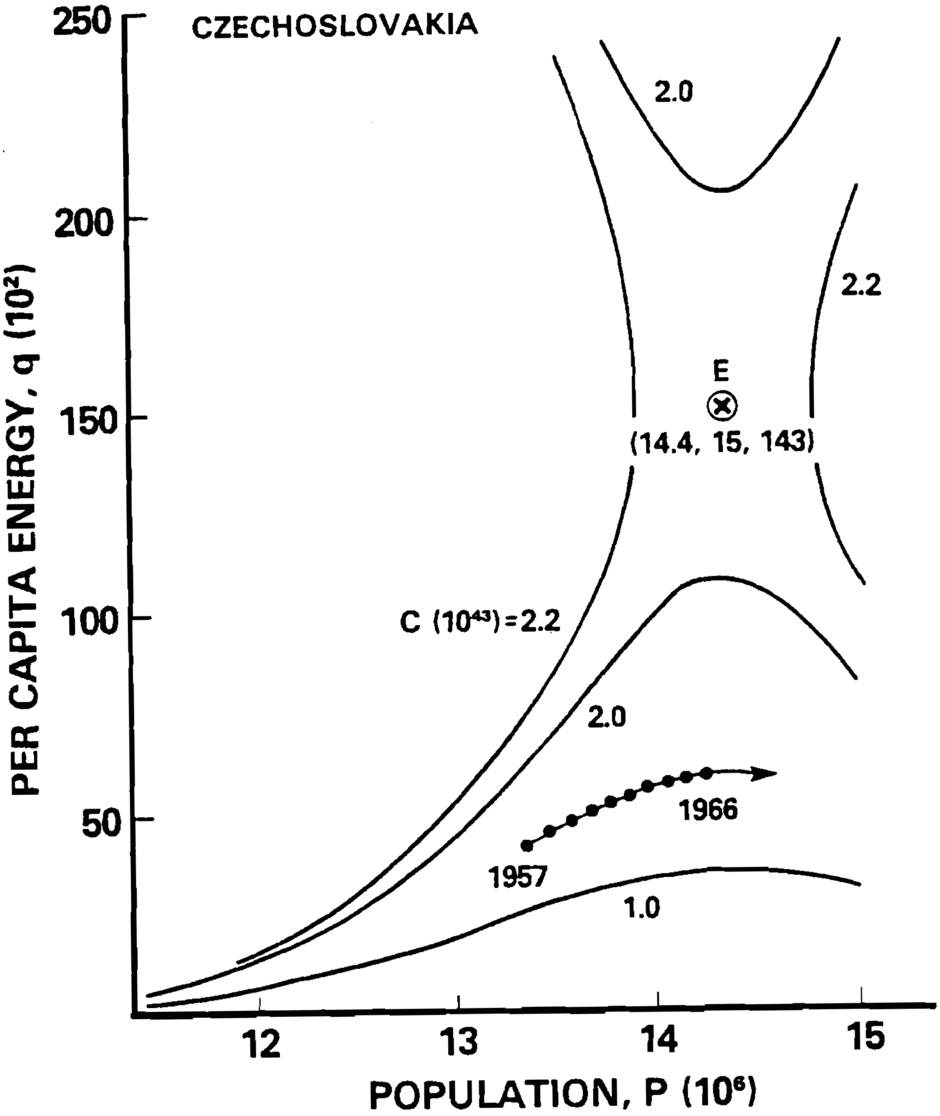

Figure 6: Population-energy phase-portrait and trajectory, Czechoslovakia, 1957-1966.

