SIMPLIFIED MULTIPLE CONTINGENCY CALCULATIONS

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FOREWORD

The mathematical models of multiregional demography, adopted by the former Migration and Settlement Task, have been found to be applicable to a wide range of demographic topics. In this paper, the authors illustrate the application of these multidimensional models to multiple contingency calculations used in actuarial practice.

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ABSTRACT

This paper expresses a number of standard life contingency formulas in matrix form, thereby generalizing them to include multiple contingency situations such as moves in and out of employment, insurance, marriage, sickness, and retirement.

CONTENTS

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INTRODUCTION

To proceed from life contingencies involving a single decrement of death to the many applications with more than one decrement has customarily involved special treatment for each case. Disability insurance, withdrawals from a life insurance plan, pensions, multiple lives, each have their own section in Jordan (1975) and other actuarial texts, with distinct notations and formulas. Special approximations are introduced for each, some of them awkward. The purpose of this paper is to show how all such multiple contingencies, along with marriage, labor force, and other tables treated by demographers, may be handled as special cases of a theorem due to Kolmogorov, with simple matrix formulae, uniform calculations, and consistent approximations.

The simplification begins by expressing all instances as individuals moving from one state to another with specified probability. Thus a person moves in and out of employment, in and out of insurance, of marriage, of sickness, of retirement. As long as the probability of moving is known and depends only on the state in which one is at the time, we can ask and answer questions on the annual premium for disability insurance (the premium to be paid only while the person is working); the premium for a pension to a widow if her husband predeceases her (that premium to be paid while the husband is working). These and many other premiums are readily calculable, along with the reserve at each duration of the policy. We are by no means restricted to moves in one direction only; a person can be employed this year, retired next year, and come back into the labor force again the following year. Thus we handle not only decrement tables, but also the more general increment-decrement tables (Schoen and Land 1978).

The formulas that turn up will be simple matrix analogues of those familiar in ordinary life contingencies. As such they not only are easily remembered, in contrast to much of the usual theory of multiple decrement, but also are readily computed. Approximations to handle data given in one-year or five-year intervals are straightforward. To make all these advantages accessible, we need the matrix differential equation due to Kolmogorov that generalizes the familiar

$$\frac{d\ l(x)}{dx} = -\ \mu(x)\ \ l(x) \tag{1}$$

whose solution is

$$l(x) = \exp\left(-\int_0^x \mu(a)\ da\right) \tag{2}$$

where $\mu(x)$ is the force of mortality and $l(x)$ the probability of surviving to age x.

To generalize we deal not only with the movement from life to death represented by the scalar rate $\mu(x)$ but with the matrix $\underset{\sim}{\mu}(x)$, standing for the instantaneous rates of movement between any pair of states that are to be included in the model—between life and death, between two regions of a country, between work and unemployment, between being married and being divorced, between work and disability. Once the basic rates (strictly forces or intensities) of movement for the $\underset{\sim}{\mu}(x)$ matrix are known, all else can be found: the chance that a man of 30 who is married will be alive 10 years later and divorced or the chance that a blue-collar person of 25 will be alive and doing white-collar work 30 years later.

The $\underset{\sim}{\mu}(x)$ matrix is to be constructed from actual data to a suitable approximation. The off-diagonal elements of $\underset{\sim}{\mu}(x)$ are each the corresponding observed rate of movement in a small time interval with sign reversed. Thus $-\mu_{ij}(x)dx$ is minus the chance that a person in state j transfers to state i during the short period of time and age dx. Each diagonal element of $\underset{\sim}{\mu}(x)$ contains the rate $\mu_{\delta i}$ of dying, with positive sign, along with the total of the off-diagonal elements of the column, $\sum_{i\neq j}\mu_{ij}$, also with positive sign. The reason for this is that the column total has to be conservative—that is, to add to zero with respect to movements among units. In short, quantities from the jth state added into the ith state must be subtracted from the jth, so any increment to μ_{ij}, $i \neq j$, has to be subtracted from μ_{jj}. The net total of each column is the death rate.

