REVELATION OF INFORMATION IN A NASH EQUILIBRIUM

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CONTENTS

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1. INTRODUCTION

Consider an extensive game in which players have disparate information about moves of nature. The question is: To what extent is this revealed by them to one another in a Nash Equilibrium? Suppose (i) nature's moves are finite* (ii) players' moves and payoffs are "smooth," and each can observe some "non-degenerate signal" based on others' moves. Construct the fictitious game $\tilde{\Gamma}$ from the original Γ by modifying information conditions in the following way: a player observes nothing of others' moves in $\tilde{\Gamma}$ but, at the same time, finds out everything about nature's moves that could have been revealed to him in Γ via the signals. If the set of players is non-atomic, we show that generically (in the space

*This is tantamount, in our context, to: players' information partitions on nature's moves are finite of payoffs) the Nash outcomes of Γ coincide with those of $\tilde{\Gamma}$ (proposition 1). In this sense, information is "fully revealed" in a Nash Equilibrium (N.E.). This may be of use in proving the existence of N.E.'s in Γ, since they exist in $\tilde{\Gamma}$ under standard concavity assumptions on payoffs. Indeed, with concavity, a diluted version of this result carries over also to the case of finite players. Here we subject the N.E.'s of Γ to a sensitivity analysis on information. The idea behind this is to limit the fineness of observation of others' moves, by introducing a bound of size ε (for small ε), and is taken from [3]. Then we find that the N.E.'s which subsist in Γ are also fully revealing (proposition 2).

2. THE NON-ATOMIC CASE

Consider an extensive game with non-atomic players as described in [2]. To make our point succinctly, we shall work with a host of simplifying assumptions, some of which can be obviously generalized (see the remarks).

Let Y be the set of positions (nodes) in the game tree, $x_0 \in Y$ its root, and $Y^* \subset Y$ the subset of terminal positions. Define the l^{th} layer $Y_l \subset Y$, for $l = 1,2,\dots$, by:

$$Y_l = \{x \in Y : \ x \text{ is at a distance } l \text{ from } x_0\}$$

(By this distance we mean the number of arcs on the unique path from x_0 to x.) The player-set is the interval $[o,n)$ equipped with the Lebesgue measure. The subinterval $[i-1,i)$ corresponds to players of type i, $1 \leq i \leq n$. $N = \{1, \dots, n\}$ is the set of all types. 0 denotes a move of nature. For any $x \in Y \setminus Y^*, \pi(x) \in \{0,1, \dots, n\}$ indicates who is to move at x and $S(x)$ is the set of moves available. (Note that all players of any fixed type are placed symmetrically in the tree.) Assume that the game is *layered* as defined in [3], i.e.

(i) π and S are constant on $Y_l \subset Y \setminus Y^*$

In view of (i) we can talk of $\pi(l), S(l)$ associated with layer l. Let $L_i = \{l : \pi(l) = i\}$, $L_* = \bigcup_{i=1}^{n} L_i$ and further assume

(ii) the game has finite length, i.e., $Y_L = Y^*$ for some L,

(iii) $S(l)$ is finite if $l \in L_0$,

(iv) $S(l)$ is the unit simplex in $R_+^{k(l)}$ if $l \in L_*$, i.e., it is the convex hull of the unit vectors and the origin in $R^{k(l)}$.

Thus, if $l \in L_i$ and $i \in N$, the arcs out of any $x \in Y_l$ correspond to the set of all measurable functions from $[i-1,i)$ to the unit simplex $S(l)$.

Put $k = \prod_{l \in L_0} |S(l)|$ ($\prod$ is for product, $|\cdot|$ for cardinality). Since nature picks all of its moves (states) and players pick only one (at the nodes labeled by their type), any such choice produces* k paths from x_0 to Y^*, called an *outcome* of the game.

For simplicity we will assume that players can observe the integral of the moves picked at any position. (This can be relaxed — see remark II.) Then any outcome in the tree has a list of integrals associated with it, which will be called a ***signal***. These signals can be identified with $\underset{l \in L_*}{X} \left[S(l) \right]^{|C_l|}$, for $C_l = \underset{\substack{t \in L_0 \\ t<l}}{X} S(t)$, and viewed as a subset of Euclidean space R^q of dimension $q = \sum_{l \in L_*} |C_l| k(l)$ (where $|C_l|$ is understood to be 1 if the Cartesian product is over an empty set). For any outcome p in the tree, let $\varphi(p) \in \underset{l \in L_*}{X} \left[S(l) \right]^{|C_l|}$ be the signal produced by it; and for a player $t \in [i-1,i)$ of type i, let $\psi_t(p) \in \underset{l \in L_i}{X} \left[S(l) \right]^{|C_l|}$ be the vector of moves picked by t at positions on this play that are labeled by his type. Choose convex neighborhoods $\underline{S}(l)$ of $S(l)$ and denote $\underset{l \in L_*}{X} \left[\underline{S}(l) \right]^{|C_l|}$ by Σ, $\underset{l \in L_i}{X} \left[\underline{S}(l) \right]^{|C_l|}$ by Σ^i. Let U^i be the space** of all C^2 functions from $\Sigma^i \times \Sigma$

* We consider only those choices by players that are jointly measurable.

** Other spaces of payoff functions can also be considered. See remark I.

to the reals, endowed with the C^2-norm. Denote $U^1 \times \ldots \times U^N$ by U. Any point $u = (u_1, \ldots, u_n) \in U$ yields a payoff function $\prod_u^t$ to each player t by the rule: if $t \in [i-1, i)$, then $\prod_u^t(p) = u_i\left[\psi_t(p), \varphi(p)\right]$ for any outcome p.

To complete the description of the game we must specify the information partition I^i on $\bigcup_{l \in L_i} Y_l = Y(i)$ for each type i. This is accomplished in two steps. We will first describe an auxiliary partition E^i and then specify how it is refined to get I^i. For $l \in L_i$, a partition J_l of C_l is given which tells us i's *a priori* information about chance moves. Every $x \in Y_l$ has an $\alpha \in C_l$ linked to it on the unique path from x_0 to x. Thus J_l induces a partition E_l of Y_l in the obvious way: expand each $v \in J_l$ to $\{x \in Y_l : x$ is linked to some $\alpha \in v\}$. (If $C_l = \phi$ we take $E_l = \{Y_l\}$.) Putting together the E_l, for all l in L_i, we have E^i. E^i will need to be refined to express the fact that any t in $[i-1, i)$ can, in addition, observe the integral of moves picked at some of the layers that precede his turn. This too has to be specified exogenously. For $l \in L_*$, let $P(l)$ be some subset of $\{t \in L_* : t < l\}$. Interpret this to mean: when he is at layer l in L_i, any player of type i can find out the choices made previously at the layers in $P(l)$. Take any two x and y in Y_l. Say "$x \sim y \bmod P(l)$" if the unique paths from x_0 to x, x_0 to y have identical integrals associated with them at each layer in $P(l)$. Then "$\sim \bmod P(l)$" is an equivalence relation which yields a partition K_l of Y_l. Collecting the K_l, for all l in L_i, furnishes a partition K^i of $Y(i)$.