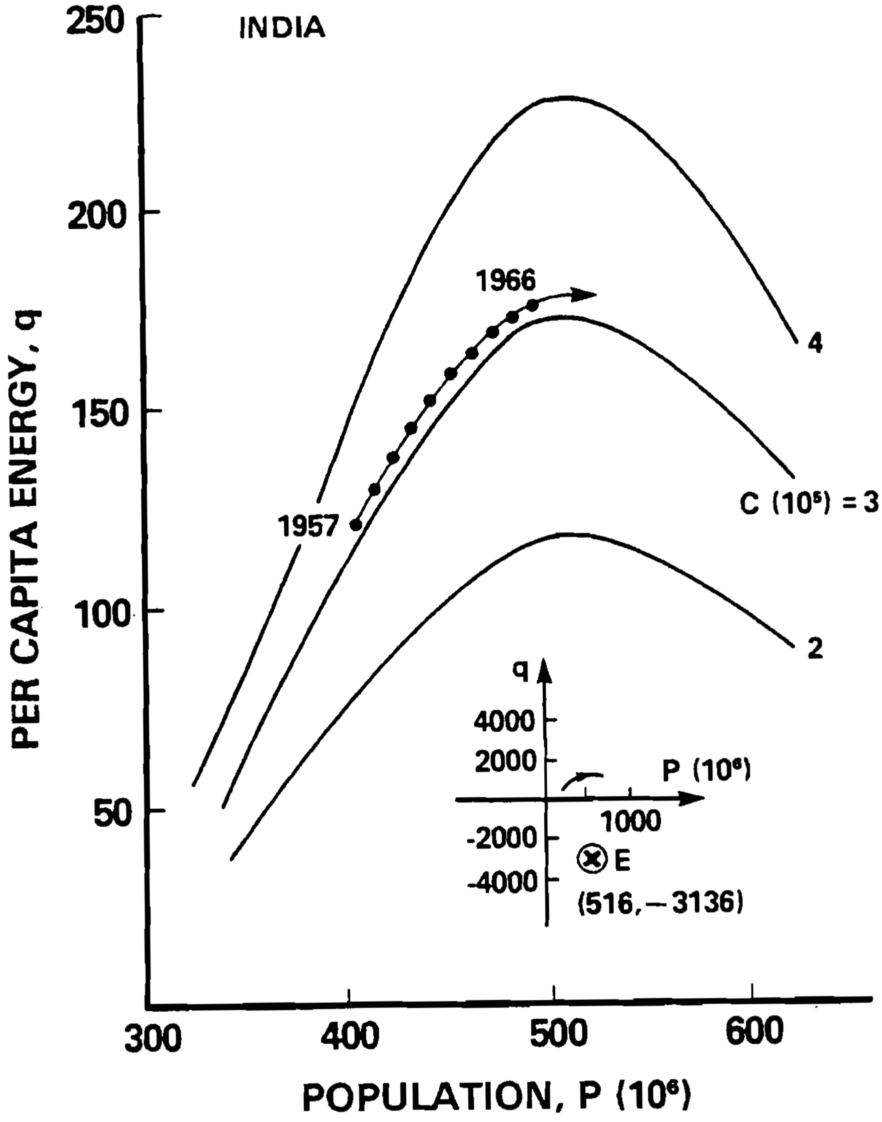

Figure 7: Population-energy phase-portrait and trajectory, India, 1957-1966.