The notation is indicated in Table 1, giving the matrix $\underset{\sim}{\mu}(x)$ in some detail. The right-hand subscript is state of origin, the left-hand subscript state of destination. Thus $\mu_{23}(x)$ is the movement from state 3 to state 2 for persons aged x. All other matrices of this article use the same subscripting, essentially that of Rogers (1975). The matrix $\underset{\sim}{M}(x)$ will be the observations that correspond to $\underset{\sim}{\mu}(x)$.

Table 1. Matrix $\underset{\sim}{\mu}(x)$ of Moves.

$$\underset{\sim}{\mu}(x) = \begin{bmatrix} \mu_{\delta 1}(x) + \sum_{i\neq 1}^{n} \mu_{i1}(x) & -\mu_{12}(x) & -\mu_{13}(x) & \cdots \\ -\mu_{21}(x) & \mu_{\delta 2}(x) + \sum_{i\neq 2}^{n} \mu_{i2}(x) & -\mu_{23}(x) & \cdots \\ -\mu_{31}(x) & -\mu_{32}(x) & \mu_{\delta 3}(x) + \sum_{i\neq 3} \mu_{i3}(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Identical with Equation (1), except that the elements are now matrices and vectors, is the basic

$$d\{\underset{\sim}{l}(x)\}/dx = -\underset{\sim}{\mu}(x)\{\underset{\sim}{l}(x)\} \qquad (3)$$

which is due originally in this application to Kolmogorov (Krishnamoorthy, 1978; Willekens, 1978). Here $\{\underset{\sim}{l}(x)\}$ is a vertical vector in which the ith element is the number of the population surviving and in the ith category at age x. In general, where people are going in and out of the several categories, we cannot say that the elements of $\{\underset{\sim}{l}(x)\}$ represent probabilities, yet probabilities are what we seek. Now suppose that in the small interval of time and age dx no one will be affected by more than one event. We would like to pass from $\underset{\sim}{\mu}(x)$ and the vector $\{\underset{\sim}{l}(x)\}$ to a matrix $\underset{\sim}{l}(x)$ whose typical element is $l_{ij}(x)$, the chance that a person born in the jth state will be in the ith state by age x.

The theory for doing this is available from standard works on linear differential equations (Coddington and Levinson, 1955; Gantmacher, 1959, vol. 2, p. 113). If there are n states, and so the matrix $\underset{\sim}{\mu}(x)$ is n X n, and if the n latent roots of that matrix are distinct, then there will be n linearly independent vectors $\{\underset{\sim}{l}(x)\}$ that satisfy Equation (3). When this is so, the matrix made by setting those vectors side by side will obviously also satisfy the equation, and it can be shown to be the complete solution. Call $\underset{\sim}{l}(x)$ the matrix made up of the several $\{\underset{\sim}{l}(x)\}$. We shall see how to obtain the elements of $\underset{\sim}{l}(x)$ so as to ensure that the ijth element is the probability that a person born in the jth category finds himself in the ith category by age x.

THE MULTIPLICATIVE PROPERTY

One mathematical property of the $\underset{\sim}{l}(x)$ will be important for the demographic application: its multiplicativity. It may be shown (though not here) that if the interval from zero to y is broken into two subintervals at any point, say $x < y$, then (Gantmacher, 1959, vol. 2, p. 127)

$$\underset{\sim}{l}(y) = \underset{\sim}{l}(y|x)\ \underset{\sim}{l}(x) \tag{4}$$

where the ijth element of $\underset{\sim}{l}(y|x)$ will in our interpretation mean the probability of being in the ith state at age y, given that the person was in the jth state at age x. When y is one or five years more than x then within the interval (x,y) our $\underset{\sim}{\mu}(x)$ may be approximated by a matrix whose elements are constants independent of age. This will be the key to the numerical solution of Equation (3).

