DIFFERENTIAL INCLUSIONS AND VIABILITY THEORY

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ABSTRACT

We present a summary of the basic results on differential inclusions and viability theory. A comprehensive exposition of these two theories is the purpose of the book on the same subject by the authors.

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INTRODUCTION

There is a great variety of motivations that led mathematicians to study dynamical systems having velocities not uniquely determined by the state of the system, but depending loosely upon it, i.e., to replace differential equations

$$x' = f(x)$$

by differential inclusions

$$x' \in F(x)$$

when F is the set-valued map that associates to the state of the system the set of feasible velocities.

A great impetus to study differential inclusions came from the development of Control Theory, i.e., of dynamical systems

$$(*) \qquad x'(t) = f(t,x(t),u(t)) \quad , \quad x(0) = x_0$$

"controlled by parameters u(t) (the "controls"). Indeed, if we introduce the set-valued map

$$F(t,x) := \{f(t,x,u)\}_{u \in U}$$

then solutions to the differential equations (*) are solutions to the "differential inclusion"

$$(**) \qquad x(t) \in F(t,x(t)) \quad , \quad x(0) = x_0$$

in which the controls do not appear explicitly.

Systems Theory provides systems of the form

$$x'(t) = A(x(t)) \frac{d}{dt}(B(x(t)) + C(x(t)) \quad ; \quad x(0) = x_0$$

in which the velocity of the state of the system depends not only upon the state $x(t)$ of the system at time t, but also on *variations of observations* $B(x(t))$ of the state. These are obviously instances of differential inclusions.

Also, differential inclusions provide a mathematical tool for studying differential equations

$$x'(t) = f(t,x(t)) \quad , \quad x(0) = x_0$$

with *discontinuous* right-hand side, by embedding $f(t,x)$ into a set-valued map $F(t,x)$ which, as a set-valued map, enjoys enough regularity to have trajectories closely related to the trajectories of the original differential equation.

But, besides this array of mathematical and physical motivations, social and biological sciences should provide many instances of differential inclusions. Indeed, if deterministic models are quite convenient for describing systems that arise in physics, mechanics, engineering and even, in microeconomics, their use for explaining the evolution of what we shall call "macrosystems" does not take in account the *uncertainty* (which, in particular, involves the impossibility of a comprehensive description of the dynamics of the system), the absence of *controls* (or the ignorance of the laws relating the controls and the states of the system) and the *variety* of available dynamics. These are reasons why usual dynamical systems, or even controlled dynamical systems, may not be suitable for describing the evolution of states of systems derived from economics, social and biological sciences.

We may expect the set of trajectories of differential inclusions to be rather large: hence an important class of problems consists naturally in devising mechanisms for selecting special trajectories.

A first class of such mechanisms is provided by *Optimal Control Theory*: it consists in selecting trajectories that optimize a given criterion, a functional on the space of all such trajectories.

This implicitly requires that:

1) there exists a decision maker who "controls" the system
2) that such a decision maker has a perfect knowledge of the future (which is involved in the definition of the criterion)
3) the optimal trajectories are chosen once and for all at the origin of the period of time.

These requirements are not satisfied by the "macrosystems" that evolve according to the laws of Darwinian evolution.

Such macrosystems appear to have neither aims nor targets nor desire to optimize some criterion. But they face a minimal requirement, called *viability*, which is to remain "alive" in the sense of satisfying given binding constraints.

For that, they use a policy, *opportunism*, that enables the system to conserve viable trajectories that its lack of determinism -- the *availability of several feasible velocities* -- allows to find.

This provides a mathematical metaphor of this deep intuition of Democritus, "Everything that exists in the universe is due to chance and necessity".

This second class of mechanisms is the object of *Viability Theory*. In particular, we shall apply Viability Theory in the framework of Control Theory for *regulating* systems through *feedback controls*, and we shall illustrate this by an application to *decentralization* by price regulation in the framework of economics.

The results presented below as well as the bibliographical references appear in a comprehensive form in the monograph "Differential Inclusions", Springer-Verlag, New York, 1983, by the authors.

1. THE PEANO AND NAGUMO THEOREMS FOR DIFFERENTIAL EQUATIONS

Throughout all this exposition, X denotes a finite dimensional space, the state space, and $K \subset X$ the subset of feasible states. We assume that K is *locally compact*; this covers two particular cases: K is *open* and K is *closed*.