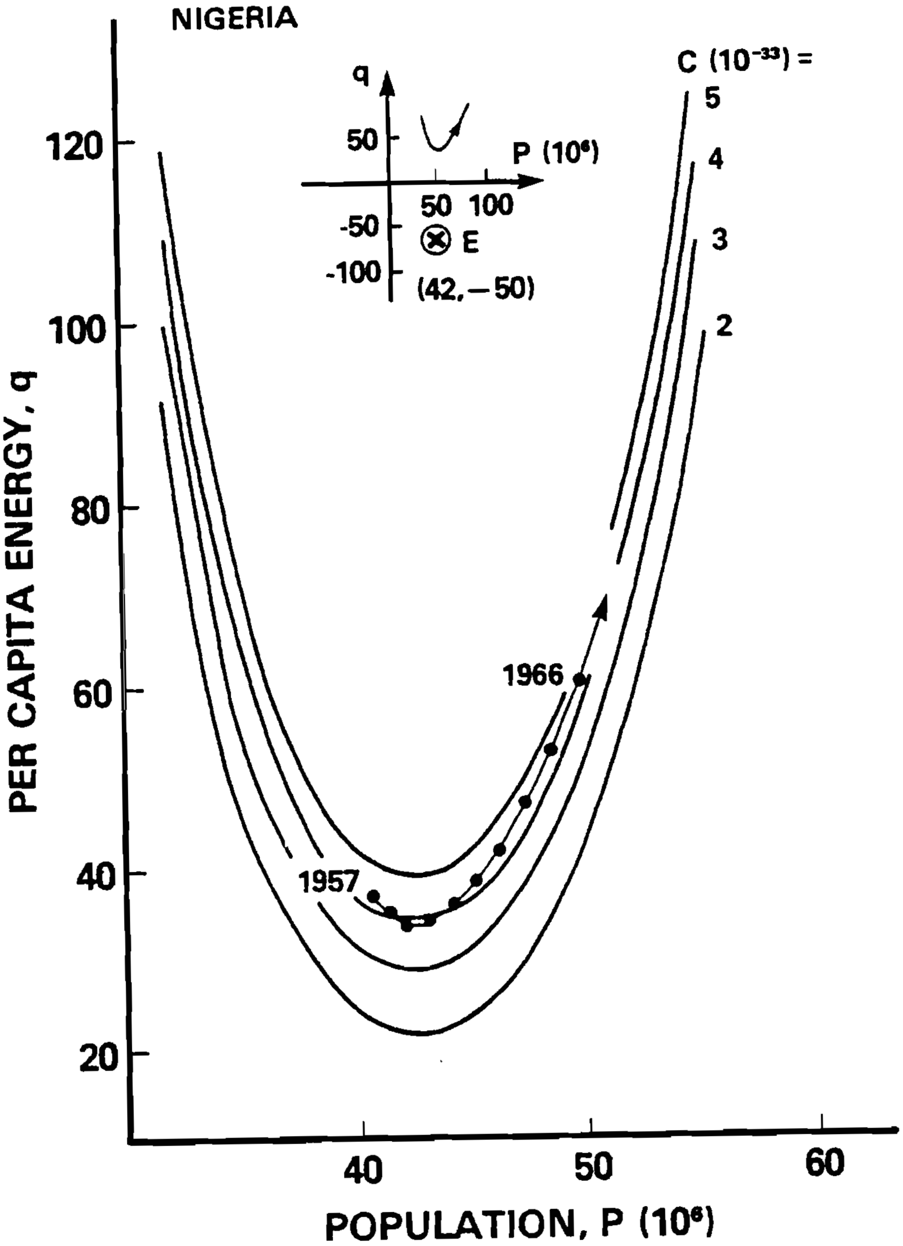

Figure 8: Population-energy phase-portrait and trajectory, Nigeria, 1957-1966.

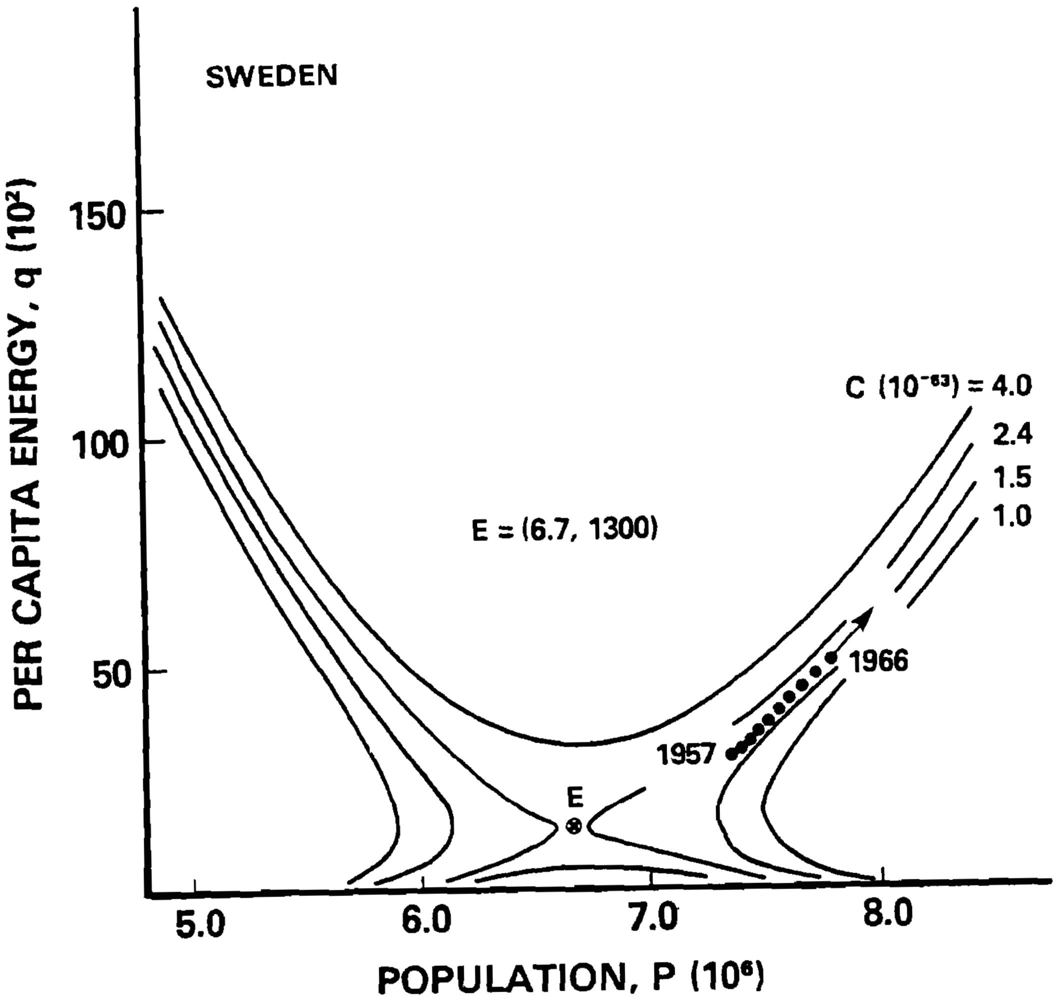

Figure 9: Population-energy phase-portrait and trajectory, Sweden, 1957-1966.