At last we are ready to define I^i. For any two partitions P_1 and P_2 of a set D, $P_1 \bigvee P_2$ is the coarsest partition of D which refines both P_1 and P_2. Then

$$I^i = E^i \bigvee K^i.$$

(For $l \in L_i$, the partition $K_l \bigvee E_l$ induced by I^i on Y_l will be denoted I_l.)

It will be useful to build another collection $\{\tilde{I}^i\}_{i \in N}$, related to $\{I^i\}_{i \in N}$. Fix layers t and l, $t<l$, and a partition Q_t of C_t. Then Q_t induces a partition Q_{tl} on C_l via the equivalence relation on C_l: $\alpha_1 \sim \alpha_2$ if α_1 and α_2 follow from the same set in Q_t. Define $\tilde{J}_l$, for all l in L_*, inductively as follows:

$$\tilde{J}_l = J_l \bigvee \left[\bigvee_{t \in P(l)} \tilde{J}_{tl} \right].$$

(Here $\bigvee_{t \in P(l)} \tilde{J}_{tl} = \{C_l\}$ if $P(l)$ is empty.) Expand each $\tilde{J}_l$ to a partition $\tilde{I}_l$ on Y_l as before. Then $\tilde{I}^i$ is obtained by putting together $\tilde{I}_l$, for all l in L_i. Given a choice of payoffs $u \in U$, we will look at the two games:

$\Gamma(u)$ with information partitions $\{I^i\}_{i \in N}$, and

$\tilde{\Gamma}(u)$ with information partitions $\{\tilde{I}^i\}_{i \in N}$.

Let G stand for any of $\Gamma, \tilde{\Gamma}$. A *strategy* of a player $t \in [i-1,i)$ in the game G consists of the choice of a move in $S(x)$ at every $x \in Y(i)$, subject to the constraint that these be identical at positions that he cannot distinguish in his information partition in G. Let $S^i(G)$ denote the set of all strategies of (any player of) type i. (Thus, for example, $S^i(\tilde{\Gamma}) = \bigtimes_{l \in L_i} \left[S(l) \right]^{|\tilde{I}_l|}$.)

A *choice of strategies* $s = \{s^t : t \in [o,n),\ s^t \in S^i(G) \text{ if } t \in [i-1,i)\}$ in the game G will be called *measurable* if it induces a measurable selection of moves at each position. If s is measurable, it gives rise to an outcome in the tree which we will denote by $p(s)$. Given s and $r^t \in S^i(G)$, $(s \mid r^t)$ is the same as $\dot{s}$ but with s^t replaced by r^t. Note that if s is measurable so is $(s \mid r^t)$, and thus our next definition makes sense.

A *Nash Equilibrium* (N.E.) of the game $G(u)$ is a choice of strategies s in G which satisfies, for all $t \in [o,n)$:

(a) s is measurable

(b) $\prod_u^t \left[p(s \mid r^t) \right] \le \prod_u^t \left[p(s) \right]$ for all $r^t \in S^i(G)$

(Here i is the type of t.) If s is an N.E. then $p(s)$ will be called a *Nash outcome* (or N.E. outcome).

Call a choice of strategies *type- symmetric* if it is constant on each $[i-1,i)$, and let $S^*(G)$ be the set of all such choices.* (Then $S^*(G) = S^1(G) \times \cdots \times S^n(G)$ in a natural way.) Denote by $\eta\left[G(u) \right]$ the set of type-symmetric N.E.'s of $G(u)$ and by $\gamma\left[G(u) \right] = \{p(s) : s \in \eta\left[G(u) \right]\}$ the set of Nash outcomes arising from $\eta\left[G(u) \right]$.

Let $P(G)$ be the set of all outcomes that arise from a type-symmetric choice of strategies in G. Suppose $z \in P(G)$ consists of the k paths $q_1, \ldots, q_k$. For $l \in L_*$, denote by $q_j(l)$ the move picked in q_j at layer l. Then it is clear from our definition of $\tilde{I}_l$ that

$$q_j,\ q_r \text{ pierce } v \in \tilde{I}_l \Rightarrow q_j(l) = q_r(l).$$

So, for $v \in \tilde{I}_l$, we can talk of the move $z(v)$ picked out by $z = (q_1, \ldots, q_k)$

*Note that any type-symmetric choice is automatically measurable.

at v. Say that z is *fully- revealing* if, for all $l \in L_{\bullet}$:

$$v_1, v_2 \in \tilde{I}_l;\ v_1 \neq v_2 \Rightarrow z(v_1) \neq z(v_2).$$

We will show that all z in $\gamma\left[\tilde{\Gamma}(u)\right]$ —equivalently* in $\eta\left[\tilde{\Gamma}(u)\right]$ — are generically fully revealing. This will be established, to begin with, for $\underline{\eta}\left[\tilde{\Gamma}(u)\right] = \{z \in \eta\left[\tilde{\Gamma}(u)\right]$: no player picks a vertex as a move in $z\}$ and we shall worry about vertices towards the end. Abbreviate $S^i(\tilde{\Gamma})$ for a while by S^i. S^i is a product of $|\tilde{I}^i|$ simplices. Partition S^i into $S^i(1), \ldots, S^i\left[r(i)\right]$ by choosing relative interiors of faces of each of these simplices. (Thus each $S^i(\cdot)$ is also a product of $|\tilde{I}^i|$ simplices.) For any n-tuple $\xi = \left[S^1\left[\alpha(1)\right], \ldots, S^n\left[\alpha(n)\right]\right]$, we will verify:

(1') There is an open dense set V_ξ of U with the property:

$$\left.\begin{array}{c} u \in V_\xi \\ z \in \underline{\eta}\left[\tilde{\Gamma}(u)\right] \\ z \in S^1\left[\alpha(1)\right] \times \cdots \times S^n\left[\alpha(n)\right] \end{array}\right\} \Rightarrow z \text{ is fully revealing}$$