The dynamics of the system are described by a (single-valued) map f from K to X, which we assume to be *continuous* and bounded. For every initial state $x_0 \in K$, we consider the initial value problem

$$x'(t) = f(x(t)) \quad , \quad x(0) = x_0 \quad .$$

We say that a solution is *viable* if

$$\forall t \geq 0 \quad , \quad x(t) \in K \quad .$$

For this problem to have viable solutions for every initial state in K, we need some consistency between the dynamics f and the viability set K.

In order to express it, we introduce the following concept of contingent cone $T_K(x)$ to K at $x \in K$, which is the tangent space when K is a smooth manifold, the tangent cone of convex analysis when K is convex, and which is the whole space at each interior point.

<u>Definition 1</u>

We say that the subset

$$(1) \qquad T_K(x) := \bigcap_{\varepsilon>0} \bigcap_{\alpha>0} \bigcup_{0<h<\alpha} \left(\frac{1}{h}(K-x) + \varepsilon B\right)$$

is the *"contingent cone"* to K at x.

In other words, $v \in T_K(x)$ if and only if

$$(2) \quad \begin{cases} \forall \varepsilon > 0, \quad \forall \alpha > 0, \ \exists u \in v + \varepsilon B, \ \exists h \in]0,\alpha] \quad \text{such that} \\ x + hu \in K \quad . \end{cases}$$

or if and only if

$$(3) \qquad \liminf_{h \to 0+} \frac{d_K(x+hv)}{h} = 0$$

or again, if and only if there exist sequences of strictly positive numbers h_n and of elements $u_n \in X$ satisfying

$$(4) \quad \text{i)} \ \lim_{n\to\infty} u_n = v, \quad \text{ii)} \ \lim_{n\to\infty} h_n = 0, \quad \text{iii)} \ \forall n \geq 0, \ x + h_n u_n \in K \ .$$

It is quite obvious that $T_K(x)$ is a *closed cone* and that the contingent cone to the closure $\overline{K}$ of K coincides with it.

When K is a convex subset and x belongs to K, then

$$(5) \qquad T_K(x) = c\ell\Big(\bigcup_{h>0} \frac{1}{h}(K-x) \Big)$$

and it is a closed convex cone.

A rather comprehensive calculus of the contingent cones has been developed.

Theorem 1 (Peano-Nagumo). Let $K \subset X$ be locally compact and $f: K \to X$ be continuous. The necessary and sufficient condition for the existence of a local viable solution of the differential equation $x' = f(x)$ for every initial state in K is

$$(6) \qquad \forall x \in K \ , \qquad f(x) \in T_K(x) \quad . \qquad \blacktriangle$$

When K is open, $T_K(x)$ is equal to X and the above reduces to Peano's Theorem.
When K is a smooth manifold, condition (6) expresses the fact that f is a *vector field*.
When K is closed, then condition (6) is no longer trivial:
The above reduces to Nagumo's theorem.
When K is closed and f is bounded, we obtain the existence of a global viable solution.

Remark:

If we drop the boundedness assumption, we obtain only the existence of a local solution.

2. THE CASE OF DIFFERENTIAL INCLUSIONS

From now on, we describe the dynamics of the system by a set-valued map F from K to X.

For every initial state $x_0 \in K$, we consider the initial value problem for the differential inclusion

$$(1) \qquad x'(t) \in F(x(t)) \quad , \quad x(0) = x_0 \quad .$$

We say also that a solution is *viable* if

$$(2) \qquad \forall t \geq 0 \quad , \quad x(t) \in K \quad .$$

For an ordinary differential equation $x' = f(x)$, it is clear what is meant by a solution. The continuity of f allows us to define a solution as a continuously differentiable function on some interval.

For differential inclusions the problem is not so easy. For instance, let F be constant, equal to $\{-1,+1\}$ and let us consider the set of solutions through 0 at $t = 0$. There are only two C^1 solutions, namely $x_1(t) = t$ and $x_2(t) = -t$, and we feel that we should accept more functions as solutions, allowing the derivative not to exist, for instance on a finite number of points, or on a countable set, or on a set of measure zero. We shall accept *absolutely continuous functions* as an adequate class of solutions.

The conditions to be imposed on the set-valued mapping F in order to have solutions are of two kinds: regularity conditions on the map (i.e. the various kinds of continuity or semi-continuity) and conditions of topological or geometric type (compactness, convexity) on the images of points. Various combinations are possible: we would not expect to obtain solutions under weak assumptions of both types, while it should be quite easy to prove existence under strong assumptions of both types. In general, the intermediate cases will be more interesting.