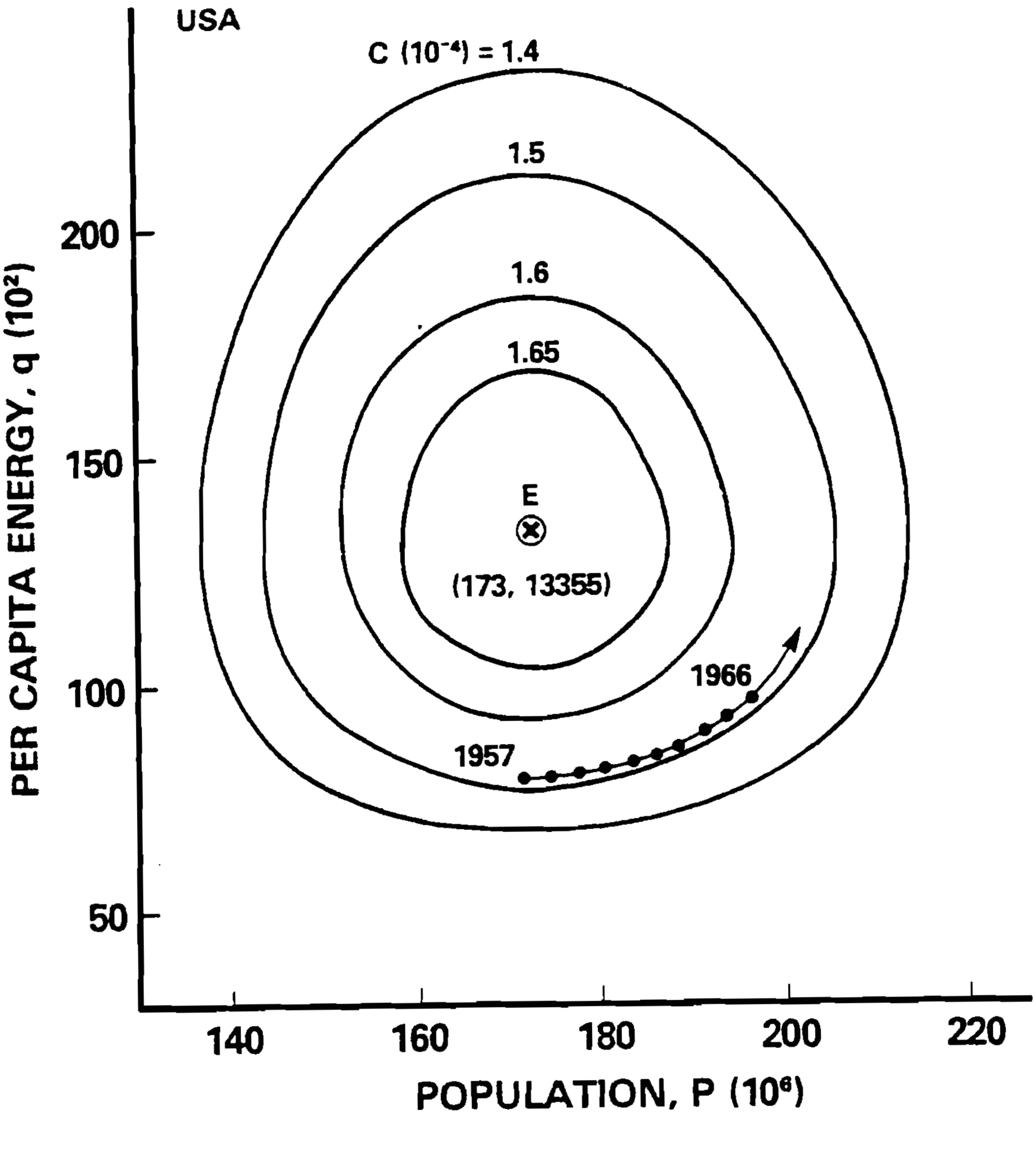

Figure 10: Population-energy phase-portrait and trajectory, U.S.A., 1957-1966.