If in the interval x, $x + h$, $\mu_{ij}(x)$ is constant, say M_{ij}, for all i and j, and $\underset{\sim}{M}_x$ is the array of the M_{ij}, then from property (4) we can write

$$\underset{\sim}{l}(x + h) = e^{-h\underset{\sim}{M}_x}\ \underset{\sim}{l}(x) \quad . \tag{5}$$

With an arbitrary radix $\underset{\sim}{l}(0)$, Equation (5) permits the construction of $\underset{\sim}{l}(x)$ step by step at intervals of h all the way to the end of life. A somewhat better approximation is obtained by multiplying both sides on the left by $e^{h\underset{\sim}{M}_x/2}$, expanding to the term in h, and multiplying by $(\underset{\sim}{I} + h\underset{\sim}{M}_x/2)^{-1}$ on the left, to obtain

$$\underset{\sim}{l}(x + h) = (\underset{\sim}{I} + h\underset{\sim}{M}_x/2)^{-1}\ (\underset{\sim}{I} - h\underset{\sim}{M}_x/2)\ \underset{\sim}{l}(x) \tag{6}$$

The approximation (6) is close enough for many kinds of data with intervals of one year or even five years. It can be improved by graduating the original data down to tenths of a year or smaller, and this was essentially what Oechsli (1971, 1975) did, using spline functions.

PROBABILITIES OVER LONG INTERVALS

The next obvious question to ask is: what is the probability that a person in the jth state at age x will find himself in the ith state at age y, where the difference $y - x$ need not be small? Without matrix methods the problem is difficult; it has to take account of not only movement out of the jth state but also movement into the ith state of persons not in the jth state at age x. The multiplicative property tells us that if $\underset{\sim}{l}(y|x)$ is the desired set of probabilities, we know that $\underset{\sim}{l}(y|x)\ \underset{\sim}{l}(x) = \underset{\sim}{l}(y)$, so multiplying on the right by $\underset{\sim}{l}^{-1}(x)$ we get

$$\underset{\sim}{l}(y|x) = \underset{\sim}{l}(y)\ \underset{\sim}{l}^{-1}(x) \qquad (7)$$

where the probability of going from the jth state at age x to the ith state at age $y > x$ is the jth element of the ith row of $\underset{\sim}{l}(y|x)$.

EXPECTED TIME IN THE SEVERAL STATES

Beyond probabilities we would like to know the expected time lived between age x and $x + h$ in the several states, where in the first instance h is small. A straight-line approximation gives, as time of residence in the ith state for those initially in the jth state, the ijth element of the matrix

$${}_h\underset{\sim}{L}_x = (h/2)(\underset{\sim}{l}_x + \underset{\sim}{l}_{x+h})$$

and a cubic gives

$${}_h\underset{\sim}{L}_x = (13h/24)(\underset{\sim}{l}_x + \underset{\sim}{l}_{x+h}) - (h/24)(\underset{\sim}{l}_{x-h} + \underset{\sim}{l}_{x+2h}) \qquad (8)$$

Adding ${}_h\underset{\sim}{L}_x$ estimates person-years over any interval of age large or small. Cumulating ${}_h\underset{\sim}{L}_x$ back from the end of the table gives the expected years in the ith state from age x to the end of life measured prospectively from birth in the jth state:

$$\underset{\sim}{T}(x) = \sum_{x}^{\omega-h} \underset{\sim}{L}(a) = \int_{x}^{\omega} \underset{\sim}{l}(a)\,da$$

For an individual just born in the jth state, the probability of being in the ith state by age x is the ijth element of $\underset{\sim}{l}(x)$. And if the ikth element of the matrix $\underset{\sim}{\mathring{e}}(x)$ is the expected number of years beyond age x in the ith state for those who survive to the kth state by age x, we must have

$$\underset{\sim}{T}(x) = \underset{\sim}{\mathring{e}}(x)\ \underset{\sim}{l}(x) \tag{9}$$

which gives for the jth state at birth the number of years for which the ith state will be occupied after age x.

Consider, for example, those in the second state at birth and let us find their expectation beyond age x in the first state. The second column of $\underset{\sim}{l}(x)$ gives the chance that the person born in the second state is in the first, the second, and so forth, state at age x. If residing in the first state at age x, he has an expected $\mathring{e}_{11}(x)$ in the first state; if residing in the second, he has an expected $\mathring{e}_{12}(x)$ in the first; and so on. In short his total expectation in the first state, given that he was born in the second, is prospectively from age zero

$$\mathring{e}_{11}(x)\ l_{12}(x) + \mathring{e}_{12}(x)\ l_{22}(x) + \mathring{e}_{13}(x)\ l_{32}(x) + \cdots$$