The justification of (1') is most simply written in the case when each $S^i\left[\alpha(i)\right]$ picks the full face of each of the $|\tilde{I}^i|$ simplices of i's moves. (The general case involves some more notation but the identical argument.) List players' strategic variables:

$$\underbrace{x_1, \ldots, x_{m(1)}}_{\text{type 1}},\ \underbrace{x_{m(1)+1}, \ldots, x_{m(2)}}_{\text{type 2}}, \ldots\ldots, \underbrace{x_{m(n-1)+1}, \ldots, x_{m(n)}}_{\text{type } n}$$

Recall that the $\underline{S}(l)$ were neighborhoods of $S(l)_{\bullet}$. Put $Z = \underset{l \in L_{\bullet}}{X}\left[\underline{S}(l)\right]^{|\tilde{I}_l|}$ and note that Z is a neighborhood of $S^1(\tilde{\Gamma}) \times \cdots \times S^n(\tilde{\Gamma})$. Consider

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*In the case $G = \tilde{\Gamma}$ we can define $s \in S^*(\tilde{\Gamma})$ to be fully revealing in exactly the same way. Then s is fully revealing $\Longleftrightarrow p(s)$ is fully revealing.

$$U \times Z \overset{D}{\longrightarrow} R^{m(n)} \times Z$$

given by:

$$\Big[\underbrace{u_1 \,\ldots\ldots, u_n}_{u};\ \underbrace{\underbrace{y_1^1 \,\ldots\ldots, y_{m(1)}^1}_{y^1},\ \underbrace{y_{m(1)+1}^2 \,\ldots\ldots, y_{m(2)}^2}_{y^2} \,\ldots\ldots,\ \underbrace{y_{m(n-1)+1}^n \,\ldots\ldots\ y_{m(n)}^n}_{y^n}}_{y}\Big]$$

$$\longmapsto$$

$$\left[\frac{\partial \prod_u^1}{\partial x_1}(y^1,z)\,,\ldots\ldots,\frac{\partial \prod_u^1}{\partial x_{m(1)}}(y^1,z),\ \frac{\partial \prod_u^2}{\partial x_{m(1)+1}}(y^2,z)\,,\ldots\ldots,\ \frac{\partial \prod_u^2}{\partial x_{m(2)}}(y^2,z)\right.$$

$$\left.,\ldots\ldots,\ \frac{\partial \prod_u^n}{\partial x_{m(n-1)+1}}(y^n,z)\,,\ldots\ldots,\ \frac{\partial \prod_u^n}{\partial x_{m(n)}}(y^n,z);\quad y\right]$$

where z abbreviates $\varphi\Big[p(y)\Big]$. Note that, by assumption, no player can affect the integral, i.e., $\dfrac{\partial \varphi_k}{\partial x_j} = 0$ for any component k of φ and $1 \le j \le m(n)$; also $\dfrac{\partial y_k^i}{\partial x_j}$ is 1 if $k = j$ and is 0 otherwise.

For fixed $u \in U$, D_u will denote the restriction of D to Z, i.e., $D_u(y) = D(u,y)$.

The set $\{\, y \in Z : y$ is not fully revealing $\}$ is a finite union of submanifolds $M_1, \ldots\ldots, M_T$ of Z, where each M_t has positive codimension in Z. It is also clear that $\{z \in \underline{\eta}\Big[\widetilde{\Gamma}(u)\Big] \cap \Big[S^1(\alpha(1)) \times \cdots \times S^n(\alpha(n))\Big] : z$ is not fully revealing$\} \subset \bigcup_{t=1}^{T} D_u^{-1}\Big[\{0\} \times M_t\Big]$, where 0 is the origin of $R^{m(n)}$.

D is clearly transverse to every submanifold of its image. By the transversal density and openness theorems (see, e.g., 18.2 and 19.1 of [1]), there is an open dense set V_ξ in $(U)^N$ such that:

$u \in V_\xi \Rightarrow D_u$ is transverse to $\{0\} \times M_t$ for $t=1, \ldots, T$ at every $y \in B$.

(Here B is any compact set chosen to ensure that $B \subset Z$ and $S^1\left[\alpha(1)\right] \times \cdots \times S^n\left[\alpha(n)\right] \subset$ Interior of B.) But then if $u \in V_\xi$(think of D_u as restricted to the interior of B from now on):

$$\text{codim } D_u^{-1}\left[\{0\} \times M_t\right] = \text{codim } (\{0\} \times M_t) > m(n), \text{for } t=1, \cdots T.$$

Therefore $D_u^{-1}\left[\{0\} \times M_t\right]$ is empty for $u \in V_\xi$. This verifies (1'). With $V = \bigcap_\xi V_\xi$, we get:

(1) There is an open dense set V of U such that

$$\left.\begin{array}{c} u \in V \\ z \in \underline{\eta}\left[\tilde{\Gamma}(u)\right] \end{array}\right\} \Rightarrow z \text{ is fully revealing}$$

We still have to take care of vertices. Drop the requirement that S is constant on layer l. In fact require it to vary on C_l. If a player is at a node where he cannot distinguish between $\alpha_1, \ldots, \alpha_t$ in C_l, then say that the set of moves available to his type is* $S(\alpha_1) \cap \cdots \cap S(\alpha_t)$, and assume that these intersections are full-dimensional polytopes. If the $S(\alpha_i)$ are in "general position", vertices of these polytopes will be distinct at distinct elements of $\tilde{I}_l$. Thus vertices will automatically be "fully revealing." Furthermore, if we fix some moves (in the polytopes corresponding to elements of $\bigcup_{l \in L_*} \tilde{I}_l$) to be vertices, and let Z' be the manifold of the remaining moves, the set $\{z \in Z' : z$ is not fully revealing $\}$ will be a finite union of submanifolds of Z' of positive codimension. Then the argument used to establish (1) can be repeated, with Z replaced by Z', to show that N.E.'s of the type given by Z' are generically fully revealing. But Z' varies over

* More generally, it could be any full-dimensional polytype contained in $S(\alpha_1) \cap \cdots \cap S(\alpha_t)$.

a finite set. (It is defined by a choice of vertices and of faces of the remaining simplices of moves in $\widetilde{\Gamma}$.) So we have:

(2): same as (1) but with $\underline{\eta}$ replaced by η.