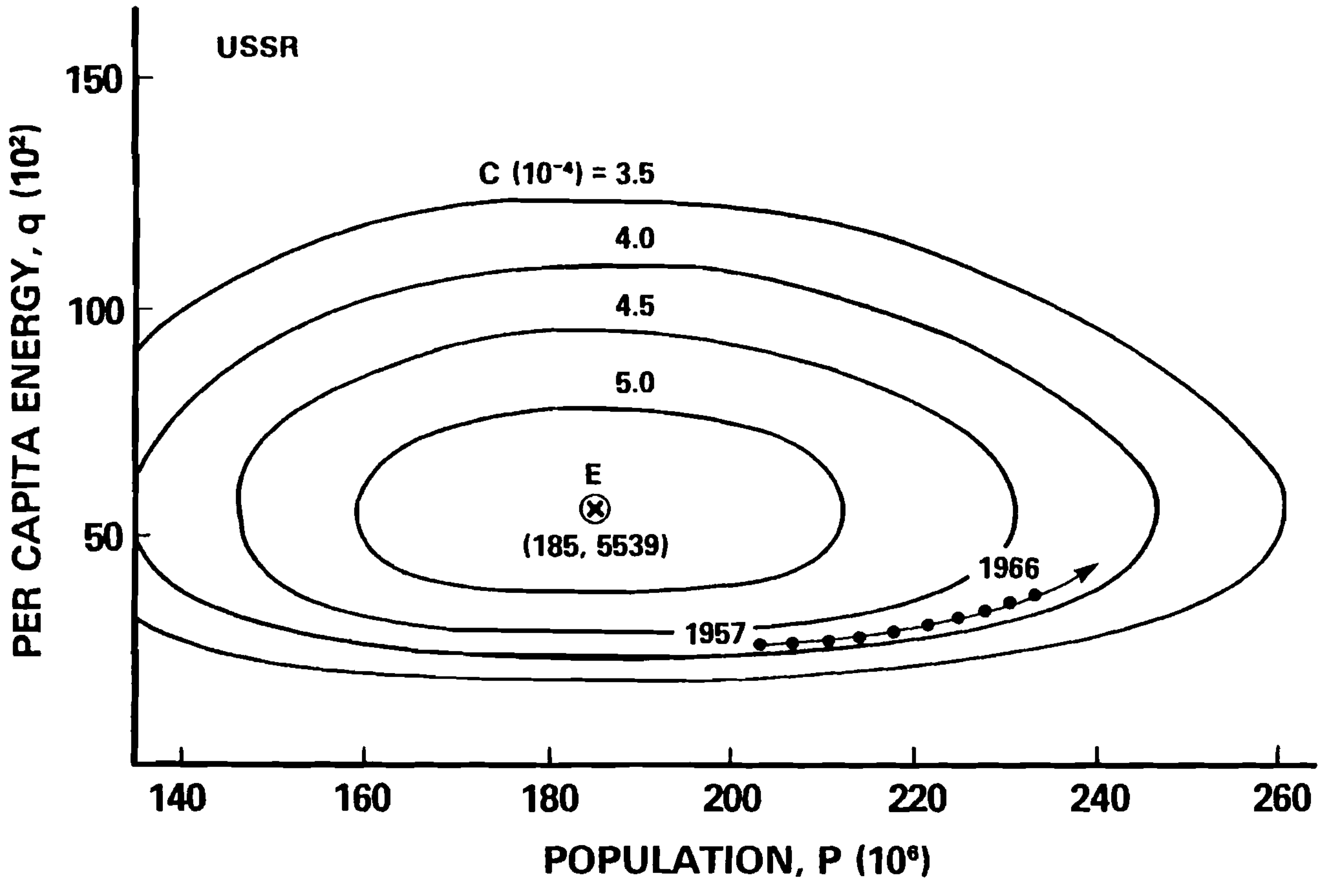

Figure 11: Population-energy phase-portrait and trajectory, USSR, 1957-1966.

Type of Economy	Country	Type of Trajectory	Equilibrium Point Location	
		Sign(μ_1/σ_1)	Sign(P_E)	Sign(q_E)
Developed with abundant resources	Canada	- (ellipse)	+	+
	U.S.A.	- (ellipse)	+	+
	U.S.S.R.	- (ellipse)	+	+
Developed with limited resources	Czechoslovakia	+ (hyperbola)	+	+
	Sweden	+ (hyperbola)	+	+
Developing	India	+ (hyperbola)	+	-
	Nigeria	- (ellipse)	+	-

Table 2

Geometric characterization of the national economies based on the historical smoothed data for the period 1957-1966