For the whole collection of states, we have

$$\underset{\sim}{\mathring{e}}(x)\underset{\sim}{l}(x) = \begin{bmatrix} \mathring{e}_{11}(x) & \mathring{e}_{12}(x) & \mathring{e}_{13}(x) & \cdots \\ \mathring{e}_{21}(x) & \mathring{e}_{22}(x) & \mathring{e}_{23}(x) & \cdots \\ \mathring{e}_{31}(x) & \mathring{e}_{32}(x) & \mathring{e}_{33}(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} l_{11}(x) & l_{12}(x) & l_{13}(x) & \cdots \\ l_{21}(x) & l_{22}(x) & l_{23}(x) & \cdots \\ l_{31}(x) & l_{32}(x) & l_{33}(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Note that here as in other expressions indexes are read right to left in order to use column vectors and the conventional subscripting of matrix elements.

Dividing Equation (9) by $\underset{\sim}{l}(x)$ on the right, we have for the expectation in the ith state for a person in the jth state at age x the ijth element of

$$\underset{\sim}{\mathring{e}}(x) = \underset{\sim}{T}(x)\ \underset{\sim}{l}^{-1}(x) \tag{10}$$

The recursive formulas familiar in the scalar life table are also available in matrix terms. Thus one way of building up the expectation of life is to start at the end of life and work backwards using

$$\underset{\sim}{\mathring{e}}(x) = \underset{\sim}{L}(x)\ \underset{\sim}{l}^{-1}(x) + \underset{\sim}{\mathring{e}}(x + 1)\ \underset{\sim}{l}(x + 1)\ \underset{\sim}{l}^{-1}(x)$$

MARITAL STATUS

Schoen and Nelson (1974), Krishnamoorthy (1978), and Willekens et al. (1980) have presented multidimensional increment-decrement life tables for women by marital status, recognizing the states of never married, married, widowed, and divorced. Working with data for the U.S. in 1970, Krishnamoorthy (1979) finds, for example, that the chance of a child just born being in the never married status at age 50 is 0.034; of being married 0.727; of being widowed (and not remarried) 0.067; and of being divorced (and not remarried) 0.090. At birth, expected number of years in the never married state is 22.56; married (including remarriages) 32.27; widowed 10.22; divorced 4.71.

Repeating Krishnamoorthy's calculations with data on Belgian women in 1970-71, Willekens et al. (1980) find some differences in the corresponding probabilities and durations. The probability that a baby girl just born is in the never married state by age 50 is 0.034; married 0.827; widowed 0.043; and divorced 0.030. Expected number of years, at birth, in the never married state is 23.27; married 40.83; widowed 8.81; divorced 1.36. An important finding is that Belgian women, with a divorce rate one fourth that of the United States, exhibit similar proportions of remarriages among divorcees and a much lower one among widows (14 percent versus two percent). Patterns of first marriage are very similar in the two countries, with about 94 percent of the girls born eventually marrying after spending roughly 0.31 of their lifetime in the single (never married) state.

LABOR FORCE STATUS

A standard application of the life table concept is to tables of working life (Wolfbein, 1949). Recently, Hoem and Fong (1976), Willekens (1978), and Schoen and Woodrow (1980) have demonstrated the superiority of the multidimensional approach. Using Danish data for 1972-74 Hoem and Fong (1976), for example, find that the chance of a baby boy just born being employed at age 20 is 0.909, and that the average number of years he can expect to be working during his entire lifetime is 40.98.* Using the same data Willekens (1978) calculates that a male unemployed at age 20 can expect to experience about 10.11 years of unemployment, whereas an employed male of the same age can expect the somewhat lower total of 8.04 years.

Repeating Willekens' calculations for females shows significant differences. The average number of future years spent in the employed state by a working 20-year-old woman is 29.82, about 11 years less than the corresponding male figure. The difference at birth is one year less and contrasts markedly with the 15 year difference in the U.S. reported by Schoen and Woodrow (1980), who comparing the conventional and the multidimensional approaches, conclude that although both give rather similar results for the proportion of total lifetime spent in the labor force, the former conceals a significant amount of movement into and out of the employed state.

> While the conventional table shows just under one entry to the labor force per male born, the increment-decrement table shows a lifetime average of 3 1/4 entries (Schoen and Woodrow, 1980, p. 319).