For any play z in $P(\Gamma)$ and $l \in L_*$ let us define the partition $I_l(z)$ on C_l which measures--so to speak--the information about nature's moves revealed by z. z induces $|C_l|$ paths from x_0 to Y_l. Denote them by $\{p_\alpha : \alpha \in C_l\}$, where p_α is the path linked to α. Say "$\alpha_1 \sim_z \alpha_2$" in C_l if the moves picked at layer t in p_{α_1} and p_{α_2} coincide for all $t \in P(l)$. Then "$\sim_z$" is an equivalence relation which produces a partition $K_l(z)$ of C_l. Set $G_l(z) = J_l \bigvee K_l(z)$. $G_l(z)$ generates a partition $I_l(z)$ on Y_l by expanding, as before, each $v \in G_l(z)$ to $\{x \in Y_l : x$ is linked to some $\alpha \in v\}$. Note

$$z \text{ fully revealing} \Rightarrow I_l(z) = \widetilde{I}_l \text{ for } l \in L_*$$

Let B be the collection of partitions $\left\{ \{ I_l(z) \}_{l \in L_*} : z \text{ is in } P(\Gamma) \right\}$. For $b = \{b_l\}_{l \in L_*} \in B$ denote by Γ_b the game with the information partition b_l of of Y_l. We claim, for any $b \in B \setminus \left\{ \{\widetilde{I}_l\}_{l \in L_*} \right\}$,

(3) There is an open dense set V_b of U such that, for every $u \in V_b$, $\gamma(u, b) = \{ z \in \gamma(\Gamma_b(u)) : \{I_l(z)\}_{l \in L_*} = b \} = \phi$.

To prove (3), fix $b = \{b_l\}_{l \in L_*}$. Let

$$l^* = \min\{l : b_l \neq \widetilde{I}_l\}$$
$$L^* = \{l \in L_* : l < l^*\}$$

Clearly $L^* \neq \phi$ (e.g. it always contains the first layer in L_*). Consider the map:

$$U \times A \xrightarrow{D'} R^m \times A$$

defined in the same way as D but now for the game Γ_b instead of $\tilde{\Gamma}$. (Thus $A = \underset{l \in L_*}{X} \left[\underline{S}(l) \right]^{|b_l|}$ and dim $A = m$ now). The set $\{x \in A : x$ is not fully revealing at *some* layer l in L^* $\}$ is a finite union of submanifolds $A_1, \ldots, A_J$ each of which has positive codimension in A. On the other hand,

$$\gamma(u, b) \subset \left\{ p(s) : s \in \bigcup_{j=1}^{J} D_u^{'-1} \left[\{0\} \times A_j \right] \right\} .$$

D' is transverse to every submanifold of its image, hence the sets in the union are generically empty, proving (3). (This argument again ignores vertices, which can be incorporated as explained earlier.)

Take $z = (q_1, \ldots, q_k)$ in $\gamma\left[\Gamma(u)\right]$, with $\{I_l(z)\}_{l \in L_*} = \{b_l\}_{l \in L_*} = b$. By the definition of $I_l(z)$ we get:

(4) q_j, q_r pierce $v \in b_l \Rightarrow q_j(l) = q_r(l)$.

By (4) we may define $\tilde{s}^i \in S^i(\Gamma_b)$ by

$\tilde{s}^i(v) =$ the common value of all $q_j(l)$ that pierce v.

(Here $v \in b_l$, $l \in L_i$.) Then $\tilde{s}^1, \ldots, \tilde{s}^n$ produce the outcome z, and (recalling the non-atomicity of the player-set) constitute an N.E. of Γ_b. In other words:

(5) $z \in \gamma\left[\Gamma(u)\right], \{I_l(z)\}_{l \in L_*} = b \Rightarrow z \in \gamma\left[\Gamma_b(u)\right]$.

Next take $\tilde{z} \in \gamma\left[\Gamma(u)\right]$ and suppose it is fully revealing. For $x \in Y(i)$ denote by $I^i(x)$ the information set in I^i that contains x. Define* $s^i \in S^i(\tilde{\Gamma})$ by:

$$s^i(v) = \begin{cases} q_j(l) \text{ if some } q_j \in \{q_1, \ldots, q_k\} = \tilde{z} \text{ pierces } v = I^i(x) \text{ at } x \\ \text{arbitrary, otherwise.} \end{cases}$$

*Since z is fully revealing, the definition makes sense.

Then $s^1, \ldots, s^n$ give rise to the play $\tilde{z}$ and, again using the non-atomicity of the player set, constitute an N.E. of Γ. Summing up:

(6) $\tilde{z} \in \gamma\left[\tilde{\Gamma}(u)\right]$, $\tilde{z}$ fully revealing $\Rightarrow \tilde{z} \in \gamma\left[\Gamma(u)\right]$.

Then (2), (3), (5), (6) (and the fact that B is finite) imply:

Proposition 1 Assume vertices to be in general position. Then there is an open dense set V in U such that:

$$u \in V \Rightarrow \begin{cases} \gamma\left[\Gamma(u)\right] = \gamma\left[\tilde{\Gamma}(u)\right] \\ \text{every } z \text{ in } \gamma\left[\Gamma(u)\right] \text{ is fully revealing.} \end{cases}$$

REMARKS

(I) It might be more natural to vary payoffs on the terminal nodes Y^* and to take the induced payoff of an outcome to be the expectation w.r.t. some fixed probability distribution $\beta: C_L \to R$, $C_L = \prod_{l \in L_0} S(l)$.

Proposition 1 remains true in this setting. For α in C_L and i in N put $H_\alpha^i = \underset{l \in L_i}{X} \underline{S}(l)$, $H_\alpha = \underset{l \in L_*}{X} \underline{S}(l)$. An outcome z produces, for any player $t \in [i-1,i)$, a point $\left[\Psi_\alpha^t(z), \psi_\alpha(z)\right]$ in each $H_\alpha^i \times H_\alpha$. Let U_i^α be the space of all C^2 functions on $H_\alpha^i \times H_\alpha$. $U_i^* = \underset{\alpha \in C_L}{X} U_i^\alpha$, $U^* = U_1^* \times \cdots \times U_n^*$.

For $u = (u_1, \ldots, u_n) \in U^*$, $u_i = \left\{u_i^\alpha\right\}_{\alpha \in C_L}$, and an outcome z put $\hat{\prod}_u^t(z) = \sum_{\alpha \in C_L} \beta(\alpha) u_i^\alpha \left[\Psi_\alpha^t(z), \psi_\alpha(z)\right]$. If we define the maps D, D' etc. using $\hat{\prod}_u^t$ in place of $\prod_u^t$, they are also easily checked to remain transverse to every submanifold of their images. Thus the same proof shows that proposition 1 is true when we replace U by U^* .

Various other spaces of payoff functions can be described which are "rich enough" to satisfy the transversality condition needed to give proposition 1.