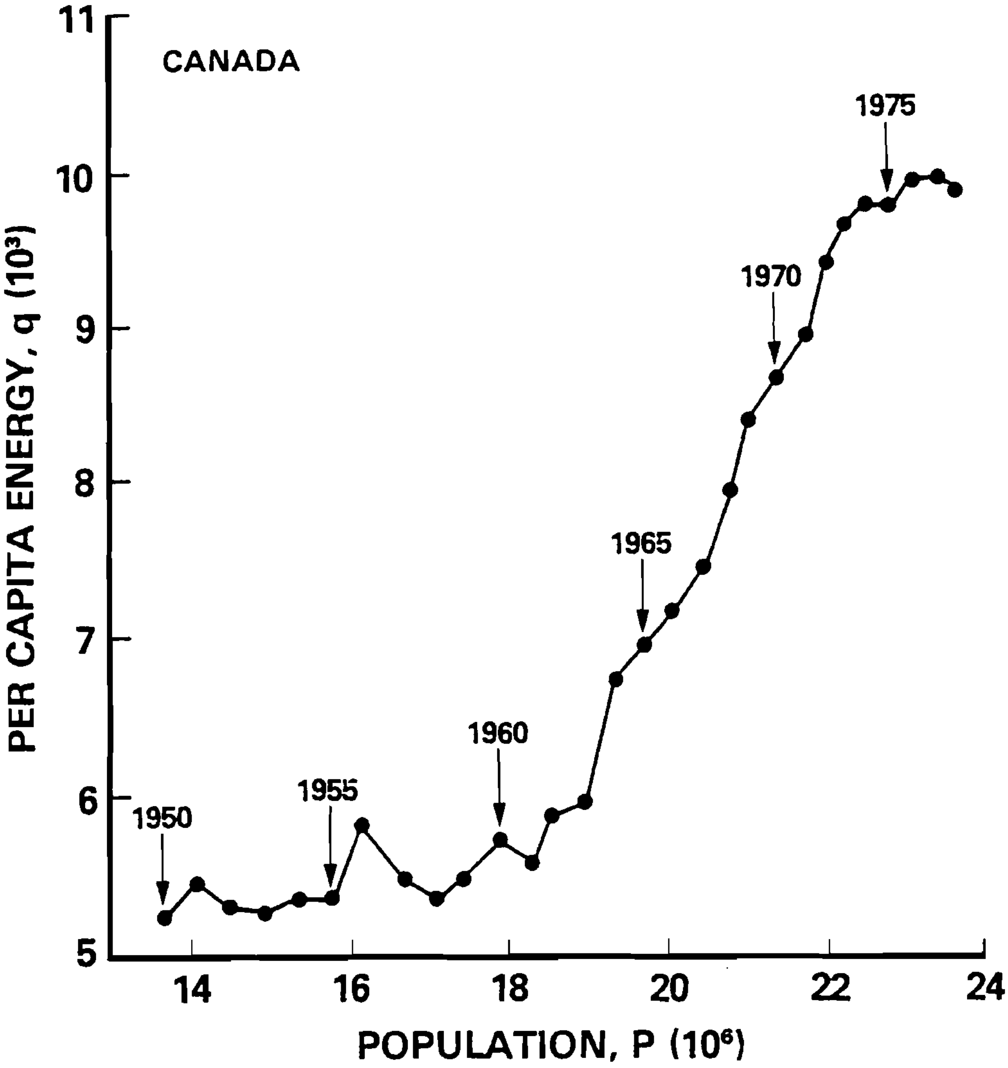

Figure 12: Historical population and per capita energy consumption, Canada, 1950-1978.

The resultant trajectory of the equilibrium point $(P_E, q_E)_i$ for the $i = 1,2,\ldots,20$ sliding overlapping time intervals is displayed in Figure 13.

The equilibrium trajectory in the Figure clearly displays some dynamic transformations and strong resonance features. The geometry starts out as a hyperbola, (1950-59), then transforms itself into an ellipse (1951-60), and returns to a hyperbola for the next two sliding time periods. From 1954-63 to 1960-69 it remains an ellipse, transforms to a hyperbola in 1961-70, and remains in that shape for the remainder of the study period. The other interesting feature is the apparent tendency to seek what may be termed resonances of its equilibrium coordinates P_E or q_E for single isolated intervals. Some simplification of the complexity of Fig. 13 is possible by eliminating the "exceptional" years 1951, 1956, and 1975; the clear separation of the elliptic and hyperbolic domains (Fig. 14) is the interesting feature.

8. POTENTIAL EXTENSIONS.

The preceding suggests that the low-order population-energy model formulated and tested here can be used to establish "temporal-average" phase-space portraits for a specified historical period. As is shown in the analysis leading to Figures 13 and 14, these "average" portraits can undergo dramatic transformations over time. The appearance of physically unattainable equilibria points is a particularly interesting aspect for it may well tell us something about the model as well as about certain features of the national economies considered.