We choose to study the following compromises

a) F is upper semicontinuous
b) the values of F are compact and convex

and

a) F is continuous
b) the values of F are compact, but not necessarily convex.

The case of Lipschitzean maps F bridges those two classes of differential inclusions, because the Relaxation Theorem states that, in this case, the set of trajectories of the differential inclusion

$$x'(t) \in F(x(t)) \quad , \quad x(0) = x_0$$

is dense in the set of trajectories of the differential inclusion

$$x'(t) \in \overline{co}\, F(x(t)) (:= \text{closed convex hull of } F(x(t))) \ .$$

The simplest approach to the existence problem for a differential inclusion would be to reduce it to the corresponding problem for an ordinary differential equation. To begin with, we would like to know whether there exists a differential equation in some sense concealed into the differential inclusion, i.e., whether there exists a continuous function $f(\cdot)$ such that for every x in some domain, $f(x) \in F(x)$. Unfortunately, continuous selections do not exist other than under very restrictive assumptions. Here, we begin by the first class of problems, when F satisfies the following assumption

(A) $\begin{cases} F \textit{ is upper semicontinuous and bounded with nonempty} \\ \textit{compact convex values.} \end{cases}$

(A map is called *upper semicontinuous* if for each $x_0 \in K$ and for each neighborhood V of $F(x_0)$, there exists a neighborhood U of x_0 such that, for all $x \in U$, $F(x) \subset V$. It is *continuous* if moreover, for every open set W intersecting $F(x_0)$, there exists a neighborhood U_1 of x_0 such that, for all $x \in U_1$, $F(x) \cap W \neq \emptyset$.)

Theorem 1 (Viability Theorem). Let $K \subset X$ be locally compact and F satisfy assumption A. The necessary and sufficient condition for the existence of a viable solution of the differential inclusion $x' \in F(x)$ for every initial state in K is

$$(3) \qquad \forall x \in K \quad , \quad F(x) \cap T_K(x) \neq \emptyset \quad . \qquad \blacktriangle$$

Let us denote by $\mathcal{T}(x_0)$ the set of viable solutions to the initial value problem (1).

We summarize the main qualitative properties of the trajectories in the two following statements.

Theorem 2. We posit assumptions of Theorem 1. The map $x_0 \to \mathcal{T}(x_0)$ is upper semicontinuous with compact values from K to the space $\mathcal{B}(0,\infty,X) := \{x \in \mathcal{C}(0,\infty;X) \mid x' \in L^\infty(0,\infty;X)\}$, when $\mathcal{C}(0,\infty,X)$ is supplied with the topology of uniform convergence on compact intervals and $L^\infty(0,\infty;X)$ is supplied with the weak-star topology. ▲

The *attainable set* $A_T(x_0)$ is defined by

$$(4) \qquad A_T(x_0) := \{x(T) \mid x(\cdot) \in \mathcal{T}(x_0)\} \quad .$$

Theorem 3. We posit the assumptions of Theorem 1. The maps $x_0 \to A_T(x_0)$ are upper semicontinuous with compact connected values from K to X. They have the fixed point property: if A_T maps a convex subset of K into itself, it has a fixed point. ▲

When K is convex and compact, the assumptions of the Viability Theorem imply the existence of an *equilibrium* (or a *stationary* solution or *rest* point) of the dynamical system, i.e., a solution $\bar{x} \in K$ to the inclusion

$$(5) \qquad 0 \in F(\bar{x}) \quad .$$

Theorem 4. Let $K \subset X$ be compact convex and F satisfy assumption (A). We posit the following tangential condition:

$$(3) \qquad \forall x \in K \quad , \quad F(x) \cap T_K(x) \neq \emptyset \quad .$$

Then

a) There exists an equilibrium $\bar{x} \in K$ of F.

b) $\forall h > 0$, $\forall x_0 \in K$, there exists a sequence of elements $x^n \in K$ such that $x^0 = x_0$ and

$$(6) \qquad \forall n \in N \quad , \quad \frac{(x^n - x^{n-1})}{h} \in F(x^n) \quad .$$

▲

Remark

We can regard the sequence of elements $x^n \in K$ as a viable *discrete* trajectory of the dynamical system. The finite-difference system is called the "implicit" finite difference scheme of the differential inclusion.