The usual table gratuitously simplifies reality in permitting movement in only one direction—from never having worked into the labor force and from the labor force into permanently retired.

Based on these mathematical and demographic preliminaries, financial calculations can be made as a direct application.

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*The data in Hoem and Fong (1976) refer to active and inactive statuses, which are not equivalent to being employed and unemployed. Nevertheless, for expositional convenience we shall use the two sets of terms interchangeably.

ANNUITIES

A life annuity of one unit per annum, payable continuously for the rest of life to a person now aged x, has the present value

$$a(x) = \frac{\int_0^{\omega - x} e^{-\delta t} \; l(x + t)dt}{l(x)} \tag{11}$$

where δ is the force of interest, i.e., the nominal annual rate of interest compounded continuously, and $l(x + t)/l(x)$ is the probability that a person now aged x will be alive t years later. Thus $a(x)$ is the present value of the annuity payable irrespective of whether the person is working or not working, in good health or bad, living in a certain region or not. It is helpful to write $a(x)$ in the form

$$a(x) = \frac{N(x)}{D(x)} = \frac{\int_x^{\omega} e^{-\delta t} \; l(t)dt}{e^{-\delta x} \; l(x)}$$

since $N(x)$ and $D(x)$ need be calculated only once for each x.

Now suppose that the person is to be paid the annuity only if he is alive and in a certain state—say, sick, retired, or unemployed. We assume as usual that the condition can be defined precisely and that it is not affected by the annuity. If, for example, the annuity is large enough that an appreciable number of people will change their residence in order to obtain it, then place of residence is not a suitable condition. If the annunity is payable only when the person is sick, then there must be some criterion of the sick condition. In application we will have in mind the value of the annuity that the person pays to the company or government in the form of a premium or tax while he is working, and the different annuity that the person receives while disabled or retired.

Let us then suppose that a person is now aged x and in the jth state, and that the annuity is payable only while he is alive and in the ith state. The expected future lifetime in state i for a person initially in state j, is provided by the ratio of two matrices given as (10) above

$$\underset{\sim}{\mathring{e}}(x) = \underset{\sim}{T}(x) \; \underset{\sim}{l}^{-1}(x)$$

If money carries no interest, (10) is the answer to the annuity problem.

If money carries interest at rate δ then we need to modify the matrix $\underset{\sim}{T}(x)$ by multiplying all of the $l_{ij}(x + t)$ of which it is made up by $e^{-\delta t}$. The discounted matrix $\underset{\sim}{l}(x)$, with elements $e^{-\delta x} \; l_{ij}(x)$, can be designated $\underset{\sim}{D}(x)$, and the corresponding cumulative integral

$$\underset{\sim}{N}(x) = \int_0^{\omega - x} e^{-\delta(t+x)} \underset{\sim}{l}(x + t)dt \qquad (12)$$

takes the place of the undiscounted $\underset{\sim}{T}(x)$. We have followed the usual actuarial form of discounting everything back to age zero. Then we have for the value of an annuity, payable to or by a person age x initially in state j, but only while he is in state i, with money discounted at rate of interest δ, the ijth element of

$$\underset{\sim}{a}(x) = \underset{\sim}{N}(x) \; \underset{\sim}{D}^{-1}(x) \qquad (13)$$

A common arrangement is a life annuity plus a condition that the annuity continues for ten (or some other number) years after the policy is taken out. This means in effect an annuity certain for ten years plus a deferred annuity beyond that. We would have as the initial value per payment of unity per annum:

$$\int_0^{10} e^{-\delta a} \, da + \underset{\sim}{N}(x + 10) \; \underset{\sim}{D}^{-1}(x) = \frac{1 - e^{-10\delta}}{\delta} + \underset{\sim}{N}(x + 10) \; \underset{\sim}{D}^{-1}(x)$$