(II) Signals were taken to be integrals for simplicity. More generally, let them depend on the measurable choice of moves at $x \in \bigcup_{l \in L_*} Y_l$ modulo null sets, i.e., two such choices that differ only on a null set yield the same signal. If we restrict to strategies in $S^*(\Gamma)$ (i.e., type-symmetric choices) then the signals produced by these at layer $l \in L_*$

may be represented by a map:

$$M_l = \underset{t \in P(l)}{X} \left[\underline{S}(t) \right]^{|\widetilde{I}_t|} \xrightarrow{\varphi_l} R^{q(l)}$$

Require that φ_l be smooth, and have full rank. Clearly the proof of Proposition 1 goes through. Even if φ_l failed to have full rank on submanifolds $M_l(1), \ldots, M_l\left[\beta(l)\right]$ of positive codimension in M_l, this would not matter since the finite N.E. set would generically miss $M_l(1), \ldots, M_l\left[\beta(l)\right]$ anyway. Finally, suppose that φ_l generically reveals ------- i.e., at each point in its domain, except possibly at a finite number of lower-dimensional submanifolds------- not all but some ***fixed subset*** of information (regarding chance moves) that existed in the layers in $P(l)$. Then $\widetilde{I}_l$ has to be defined accordingly for the results to go through.

(III) It is more natural to consider, instead of U^i, the space $\underline{U}^i$ of all C^2 maps on $D^i = \left[\underset{l \in L_i}{X} \left[S(l) \right]^{|C_l|} \right] X \left[\underset{l \in L_*}{X} \left[S(l) \right]^{|C_l|} \right]$. (A map is C^2 on D^i if it can be extended to a C^2 map on some neighborhood of D^i.) But if V is open dense in $U = U^1 \times \cdots \times U^n$ then $\underline{V} = \{(\underline{u}_1, \ldots, \underline{u}_n) : (u_1, \ldots, u_n) \in V\}$ is also open dense in $\underline{U}^1 \times \cdots \times \underline{U}^n$ (where $\underline{u}_i$ is the restriction of u_i to D^i). This follows from the well-known fact that:

(*) There is a $K>0$ such that : if $||u||_{D^i} \leq \varepsilon$ for any $u \in \underline{U}_i$, then there is an extension u^* of u from D^i to $\Sigma^i \times \Sigma$ with $||u^*||_{\Sigma^i \times \Sigma} \leq K\varepsilon$.

Thus defining payoffs on D^i would not affect the result.

(IV) We could take the sets $S(l), l \in L_*$ to be a finite union of submanifolds in $R^{k(l)}$. Proposition 1 obviously remains true. If, in addition, each $S(l)$ is "nice enough" so that (*) of Remark III holds, then again we can take V^* to be open, dense in $U^1 \times \cdots \times U^n$.

(V) Instead of defining U_i on $\Sigma^i \times \Sigma$, let it be defined on an open set G^i in Euclidean space of dimension at least that of Σ^i, and suppose that a smooth map $\Sigma^i \times \Sigma \xrightarrow{F^i} G^i$ now yields payoffs by: $\prod_u^t(p) = u_i\left[F^i\left[\psi_t(p), \varphi(p)\right], \text{ for } t \in [i-1, i)\right]$. Assume that, for every $z \in \Sigma$, the map $\Sigma^i \xrightarrow{F_z^i} G^i$ given by $F_z^i(x) = F^i(x, z)$ has full rank everywhere. Then proposition 1, and its proof, remain valid. We could also replace Σ by the appropriate space of signals taken from Remark II.

If the full rank condition fails the N.E. set will no longer be generically finite but a finite union of submanifolds of positive codimension. If the intersection of these with non-revealing strategies is transversal, then non-revealing N.E.'s will form lower dimensional submanifolds of N.E.'s ... and thus "most N.E.'s" would still remain fully revealing. We have not checked the details of this picture.

(VI) The assumption that Γ is layered can be relaxed somewhat as follows:

(a) $x \in Y_l$ and $\pi(x) = 0 \Rightarrow \pi(y) = 0$ and $S(y) = S(x)$ for all $y \in Y_l$

(b) $x, y \in Y_l$; x and y are linked to $\alpha \in C_l \Rightarrow \pi(x) = \pi(y)$, $S(x) = S(y)$.

Then Proposition 1 goes through, with $\widetilde{\Gamma}$ defined in the appropriate way, by the same arguments.

(VII) The kind of games we have considered here may be of use in the analysis of strategic market games with uncertainty. See [4] for an example, where a special case of Proposition 1 is examined.

(VIII) The information partitions I^i need not satisfy the condition of "perfect recall". But if they are refined in order to do so, the Nash outcomes of Γ would not change. More generally, take any game in extensive form (not necessarily layered) and suppose that each information set is contained in Y_l for some l. Then it is easily checked that refining information by perfect recall leaves the Nash outcomes of the game invariant.

3. THE FINITE CASE

The game tree is defined exactly as in the non-atomic case except that $N = \{1, \ldots, n\}$ is the set of players. Thus every branch out of $x \in Y(i)$ is a move of player i (i.e., an element of $S(x)$); if $\pi(l) = i \in N$, then i observes the *individual* moves made in $P(l)$ etc. We can--and will--simplify the space of payoffs somewhat. Observe that the set of outcomes in the tree is now identifiable with $\underset{l \in L_*}{X} \left[S(l) \right]^{|C_l|}$. Let $\underline{U}$ be just the space of all C^2 functions on $\underset{l \in L_*}{X} \left[\underline{S}(l) \right]^{|C_l|}$. As before, for

$u \in (\underline{U})^N$, $\eta(G(u))$ (or $\gamma(G(u))$) is the set of all N.E.'s (or N.E. outcomes) of the game $G(u)$ where again $G = \Gamma, \tilde{\Gamma}$. First we note that there is an open dense set $\underline{V}$ in $(\underline{U})^N$ such that:

(7) $u \in \underline{V} \Rightarrow$ every z in $\gamma\left[\tilde{\Gamma}(u)\right]$ is fully revealing.