Example

Let $X = R^n$ be the space of states of the system we wish to describe and $Y = R^m$ be the space of "observations". We denote by $g : X \to Y$ the "observation map" of the system and by $C : Y \to X$ the "feedback map".

In this model, we assume that the evolution law is

$$(7) \qquad x'(t) \in F(x(t)) + C\frac{d}{dt}g(x(t)) \quad ; \quad x(0) = x_0 \quad .$$

In other words, we assume that the velocity depends not only upon the state of the system but also upon the variations of observations of the state.

We assume that

(8)
- i) $C \in L(Y,X)$ is continuous and linear
- ii) $g : X \to Y$ is continuously differentiable on an open subset Ω containing K.

We set

$$(9) \qquad A(x) := I - C\nabla g(x) \quad .$$

So, the system can be written

(10) $A(x(t))x'(t) \in F(x(t))$.

Corollary 1

Assume that $K \subset X = R^n$ is a closed subset, F is an upper semicontinuous set-valued map from K into X with nonempty closed convex values. Let $C \in L(Y,X)$ and $g \in C^1(\Omega,Y)$. We suppose that there exists $c > 0$ such that,

(11) $\forall x \in X,\ \exists v \in F(x) + C\nabla g(x)v$ such that $v \in T_K(x) \cap cB$.

Then, for any initial state $x_0 \in K$, there exists a viable solution to the differential inclusion (7). ▲

Application: Regularization of differential equations with discontinuous right-hand side.

In order to provide existence for solutions of differential equations

(12) $x'(t) = f(x(t))$

when $f : R^n \to R^n$ is not continuous, the easiest way is to consider the smallest upper semicontinuous convex valued map F where graph contains the graph of f. When f is locally bounded, this set-valued map F is defined by

(13) $$F(x) := \bigcap_{\varepsilon>0} \overline{co}\, f(x + \varepsilon B) \quad .$$

It is clear that

(14)
- i) $\forall x,\ f(x) \in F(x)$
- ii) the map $x \to F(x)$ is upper semicontinuous with convex values
- iii) whenever f is continuous at x, $F(x) = \{f(x)\}$.

Certainly, any solution to the differential equation (12) is a solution to the differential inclusion

(15) $x'(t) \in F(x(t))$.

We stress the point that whenever f is continuous at x(t), then a solution to the differential inclusion (15) satisfies the equation $x'(t) = f(x(t))$.

In order to obtain this result, we do not need property (14) i) at points when f is not continuous. We can look for "smaller" set-valued maps ϕ which still satisfy properties (14) ii) and iii) so that differential inclusions

(16) $x'(t) \in \phi(x(t))$

yield trajectories $x(\cdot)$ satisfying the equation $x'(t) = f(x(t))$ whenever f is continuous at x(t).

We describe one such map ϕ.

Proposition 1

Let f be a single-valued map from an open subset $\Omega \subset R^n$ to R^n which is locally bounded. We set

(17) $$\phi(x) := \bigcap_{\varepsilon>0} \bigcap_{meas(N)=0} \overline{co}\, f(((x+\varepsilon B)\cap\Omega)\backslash N) .$$

Then

(18)
- i) the map $x \to \phi(x)$ is upper semicontinuous with nonempty convex values
- ii) whenever f is continuous at x, $\phi(x) = \{f(x)\}$.

Assume moreover that f is measurable on Ω. Then

(19) iii) f(x) belongs to F(x) at almost every x in Ω.

When F is continuous, but with nonconvex values, we have a similar but weaker viability theorem, an adaptation using a construction of Filippov.

Theorem 5. Let $K \subset X$ be closed and F be a bounded continuous map from K to the nonempty compact subsets of X satisfying

(20) $\forall x \in K \quad , \quad F(x) \subset T_K(x) \quad .$

For every initial state, there exists a viable trajectory of the differential inclusion $x' \in F(x)$. ▲

3. THE RELAXATION THEOREM

The Relaxation Theorem plays a fundamental role in the qualitative theory of differential equations, and concerns the relations between the set of solutions of the two problems $x' \in F(x)$ and $x' \in \overline{co}\ (F(x))$. Certainly solutions of the first are also solutions of the second: we wish to study, however, to what extent the operation of convexifying the right-hand side really introduces new solutions. In other words, under what conditions will the set of solutions to the differential inclusion

(1) $x'(t) \in F(x(t))$

be *dense* in the set of solutions of the "convexified" differential inclusion

(2) $x'(t) \in \overline{co}\ (F(x(t)))$?