In practice annuities and insurance are covered not by single payments but monthly or annually. Let us suppose continuous payments (any other form can easily be approximated from this) and that payments are made while the person is in state i and an annuity received while the person is in state h. In the obvious applications i will be working and h retired; or else i in good health and h in hospital; or i working and h unemployed. Now a premium of unity from a person in the jth state at age x, payable as long as he is in the ith state, has an initial discounted value given by the ijth element of the matrix $\underset{\sim}{a}(x) = \underset{\sim}{N}(x)\ \underset{\sim}{D}^{-1}(x)$, so a premium of p_{hi}, say, will have an initial discounted value equal to the ijth element of

$$p_{hi}\ \underset{\sim}{a}(x) = p_{hi}\ \underset{\sim}{N}(x)\ \underset{\sim}{D}^{-1}(x)$$

The benefit is another annuity, say of unity, paid each year during tenure in the hth state. For a person in the jth state this must have a present value of the hjth element of $\underset{\sim}{a}(x)$, say $a^{(hj)}(x)$. Then the required premium is obtained by equating payment and benefit, i.e.,

$$p_{hi}\ a^{(ij)}(x) = a^{(hj)}(x) \tag{14}$$

or

$$p_{hi} = a^{(hj)}(x)\ /\ a^{(ij)}(x) \tag{15}$$

Other forms are straightforward enough to be written out immediately. Thus a single-payment endowment, on the life of a person in the jth state, due at age $x + h$, if the person is then in the ith state, is the ijth element of the matrix

$${}_h\underset{\sim}{E}_x = \underset{\sim}{D}_{x+h}\ \underset{\sim}{D}_x^{-1}$$

It is often desired to pay only for a certain term at most, without changing the benefit, as in a 20-payment life policy. A 20-payment life policy for disablement taken out by a person in state j, with premiums payable in state j and with benefits receivable in state i, would be the ijth elements of $\underset{\sim}{N}(x)$ divided by the jjth element of $\underset{\sim}{N}(x) - \underset{\sim}{N}(x + 20)$.

As the simplest of the innumerable examples possible, consider the table of working life calculated by Frans Willekens (1979). His expectations matrix, recognizing only working as state 2 and not working as state 1, is

$$\overset{\circ}{\underset{\sim}{e}}(17) = \begin{bmatrix} 10.09 & 8.71 \\ 41.59 & 42.97 \end{bmatrix}$$

Thus a male employed at age 17 would have an expected 8.71 years unemployed; one unemployed at age 17 would have an expected 10.09 years of unemployment. If we discount throughout at three percent per annum, then the same matrix becomes

$$\underset{\sim}{a}(17) = \begin{bmatrix} 4.54 & 3.22 \\ 21.28 & 22.60 \end{bmatrix}$$

so the payment of $1 per year while the person is unemployed would have a prospective cost of $3.22 for a person initially employed and of $4.54 for a person initially unemployed.

BENEFITS CERTAIN

People who pay for a pension over many years wish to avoid the contingency that they might die early in the benefit period, or indeed before it starts, and thus seem to obtain little or none of what they have paid for. To avoid the appearance of a loss, they prefer to have a term certain of benefit that would go to their beneficiaries and that usually adds little to the premium. This also can be readily worked out for the general multi-dimensional case.

FUNCTIONS OF MULTIPLE LIVES

If both father and mother have jobs they may wish to insure their children against the possibility that both fall sick or both die. Matrix expressions can be worked out for such cases.

A GENERALIZED FORM OF INSURANCE

In ordinary insurance practice where living and dead are the only states recognized, the value of an assurance of unity on a life now aged x is

$$A(x) = \frac{\int_x^\omega e^{-\delta t} \mu(t) l(t) dt}{e^{-\delta x} l(x)} = \frac{M(x)}{D(x)}$$

We proceed to the general case of insuring the person against passing from the ith to the hth state, given that he is of age x and in the ith state at the time of the insurance. The probability discounted back to time x that the transition will take place at time y is the hith element in

$$- \underset{\sim}{\bar{\mu}}(y) \underset{\sim}{l}(y) \underset{\sim}{l}^{-1}(x) e^{-\delta(y-x)} dy \tag{17}$$

where $\bar{\mu}^{(hi)}(y)$ is the probability of transition from i to h, given that the person has arrived at state i by age y.* This has to be added through all $y > x$. Defining $\underset{\sim}{M}(x)$ as the sum

$$\underset{\sim}{M}(x) = \int_x^\omega \underset{\sim}{\bar{\mu}}(y) \underset{\sim}{l}(y) e^{-\delta y} dy$$

there being no danger of confusion with the quite different $\underset{\sim}{M}(x)$ matrix containing average rates in a small finite interval, we have for the single payment