One possible and pertinent use of this methodology could be a comparative analysis of national population-energy patterns for specified time intervals using the temporal average portraits as indicators. The distinction between "prediction" and "descriptions" seems most pertinent here.

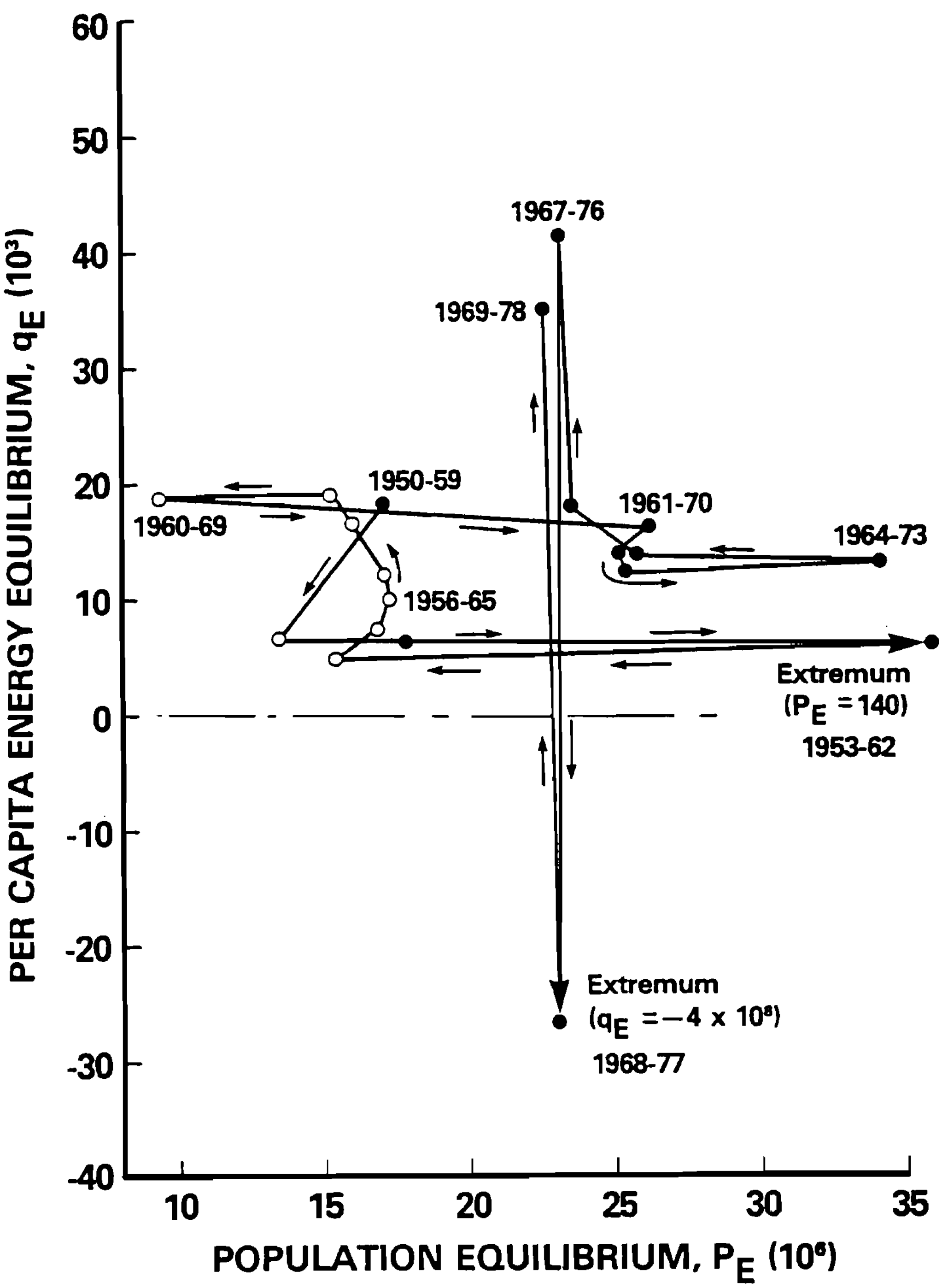

Figure 13: Illustration showing movement of equilibrium point (P_E,q_E) based on 10-year sliding period. The open circles characterize elliptic P-q phase-portraits while the solid circles characterize hyperbolic P-q phase-portraits. Data is for Canada, 1950-1978.