This problem is particularly relevant in control theory; solutions to the convexified problem are often called *relaxed solutions*, and the problem we have mentioned, the *problem of relaxation*. We shall prove that the relaxation property holds when F is Lipschitzean with compact values, while it does not necessarily hold when F is only continuous.

In the theory of control, one encounters the following problem. It is given an affine differential equation,

$$x' = Ax + bu$$

a set of controls $\mathcal{U}$, a compact convex subset of R^n, an initial condition and a time interval [0,T]. One considers the attainable set A_T at time T, the images at T of all solutions issued at t = 0 from ξ using all controls $u(\cdot)$, measurable on [0,T], such that $u(t) \in U$ a.e. on [0,T].

The question can then be raised as to whether it would be possible to have the same attainable set economizing on the set of controls, hence to have the same results with controls that are much simpler to build. It is a famous result (the Bang-Bang principle) that one can actually reduce the set U to extr(U), the set of its extremal points, even when this set is not closed.

For a differential inclusion one focuses on the set of solutions through an initial point. Since for a nonconvex right-hand side this set is, in general, not closed (even if its section at any given time might be), we should consider the problem of the possible equivalence of the closure of the set of solutions of a nonconvex problem with the set of solutions of the convexified one. In other words one can look at the set of solutions of a nonconvexified problem and ask for conditions to insure the equivalence of its closure with the set of solutions of the convexified. A second way of looking at the question, more related to the Bang-Bang principle, would be to begin with a differential inclusion $x' \in F(x)$ with compact convex values and to ask for a subset of F(x) in order essentially to retain the solutions of the original problem. This second question is by far more difficult and so far has no complete answer.

We begin by presenting an inequality for a Lipschitzean differential inclusion, an analog of Gronwall's inequality: given an almost solution y, we shall state the existence of at least one solution x satisfying the desired inequality (other solutions with the same initial data need not, obviously, satisfy any reasonable inequality). Hence the following is also an existence result.

Theorem 1. Let there be given an interval I := [a,b], an absolutely continuous function $y : I \to R^n$, a positive constant β, and call Q the subset of $I \times R^n$ defined by $(t,x) \in Q$ if $t \in I$ and $\|x-y(t)\| \leq \beta$. Assume that F, from Q into the nonempty and closed subsets of R^n, is continuous and satisfies the Lipschitz condition:

(3) $\forall t,x,y \quad , \quad F(t,y) \subset F(t,x) + k(t)\|x-y\| B \quad ,$

where B is the unit ball.

Assume moreover that

(4) $\|y(a) - x^0\| = \delta \leq \beta \quad , \quad d(y'(t), F(t,y(t))) \leq p(t)$ a.e.

with $p \in L^1(I)$. Set

(5) $$\xi(t) := \delta e^{\int_a^t k(s)ds} + \int_a^t e^{\int_s^t k(u)du} p(s)ds \quad ,$$

and let $J := [a,\omega]$ be a nonempty interval such that $t \in J$ implies $\xi(t) \leq \beta$. Then there exists a solution $x(\cdot)$ on J to the problem

(6) $x'(t) \in F(t,x(t)) \quad , \quad x(a) = x_0$

such that

(7) $|x(t) - y(t)| \leq \xi(t)$

and

(8) $|x'(t) - y'(t)| \leq k(t)\xi(t) + p(t)$ a.e. ▲

Remark

Let x_0, y_0 two initial points, $|x_0 - y_0| = \delta \leq b$ and take $p = 0$ in the preceding theorem. Then to any solution $y(\cdot)$ such that $y(0) = y_0$ we can associate a solution $x(\cdot)$ such that $x(0) = x_0$ and $|x(t) - y(t)| \leq |x_0 - y_0| e^{\int_0^t k(s)ds}$. Hence we have the following

Corollary 1

The map T_I from R^n to nonempty subsets of $C(I,R^n)$ that associates to an initial point the set of solutions on I issued from that point, is Lipschitzean with constant $e^{\int_I k(s)ds}$. ▲