*The numerical approximation of the matrix $\underset{\sim}{\bar{\mu}}(y)$ that has been used is $\underset{\sim}{R}(y)\underset{\sim}{L}(y)\underset{\sim}{l}(y)^{-1}$, where $\underset{\sim}{R}(y)$ is a diagonal matrix with interstate transition rates or death rates in the diagonal.

$$\underset{\sim}{A}(x) = \underset{\sim}{M}(x)\ \underset{\sim}{D}^{-1}(x) \tag{18}$$

The insurance of unity taken out by a person in the ith state aged x, against arriving into the hth state, is worth the hith element of $\underset{\sim}{A}(x)$. To allow for insurance against the hth or the kth state we would add two such expressions.

If the insurance is to be paid for by an annuity then to find the premium we must divide the required element of $\underset{\sim}{A}(x)$ by an element of $\underset{\sim}{a}(x)$. Suppose the person aged x and now in the ith state, and insured against passage from the ith to the hth state, to be covered by equal payments while he is in the kth state. Then the annual premium is

$$A^{(hi)}(x)\ /\ a^{(ki)}(x) \tag{19}$$

A combined insurance and endowment provides for the payment of a sum to the beneficiary if the person dies within 10 years, and to the person if he survives to the end of the 10-year period. The present value of such a policy taken out in state j, and insured against falling into state i, is the ijth element of

$$\left[\underset{\sim}{M}(x) - \underset{\sim}{M}(x + 10) + \underset{\sim}{D}(x + 10)\right] \underset{\sim}{D}^{-1}(x)$$

In another type of policy the insured gets his premiums back without interest if he survives and keeps out of state i for the 10 years. If the premium is p then we have to solve

$$p\left[\underset{\sim}{N}(x) - \underset{\sim}{N}(x + 10)\right] \underset{\sim}{D}^{-1}(x) = \left[\underset{\sim}{M}(x) - \underset{\sim}{M}(x + 10) + 10p\ \underset{\sim}{D}(x + 10)\right]\underset{\sim}{D}^{-1}(x)$$

except that this is not to be interpreted as a matrix equation, but only as the jjth element on the left for $\underset{\sim}{D}(x + 10)\ \underset{\sim}{D}^{-1}(x)$, and as the ijth element for the $\underset{\sim}{M}$'s. Thus we would write $n^{(jj)}(x)$ for the jjth element of $[\underset{\sim}{N}(x) - \underset{\sim}{N}(x + 10) - 10\underset{\sim}{D}(x + 10)]\underset{\sim}{D}^{-1}(x)$, and $m^{(ij)}(x)$ for the ijth element of $[\underset{\sim}{M}(x) - \underset{\sim}{M}(x + 10)]\underset{\sim}{D}^{-1}(x)$, and the premium is

$$p = m^{(ij)}(x) / n^{(jj)}(x)$$

RESERVES

The contingent liability, say when the person has moved along from age x to age y, is available for all such policies. For a single premium annuity taken out at age x and state j, and payable when the person is in state i, first suppose it is known that the person has moved to the kth state. The reserve needed will be the ikth term of

$$\underset{\sim}{V}(y) = \underset{\sim}{a}(y) = \underset{\sim}{N}(y)\ \underset{\sim}{D}^{-1}(y) \qquad (20)$$

If one does not have the information on what the present state of the individual is then the reserve might take account of the probability of the person being in the kth state at age y, given that he was in the jth state at age x, i.e., the kjth element of

$$\underset{\sim}{l}^{*}(y|x) = \underset{\sim}{l}^{*}(y)\ \underset{\sim}{l}^{*-1}(x)$$

where the asterisk * means that the $\underset{\sim}{l}$ matrix has been recalculated omitting the contingency of death—we know at least that the person is alive. The reserve is now

$$\underset{\sim}{V}(y|x) = \underset{\sim}{N}(y)\ \underset{\sim}{D}^{-1}(y) \left[\underset{\sim}{l}^{*}(y|x)\right] \qquad (21)$$

In practice it is not of great interest to weight according to the proportions $\underset{\sim}{l}^{*(ki)}(y|x)$ that would be in the several states k, given that they were in a certain distribution of state j when they took out the policy. Since we do not care about those who dropped their policies prior to the time of valuation, it is better to weight the several states k according to the business in hand at the time the reserve is being calculated.