This can be checked by repeating without change the argument for (2), and using in place of D the map from $(\underline{U})^N \times Z$ to $R^{m(n)} \times Z$ given by:

$$\Big[u_1, \ldots\ldots, u_n;\ \underbrace{y_1, \ldots, y_{m(1)}, y_{m(1)+1}, \ldots, y_{m(2)} \ldots\ldots y_{m(n-1)+1}, \ldots, y_{m(n)}}_{y} \Big]$$

$$\longmapsto$$

$$\left[\frac{\partial u_1}{\partial x_1}(y), \ldots, \frac{\partial u_1}{\partial x_{m(1)}}(y), \frac{\partial u_2}{\partial x_{m(1)+1}}(y), \ldots, \frac{\partial u_2}{\partial x_{m(2)}}(y), \right.$$

$$\left. \ldots\ldots, \frac{\partial u_n}{\partial x_{m(n-1)+1}}(y), \ldots, \frac{\partial u_n}{\partial x_{m(n)}}(y), y_1, \ldots\ldots, y_{m(n)} \right].$$

Similarly the following analogue of (3) holds:

(8) For any $b \in B \setminus \left\{ \{\widetilde{I}_l\} \right\}_{l \in L_*}$, there is an open dense set $\underline{V}_b$ of $(\underline{U})^N$ such that:

$$u \in \underline{V}_b \Rightarrow \left\{ z \in \gamma\left[\Gamma_b(u)\right] : \left\{ I_l(z) \right\}_{l \in L_*} = b \right\} = \phi \quad .$$

For the rest of this section we will need a standard concavity assumption on payoffs. Put $U_c = \{ (u_1, \ldots, u_n) \in (\underline{U})^N$: each u_i is strictly concave on $\underset{l \in L_i}{X} \left[\underline{S}(l) \right]^{|C_l|}$. for every fixed choice of the other variables $\}$ and note that it is open in $(\underline{U})^N$ By a well-known theorem, $\eta(\widetilde{\Gamma}(u)) \neq \phi$ for $u \in U_c$.

Let us recapitulate the notion of an ε-N.E. introduced in [3]. Consider the sequence of moves on the path from x_0 to x, replace chance moves by the number 0 and call the resulting vector $M(x)$. For x and y in Y_l, $l \in L_*$, define:

$$d_\Gamma(x,y) = \begin{cases} \infty \text{ if } x \text{ and } y \text{ are in distinct elements of } E_l \\ ||M(x) - M(y)|| \text{ otherwise} \end{cases}$$

where $||\ \ ||$ is the Euclidean norm.

An $s = (s^1, \ldots, s^n) \in S^1(\Gamma) \times \cdots \times S^n(\Gamma)$ gives rise to an outcome $p(s) = \left[p_1(s), \ldots, p_k(s) \right]$ consisting of k paths in the tree. These pierce each layer Y_l in $|C_l|$ points which we will denote by $x_\alpha(s)$, $\alpha \in C_l$. Here the point $x_\alpha(s)$ has $\alpha \in C_l$ linked to it, i.e., α
occurs on the unique path from x_0 to $x_\alpha(s)$. Put
$N_\alpha(s,\varepsilon) = \{ x \in Y_l : d_\Gamma(x, x_\alpha(s)) < \varepsilon \}$, and

$\varepsilon(s) = \frac{1}{2} \min \left\{ d_\Gamma(x_\alpha(s), x_\beta(s)) : \alpha, \beta \in C_l,\ M(x_\alpha(s)) \neq M(x_\beta(s)),\ l \in L_* \right\}$. If

$\varepsilon < \varepsilon(s)$, then the sets $N_\alpha(s,\varepsilon))$ and $N_\beta(s,\varepsilon))$ are disjoint whenever $M(x_\alpha(s)) \neq M(x_\beta(s))$. Thus for $\varepsilon < \varepsilon(s)$ we can define s_ε^i on $Y(i)$ as follows:

$$s_\varepsilon^i(x) \begin{cases} s^i(x_\alpha) \text{ if } x \,\varepsilon N_\alpha(s,\varepsilon) \\ s^i(x) \text{otherwise} \end{cases}$$

Note that $s_\varepsilon^i \in S^i(\Gamma)$ for $\varepsilon < \varepsilon(s)$.

We will say that s is an ε-N.E. of $\Gamma(u)$ if, for each player i,

$$u_i\left[\varphi\left[p\left[s_\varepsilon^1, \ldots, s_\varepsilon^{i-1}, t, s_\varepsilon^{i+1}, \ldots, s_\varepsilon^n\right]\right]\right] \leq u_i\left[\varphi\left[p(s)\right]\right],$$

for all $t \in S^i(\Gamma)$. This is intuitively the same as an N.E. except that unilateral deviations by a player are taken to be unobserved by others if they are of very small size. Let

$$\eta^*(\Gamma(u)) = \{s \in S^1(\Gamma) \times \cdots \times S^n(\Gamma) : \ s \text{ is an } \varepsilon\text{-N.E. of } \Gamma(u) \text{ for some } \varepsilon>0\}$$

and denote by $\gamma^*(\Gamma(u))$ the set of outcomes induced by $\eta^*(\Gamma(u))$. Our aim is to establish:

Proposition 2 There is an open dense set V_c of U_c such that

$$u \in V_c \Rightarrow \begin{cases} \gamma^*(\Gamma(u)) \subset \gamma(\tilde{\Gamma}(u)); \\ \text{if } \tilde{z} \in \gamma^*(\Gamma(u)),\ \tilde{z} \text{ is fully revealing.} \end{cases}$$

First we show:

$$\text{(13)} \qquad \left.\begin{array}{r} b \in B \\ u \in U_c \\ z \in \gamma^*(\Gamma(u)) \\ \{I_l(z)\}_{l \in L_*} = b \end{array}\right\} \Rightarrow z \in \gamma(\Gamma_b(u)).$$

Fix $\varepsilon>0$ such that z is an ε-N.E. outcome of $\Gamma(u)$. Let $(q_1, \ldots, q_k)$ be the k paths in z. (Recall that $q_t(l)$ is the move picked at layer l along q_t.) From the definitions of $I_l(z) = b_l$ and the strategy-sets in Γ, we get

$$(14) \qquad \left.\begin{array}{r} v \in b_l \\ q_i \text{ pierces } v \\ q_j \text{ pierces } v \end{array}\right\} \Rightarrow q_i(l) = q_j(l)$$

Also, it is clear that each v in b_l is pierced by at least one q_α. Thus we may define $\tilde{s}^i \in S^i(\Gamma_b)$ by:

$$(15) \qquad \tilde{s}^i(x) = q_j(l) \text{ if } \begin{cases} x \in Y(i) \\ x \in v \in b_l \\ q_j \text{ pierces } v \end{cases}$$

Put $\tilde{s} = (\tilde{s}^1, \ldots, \tilde{s}^n)$ and observe that $(p_1(\tilde{s}), \ldots, p_k(\tilde{s})) = (q_1, \ldots, q_k) = z$. It remains to show that $\tilde{s} \in \eta(\Gamma_b(u))$ to verify (13).