Here is the *Relaxation Theorem.*

Theorem 2 (Filippov-Ważewski). Let F, from $Q := \{x \in R^n | \|x - \xi_0\| \leq b\}$ into the compact subsets of R^n, be Lipschitzean. Set $I := [-T,+T]$ and let $x : I \to Q$ be a solution to

$$(9) \qquad x'(t) \in \overline{co}\,(F(x(t)) \quad , \quad x(0) = \xi_0$$

such that, for $t \in I$, $\|x(t) - \xi_0\| \leq b$. Then for every positive ε, there exists $y : I \to Q$, a solution to

$$(10) \qquad y'(t) \in F(y(t)) \quad , \quad y(0) = \xi_0$$

such that for $t \in I$, $\|y(t) - x(t)\| \leq \varepsilon$. ▲

4. THE TIME DEPENDENT AND STATE DEPENDENT VIABILITY THEOREMS

We shall study now time dependent differential inclusions

$$(1) \qquad x' \in F(t,x) \quad ; \quad x(0) = x_0 \quad .$$

We shall look for time-dependent viable trajectories, i.e., trajectories $x(\cdot)$ defined on $[0,T[$ and satisfying:

$$(2) \qquad \forall t \in [0,T[\quad , \quad x(t) \in K(t)$$

where $t \to K(t)$ is a set-valued map from $[0,\infty[$ to X.

In order to state the necessary and sufficient condition of viability, we need to define the concept of *contingent derivative* of the set-valued map $t \to K(t)$.

We adapt to the case of a set-valued map the intuitive definition of a derivative of a function in terms of the tangent to its graph.

Let F *be a strict set-valued map from* $K \subset X$ to Y and (x_0,y_0) belong to the graph of F.

We denote by $DF(x_0,y_0)$ *the set-valued map from* X *to* Y *whose graph is the contingent cone* $T_{\text{graph}(F)}(x_0,y_0)$ *to the graph of* F *at* (x_0,y_0). ■

In other words

(3) $v_0 \in DF(x_0,y_0)(u_0)$ if and only if $(u_0,v_0) \in T_{\text{graph}(F)}(x_0,y_0)$.

Definition 1

We shall say that the set-valued map $DF(x_0,y_0)$ from X to Y is the "contingent derivative" of F at $x_0 \in K$ and $y_0 \in F(x_0)$. ▲

We point out that

(4) $$\forall x_0 \in K \quad , \quad \forall y_0 \in F(x_0) \quad , \quad DF(x_0,y_0)^{-1} = D(F^{-1})(y_0,x_0) \quad .$$

The contingent derivatives allow the derivative of restrictions of functions to subsets with empty interior. If F is a map from X to Y, we denote by $F|_K$ its restriction to K defined by

$$F|_K := \begin{cases} F(x) & \text{when } x \in K \\ \emptyset & \text{when } x \notin K \end{cases}$$

Let F be a continuously differentiable single-valued map on a neighborhood of K and $F|_K$ be its restriction to K. Then

(5) $$\forall x_0 \in K \quad , \quad DF|_K(x_0)(u_0) = \begin{cases} \{\nabla F(x_0) \cdot u_0\} & \text{when } u_0 \in T_K(x_0) \\ \emptyset & \text{when } u_0 \notin T_K(x_0) \end{cases} .$$

We give an analytical characterization of $DF(x_0,y_0)$, which justifies that the above definition is a reasonable candidate for capturing the idea of a derivative as a (suitable) limit of differential quotients.

(6) $$\begin{cases} v_0 \in DF(x_0,y_0)(u_0) \quad \text{if and only if} \\ \liminf\limits_{\substack{h \to 0+ \\ u \to u_0}} d\left(v_0, \dfrac{F(x_0+hu) - y_0}{h}\right) = 0 \end{cases}$$

Naturally, a reasonable calculus (including chain rule formulas) is available.

We come back to the *time dependent Viability Theorem.*

Theorem 1. Let K be a set-valued map from $[0,\infty[$ to a Hilbert space X with closed graph and F be a bounded upper semicontinuous map from graph (K) to $R \times X$ with nonempty compact convex values. We posit the condition:

(7) $\quad \forall t \geq 0 \quad , \quad \forall x \in K(t) \quad , \quad F(t,x) \cap DK(t,x)(1) \neq \emptyset \quad .$

Then for all $t_0 > 0$ and for all $x_0 \in K(t_0)$, there exists a viable trajectory on $[t_0,\infty[$ of the differential inclusion (1). ▲

We consider now the state-dependent case, where the viability set depends upon the state.