Annuity calculations (13) or (20) give the value of a policy that pays one unit per year of unemployment. For example, if money carries no interest and if the policy is issued to an employed 20-year-old Danish male, then 8.04 units is the value of the policy. If the interest rate is three percent, the value declines to 2.84 units and increasing the rate to six percent lowers the value of the policy to 1.40. We may calculate the reserve still needed to cover the expected payments to, say, a 30-year-old youth. Equation (21) gives the answer, and for the Danish data the figure is 2.54 units, if the prevailing rate of interest is three percent, and 1.04 units if it is six percent.

We may leave this somewhat improbable kind of policy and go on to one in which the payments are made by an annuity while the person is working and benefits obtained while the person is disabled or unemployed. In general to obtain the reserve that would be needed in respect of a person in state j at age $y > x$ we recall that the future payments are an annuity of p_{hi} with expected value per unit equal to the ijth term in $\underset{\sim}{a}(y)$, i.e., $p_{hi}\, a^{(ij)}(y)$ if the person is still in state j. The future benefits are an annuity of unity while a person now in state j is in state h, i.e., $\underset{\sim}{a}^{(hj)}(y)$. Hence the reserve must be

$$V(y|x) = a^{(hj)}(y) - p_{hi}\, a^{(ij)}(y)$$

APPROXIMATIONS

If precise information on transitions is available for single years of age, and in the right form, then no question of approximations need arise. Thus when we know that the probability that a man of exact age 31 who is now working will be unemployed one year later, along with similar information for all other possible transitions, then we need go no further in order to construct the $\underset{\sim}{l}$-matrix. For such ratios gives us directly what might be called a $\underset{\sim}{p}$-matrix covering each year separately, and the cumulative product of the $\underset{\sim}{p}$-matrices starting at the beginning of life or any other point, gives the probabilities of transitions over long periods of time.

Rarely is information available in such convenient form. At best we might have exposed population and deaths or other transitions over a period of a year. The ratio of cases of transition to exposure we have called the $\underset{\sim}{M}$ matrix, and to go from these to the $\underset{\sim}{l}$-matrix has been discussed above, culminating in (6). By integrating the $\underset{\sim}{l}$-matrix the stationary population, or $\underset{\sim}{L}$-matrix, can be calculated, and from that one can go on to expectations of time spent in the various states. By extension of our recursive formula for $\underset{\sim}{\mathring{e}}(x)$ we could estimate the integral for financial calculations in one-year intervals from the $\underset{\sim}{L}$-matrix,

$$\underset{\sim}{a}(x) = \underset{\sim}{e}^{-\delta/2}\, \underset{\sim}{L}(x)\, \underset{\sim}{l}(x)^{-1} + \underset{\sim}{\mathring{e}}^{-\delta}\, \underset{\sim}{a}(x+1)\, \underset{\sim}{p}(x) \qquad (22)$$

but this seems to add an unnecessary approximation to the process. What we should do instead is to multiply the $\underset{\sim}{l}$-matrix at each age by $e^{-\delta x}$, where i is the rate of interest, and then carry out the integration for $\underset{\sim}{N}(x)$ just as though we were finding expected times $\underset{\sim}{T}(x)$. When we calculate $\underset{\sim}{a}(x)$ for a man aged 40 not in the labor force on (22) we find the present value of a life insurance of unity at three percent to be 0.13295. The alternative more precise calculation is 0.13302 or about one part in 2000 higher.

CONCLUSION

To rewrite standard actuarial formulas in matrix terms is an extensive and potentially useful task. This paper points the way and does a small part of the work. Issues arise that we find challenging and have not yet resolved, including the retrospective valuation of policies in the light of various degrees of knowledge of what states the person has been through. Prospective valuation raises no such subtle issues. We have stopped short of any consideration of variance or of risk; our formulas are deterministic.

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