Let $\hat{s} = (\hat{s}^1, \ldots, \hat{s}^n)$ be an ε-N.E. of Γ which produces the play z, i.e., $p(\hat{s}) = z$, and let N_α, for $\alpha \in \bigcup_{l \in L_*} C_l$ be the neighborhoods associated with it. (Recall: N_α has center $x_\alpha(\hat{s})$ and radius $\varepsilon < \varepsilon(\hat{s})$ using the distance d_Γ.) Fix player i, *w.l.o.g.* $i=1$. Put $\tilde{s}^1(\delta) = \{t \in S^1(\Gamma_b) : ||t - \tilde{s}^1|| < \delta\}$. ($S^1(\Gamma_b)$ is a product of simplices* and $||\cdot||$ the Euclidean norm on it). Also, for $t \in S^1(\Gamma_b)$, let $(\tilde{s}|t)$ stand for $(t, \tilde{s}^2, \ldots, \tilde{s}^n)$.) Note that, by the definition of b_l

$$(16) \qquad \left.\begin{array}{r} l \in L_* \\ \alpha, \beta \in u \in J_l \\ x_\alpha(\tilde{s}) \in v_1 \in b_l \\ x_\beta(\tilde{s}) \in v_2 \in b_l \\ v_1 \neq v_2 \end{array}\right\} \Rightarrow M(x_\alpha(\tilde{s})) \neq M(x_\beta(\tilde{s}))$$

Pick $\delta_1 > 0$ to ensure

---

*In fact, $S^1(\Gamma_b) = \underset{l \in L_1}{X} \left[S(l) \right]^{|b_l|}$

$$(17)\qquad \left.\begin{array}{r} l\in L_1 \\ v_1,v_2\in b_l \\ \tilde{s}^1(v_1)\neq\tilde{s}^1(v_2) \\ t\in\tilde{S}^1(\delta_1) \end{array}\right\} => t(v_1)\neq t(v_2)$$

where $\tilde{s}^1(v_1)$ is the move picked by $\tilde{s}^1$ at (any point in) v_1, etc. Since $x_\alpha(\tilde{s})\in v\in b_l$ implies $x_\alpha(\tilde{s}\,|\,t)\in v\in b_l$ (for $\alpha\in C_l$), and $\tilde{s}^i(x)$ is constant on each relevant $v\in b_l$, (16) and (17) yield:

$$(18)\qquad \left.\begin{array}{r} l\in L_1 \\ \alpha,\beta\in u\in J_l \\ x_\alpha(\tilde{s})\in v_1\in b_l \\ x_\beta(\tilde{s})\in v_2\in b_l \\ v_1\neq v_2 \\ t\in\tilde{S}^1(\delta_1) \end{array}\right\} => \left\{\begin{array}{c} M(x_\alpha(\tilde{s}\,|\,t)) \\ \text{and} \\ M(x_\beta(\tilde{s}\,|\,t)) \\ \text{are unequal} \end{array}\right\} => \left\{\begin{array}{l} x_\alpha(\tilde{s}\,|\,t)\text{and} \\ x_\beta(\tilde{s}\,|\,t)\text{are} \\ \text{in distinct} \\ \text{elements of } I^1 \end{array}\right.$$

(The second implication of (18) follows from the definition of I^1.) Finally, by the continuity of M on $\underset{l\in L_*}{X}\left[S(l)\right]^{|C_l|}$ (and recalling the definitions of $d_\Gamma, S^1(\Gamma_b), \ldots, S^n(\Gamma_b)$) , there is a $\delta_2>0$ such that:

$$(19)\qquad t\in\tilde{S}^1(\delta_2) => x_\alpha(\tilde{s}\,|\,t)\in N_\alpha \text{ for all } \alpha$$

Put $\delta=\min\{\delta_1,\delta_2\}$. Since $\hat{s}^i_\varepsilon(x)$ is constant on each relevant N_α we get, by (19),

$$(20)\qquad \left.\begin{array}{c} \alpha\in C_l \\ i=\pi(l)=2,\ldots,n \\ t\in\tilde{S}^1(\delta) \end{array}\right\} => \left\{\begin{array}{l} \hat{s}^i_\varepsilon(x_\alpha(s\,|\,t)) = \hat{s}^i_\varepsilon(x_\alpha(s)) \\ = \tilde{s}^i(x_\alpha(\tilde{s})) = \tilde{s}^i(x_\alpha(\tilde{s}\,|\,t)) \end{array}\right.$$

((19) is only used for the first equality; the last two are obvious from the definitions). By (18), we can find a $\hat{t}\in S^1(\Gamma)$, for every $t\in\tilde{S}^1(\delta)$, to satisfy:

(21) $\hat{t}(x_\alpha(\tilde{s}\mid t))=t(x_\alpha(\tilde{s}\mid t))$

for all $\alpha\in\bigcup\{C_l:\pi(l)=1\}$. (20) and (21) give:

(22) $t\in\tilde{s}^1(\delta)\Rightarrow p(\hat{t},\hat{s}_\varepsilon^2,\ldots,\hat{s}_\varepsilon^n)=p(t,\tilde{s}^2,\ldots,\tilde{s}^n)$

But $\hat{s}$ is an ε-N.E. of Γ, so we deduce

(23) $u_1\Big[\varphi(p(\tilde{s}\mid t))\Big]\le u_1[\varphi(p(\tilde{s}))]$ for $t\in\tilde{s}^1(\delta)$.

Since $u_1\Big[\varphi(p(\tilde{s}\mid t))\Big]$ is concave in t, a local maximum must in fact be global, so (23) holds for all $t\in S^1(\Gamma_b)$, which verifies $s\in\eta(\Gamma_b(u))$ and thereby (13).

Noting that B is finite and U_c is open in $(\underline{U})^N$, Proposition 2 follows immediately from (7), (8), and (13).

FURTHER REMARKS

(IX) The variations mentioned in Remarks I - VIII go through in the finite case.