Let P be a set-valued map satisfying

(8) $\begin{cases} \text{i)} \quad \forall x \in K \quad , \quad x \in P(x) \quad \text{(reflexivity)} \\ \text{ii)} \quad \forall x \in K \quad , \quad \forall y \in P(x), \text{ we have } P(y) \subset P(x) \quad \text{(transitivity)} \end{cases}$

Then the map P defines a preorder $\preccurlyeq$ by: $y \preccurlyeq x$ if y belongs to P(x). We shall say that a solution of the initial value problem

(9) $\quad x'(t) \in F(x(t)) \quad , \quad x(0) = x_0$

is *monotone* if and only if

(10) $\quad \forall t \geq s \geq 0 \quad , \quad x(t) \in P(x(s))$

or, equivalently,

(11) $\quad \forall t \geq s \geq 0 \quad , \quad x(t) \preccurlyeq x(s) \quad .$

The typical example of a preorder is the one defined by m real-valued functions $V_j : K \to R$ $(j = 1,\ldots,m)$:

(12) $\quad \forall x \in K \quad , \quad P(x) := \{y \in K \mid \forall j = 1,\ldots,m \, , \; V_j(y) \leq V_j(x)\}$.

For this preorder, a trajectory $x(\cdot)$ is monotone if and only if

(13) $\forall j = 1,\ldots,m \, , \; \forall s,t \in [0,T[\, , \; s \geq t$, then $V_j(x(s)) \leq V_j(x(t))$.

(In this case, the functions V_j play the role of Liapunov functions.)

Theorem 2. Let K be a locally compact subset of X, F be a bounded upper semicontinuous map from K to X with compact convex values and $P : K \to K$ be a continuous map.

We posit the following tangential condition:

(14) $\quad \forall x \in K \quad , \quad F(x) \cap T_{P(x)}(x) \neq \emptyset$.

Then there exists a monotone solution for every initial state. ▲

5. REGULATION OF CONTROLLED SYSTEMS THROUGH VIABILITY

Let us translate the Viability Theorem in the language of Control Theory. The dynamics of the system are described by a map

(1) $\quad f : K \times U \to X$

where U is the "control set". The state of the system evolves according to the differential equation

(2) $$\begin{cases} \text{i)} & x'(t) = f(x(t) \, , \; u(x(t))) \\ \text{ii)} & x(0) = x_0 \in K \end{cases}$$

The regulation problem can be expressed in the following way:

a) Does there exist a function $t \to u(t)$ (*open loop control*) such that the differential equation (2) has viable trajectories?

b) Does there exist a continuous single-valued function $\tilde{u}$ from K to U (*closed loop control* or *feedback control*) such that the differential equation

$$(3)\quad \begin{cases} \text{i)} & x'(t) = f(x(t),\tilde{u}(x(t))) \\ \text{ii)} & x(0) = x_0 \in K \end{cases}$$

has viable trajectories?

c) Does there exist an *equilibrium* $(\bar{x},\bar{u}) \in K \times U$, a solution to the nonlinear equation

$$(4)\quad f(\bar{x},\bar{u}) = 0 \quad .$$

We introduce the *feedback map* C defined by

$$(5)\quad \forall x \in K \quad , \quad C(x) := \{u \in U \mid f(x,u) \in T_K(x)\} \quad .$$

We shall assume that:

$$(6)\quad \begin{cases} \text{i)} & U \text{ is compact} \\ \text{ii)} & f : K \times U \to Y \text{ is continuous} \end{cases}$$

so that the set-valued map F defined by

$$(7)\quad F(x) := \{f(x,u)\}_{u \in U}$$

is continuous with compact values.

We summarize in the following statement the consequences of the Viability Theorem.

Theorem 1. Let K be a closed subset of X, U be a compact set and $f : K \times U \to X$ be a continuous map.

We assume that:

$$(8)\quad \forall x \in K \quad , \quad C(x) := \{u \in U \mid f(x,u) \in T_K(x)\} \neq \emptyset$$

and that there exists a bounded set Q such that

(9) $\forall x \in K, \quad f(x,U) := \{f(x,u)\}_{u \in U}$ is convex and contained in Q.

Then

$$(10) \quad \begin{cases} \forall x_0 \in K \quad , \text{ there exists a measurable function } u(\cdot) \text{ and} \\ \text{a viable trajectory of the differentiable equation (2)} \end{cases}$$

which are related by

$$(11) \qquad \text{for almost all } t \geq 0 \quad , \quad u(t) \in C(x(t)) \quad .$$

If we assume moreover that K is convex and compact, we infer the existence of an equilibrium $(\bar{x},\bar{u}) \in K \times U$.