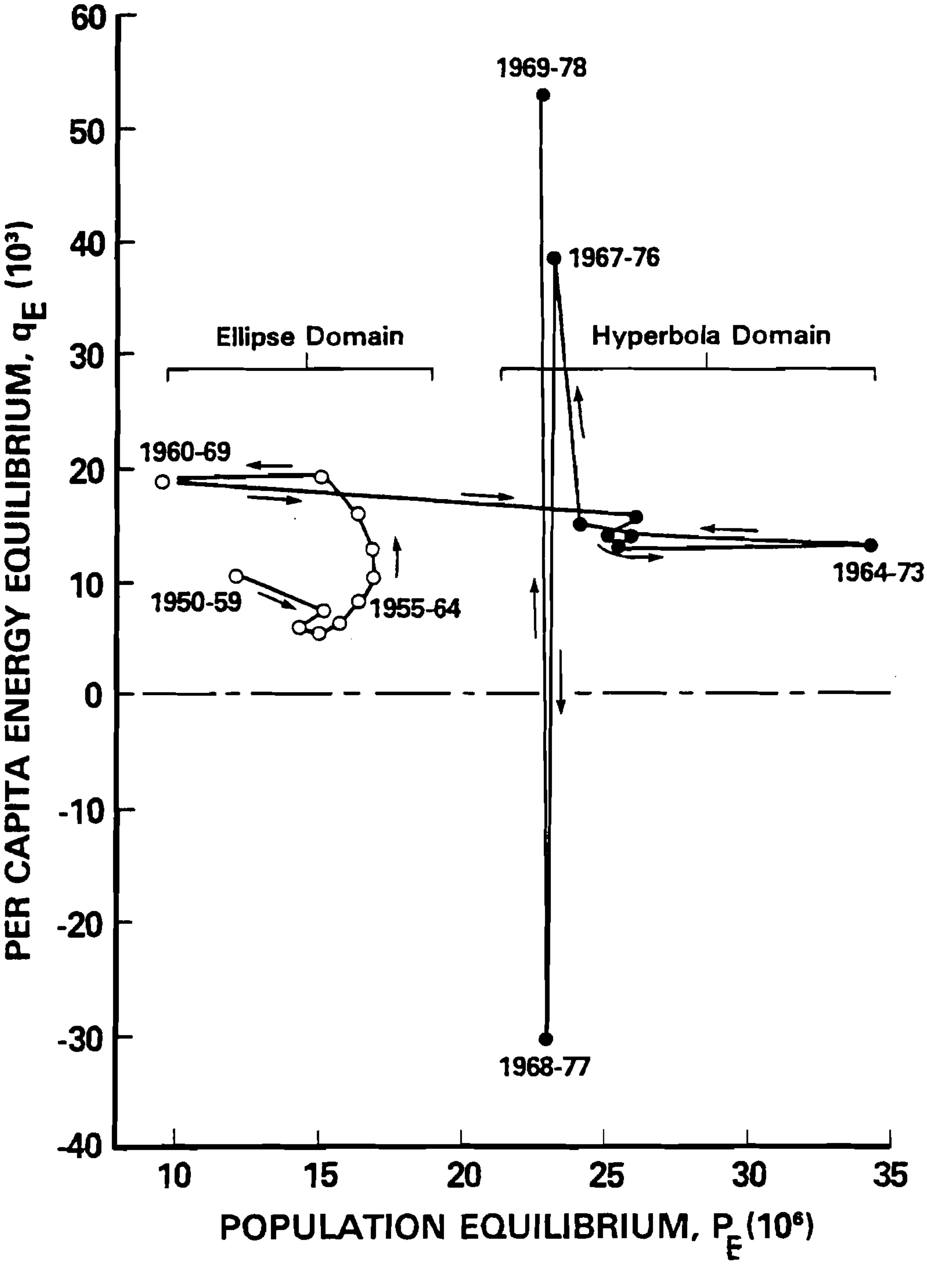

Figure 14: Movement of equilibrium point (P_E, q_E) with the "exceptional" years 1951, 1956, and 1975 removed.

More substantially, the phase-portraits might be used to identify trajectories of future national trends and tendencies. Two ways seem possible. One particular approach would be to obtain a coefficient-mapping for different societal domains by way of characterizing the coefficients--which are evidently slow functions of time--in terms of other determining factors. The other approach would involve weighting of the historical data with more weight on more recent data and less weight on more distant historical information.

Finally, one may wish to remove the linearity imposition of Equ.(1) and Equ.(5) and add higher order terms recognizing, however, that the coefficients added may introduce a further degree of complexity.

9. CONCLUDING COMMENT.

It appears that the dynamics of national population-energy characteristics can be investigated by using methodologies that are based on the formulation of coupled low-order equations. The model developed and the results established here suggest some new directions for further research.

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