(X) Let s be an $\varepsilon-N.E.$ of $\Gamma(\text{for } \varepsilon<\varepsilon(s))$, and suppose that the outcome $p(s)$ produced by s is fully revealing. Then the sets $N_\alpha(s,\varepsilon), N_\beta(s,\varepsilon)$ are disjoint whenever α and β are in distinct elements of $\tilde{J}_l$, $l \in L_*$. Furthermore, each $N_\alpha(s,\varepsilon)$ is a *union of information sets in* Γ (and thus the game Γ_ε^s below is a coarsening of Γ). For $x \in Y(i)$, let $I_\varepsilon^i(x)$ be the information set in Γ_ε^s that contains x . Now define Γ_ε^s by:

$$I_\varepsilon^i(x) = \begin{cases} I^i(x) \text{ if } x \notin \bigcup_\alpha N_\alpha(s,\varepsilon) \\ \bigcup_{\alpha \in \tilde{J}_l} N_\alpha(s,\varepsilon) \text{if } x \in N_\beta(s,\varepsilon) \text{ for some } \beta \text{ in } \tilde{J}_l . \end{cases}$$

Clearly the pair Γ_ε^s , Γ satisfies the conditions of the proposition in the Appendix. Then any $N.E.$ of Γ_ε^s is also an $N.E.$ of Γ by that proposition. On the other hand, it is immediate than an $\varepsilon-N.E.$ of Γ is an $N.E.$ of Γ_ε^s . We conclude: s is an $\varepsilon-N.E.$ of $\Gamma, p(s)$ is fully revealing $\Rightarrow s$ is an $N.E.$ of Γ . Therefore, (recalling proposition 2) for $u \in V_c$, $\gamma^*(\Gamma(u)) \subset \gamma(\Gamma(u))$.

(XI) Suppose s is a fully revealing $\varepsilon-N.E.$ of Γ . Consider $\varepsilon'<\varepsilon$. Then $\Gamma_{\varepsilon'}^s$ is a refinement of Γ_ε^s ; and the pair $\Gamma_{\varepsilon'}^s, \Gamma_\varepsilon^s$ meets the requirements of the proposition in the Appendix. Thus

$$\left.\begin{array}{r} u \in V_c \\ s \text{ is an } \varepsilon-N.E. \text{ of } \Gamma(u) \\ \varepsilon'<\varepsilon \end{array}\right\} \Rightarrow s \text{ is an } \varepsilon'-N.E. \text{ of } \Gamma(u) .$$

So if we let

$\gamma^{**}(\Gamma(u)) = \Big\{ s \in S^1(\Gamma) \times \cdots \times S^n(\Gamma) : s$ is an $\varepsilon - N.E.$ of $\Gamma(u)$ for all sufficiently small $\varepsilon \Big\}$, we have:

$$u \in V_c \Rightarrow \gamma^{*}(\Gamma(u)) = \gamma^{**}(\Gamma(u)) \ .$$

(XII) The notion of ε−N.E.'s enables us to give a concavity-free asymptotic version of the non-atomic result in proposition 1, along the lines spelled out in [3].

APPENDIX

Let T be any game tree. Denote by P the set of all outcomes in T. An outcome is now any collection of paths in T that could accrue from the choice of a move in every $S(x)$, $\pi(x) \in N$. Fix payoff functions $\prod^i : P \to R$ for each player i. Consider two information patterns $I = \{I^i\}_{i \in N}$, $\tilde{I} = \{\tilde{I}^i\}_{i \in N}$ on T which satisfy the normal conditions (where $J^i = I^i$ or $\tilde{I}^i$) :

(i) J^i partitions $Y(i) = \{x \in Y : \pi(x) = i\}$

(ii) $x, y \in v \in J^i \Rightarrow S(x) = S(y)$

(iii) No path in T pierces any v in J^i more than once.

Assume

(iv) every $\tilde{I}^i$ is a refinement of I^i.

Denote by $G, \tilde{G}$ the games with information partitions $I, \tilde{I}$ on T.

Finally, we will require that information regarding *chance* moves is identical in I^i and $\tilde{I}^i$. To make this precise let $F(x)$, for $x \in Y$, be the sequence of nature's moves on the path from the root to x. For any $v \in \tilde{I}^i$, let $\underline{v}$ be the unique element in I^i such that $v \subset \underline{v}$. Given a position x and an outcome z, say " $x \in z$ " if x occurs on one of the paths induced by z ; and say "z is an outcome in $\tilde{G}$ " if it accrues from a choice of strategies in $\tilde{G}$. Assume:

$$\text{(v)} \qquad \left.\begin{array}{r} z \text{ is an outcome in } \tilde{G} \\ x_1 \in z \,,\ x_2 \in z \\ x_1 \in v_1 \in \tilde{I}^i \\ x_2 \in v_2 \in \tilde{I}^i \\ \underline{v}_1 \neq \underline{v}_2 \\ F(x_1) \neq F(x_2) \end{array}\right\} \Rightarrow v_1 \neq v_2$$

Propositon $\eta(G)\subset\eta\ (\tilde{G})$.

Since $S^i(G)\subset S^i(\tilde{G})$ the claim makes sense. To verify it, take $s=\{s^i:\ i\in N\}\in\eta(G)$ and suppose $s\notin\eta(\tilde{G})$, i.e., there is some $i\in N$ and $\tilde{t}\in S^i(\tilde{G})$ such that the outcome $p(s\,|\,\tilde{t})$ yields a higher payoff to i than $p(s)$. Let $p(s\,|\,\tilde{t})$ consist of the paths $\{p_\alpha\}_{\alpha\in A}$ in T where A is some indexing set.

Note that, since a play "splits" only at chance nodes

(vi) x lies on p_α, y lies on $p_\beta\left[\text{for } \alpha,\beta\in A\right]$; $p_\alpha\neq p_\beta => F(x)\neq F(y)$.

(Recall: $I^i(x),\tilde{I}^i(x)$ is the information set in $I^i,\tilde{I}^i$ that contains x .)

From (v) and (vi) we get:

(vii) $x\in p_\alpha, y\in p_\beta, p_\alpha\neq p_\beta, I^i(x)\neq I^i(y)\Rightarrow I^i(x)\neq I^i(y)$.

Also, by (iii),

(viii) $x\in p_\alpha, y\in p_\alpha, x\neq y\Rightarrow I^i(x)\neq I^i(y)$.

Let X be the set of positions that occur on one of the paths in $\{p_\alpha\}_{\alpha\in A}$.

In view of (vii) and (viii) we may define $t\in S^i(G)$ by:

$$t\,(u)=\begin{cases}\tilde{t}(x) \text{ if } x\in X \text{ and } u=I^i(x)\\ \text{arbitrary otherwise}\end{cases}$$

(Here $\tilde{t}(x)$ is the move picked by $\tilde{t}$ at x) . Then a moment's reflection reveals that $p(s\,|\,t)$ is precisely $\{p_\alpha\}_\alpha\in A$. This contradicts the fact that $s\in\eta(G)$, proving the proposition.

REMARKS

(1) This proposition was mentioned in [2], [5] but only by way of a verbal remark. So it seemed worthwhile to give a precise formulation here.

(2) If there are no moves of nature in T , then (v) is vacuously satisfied, and we get the proposition of [5].