▲

When

$$(12) \quad \begin{cases} \text{i)} \quad U \text{ is convex} \\ \text{ii)} \quad \forall x \in K \quad , \quad u \to f(x,u) \text{ is affine} \quad , \end{cases}$$

the sets f(x,U) are obviously convex. For this case we can obtain the existence of a *continuous feedback control* $\tilde{u}$ yielding viable trajectories.

Theorem 2. Let K and U be convex compact of finite dimensional vector spaces, f be a continuous map from $K \times U$ to X which is affine with respect to u.

We assume that there exists $\gamma > 0$ such that

$$(13) \quad \forall x \in K \quad , \quad \forall y \in \gamma B \quad , \quad \exists u \in U \,|\, f(x,u) + y \in T_K(x) \quad .$$

The conclusions of Theorem 1 hold true and there exists a continuous feedback control $\tilde{u} : K \to U$ yielding viable trajectories of the differential equation (3).

▲

6. DECENTRALIZED REGULATION THROUGH VIABILITY

We apply the Viability Theorem for giving a possible explanation to the role of price systems in decentralizing the behavior of different consumers, in the sense that the knowledge of the price system allows each consumer to make his choice without knowing the global state of the economy and, in particular, without knowing (necessarily) the choices of his fellow consumers.

There is no doubt that Adam Smith is at the origin, two centuries ago, of what we now call decentralization, i.e., the ability for a complex system moved by different actions in pursuit of different objectives to achieve an allocation of scarce resources.

We are going to propose a dynamical model that keeps the essential ideas underlying Adam Smith's proposals. For this, we slightly modify the usual definition of a consumer and regard a *price* system not as the state of a dynamical system whose evolution law is known, but *as a control which* evolves as a function of the consumptions according to a feedback law.

To take in account the dynamical nature of the behavior of a consumer i, we describe it as an *automaton* d_i which associates to each price system p and his own consumption x_i its rate of change $d_i(x_i,p)$. Therefore, when the price p(t) evolves, the consumption $x_i(t)$ of consumer i evolves according to the differential equation

(1) $$x_i'(t) = d_i(x_i(t),p(t)) \quad , \quad x_i(0) = x_i^0 \quad .$$

So, a viability problem arises: *does there exist a price function* p(t) *such that the sum* $\sum_{i=1}^{n} x_i(t)$ *of the consumptions remains available*? In other words, do the trajectories $x_i(\cdot)$ of the n coupled differential equations satisfy the viability condition

(2) $$\forall t \geq 0 \quad , \quad \sum_{i=1}^{n} x_i(t) \in M$$, where M is the set of available resources.

We also have a concept of equilibrium: It is a sequence $(\overline{x}_1,\ldots,\overline{x}_n,\overline{p})$ of n consumptions $\overline{x}_i$ and of a price system $\overline{p}$ such that

$$(3)\quad \begin{cases} \text{i)} & \forall\, i = 1,\ldots,n \;,\; d_i(\overline{x}_i,\overline{p}) = 0 \\ \text{ii)} & \sum_{i=1}^{n} \overline{x}_i \in M \end{cases}$$

It remains to check that there are sufficient conditions which have an economic interpretation. We shall prove that equilibria and viable trajectories do exist if the instantaneous demand functions d_i satisfy the "*instantaneous Walras law*"

$$(4)\qquad \forall p \;,\; \forall x_i \;,\; \langle p, d_i(x_i,p)\rangle \leq 0 \quad .$$

This is a budgetary rule that requires that at each instant, the value of the rate of change of each consumer is not positive, i.e., that each consumer *does not spend more than he earns in an instantaneous exchange of goods.* This law does not involve the subset M of available resources.

Theorem 1. We posit the following assumptions on the instantaneous demand function $d_i : L_i \times S^{\ell} \to R^{\ell}$ which sets the variation in consumer's i demand when the price is p and its consumption is x.

$$(5)\quad \begin{cases} \text{i)} & \forall\, i = 1,\ldots,n \text{ , the function } d_i : L_i \times S^{\ell} \to R^{\ell} \text{ is continuous} \\ \text{ii)} & \forall\, x \in L_i \;,\; \forall p \in S \;,\; d_i(x,p) \in T_{L_i}(x) \end{cases}$$

and

$$(6)\qquad \forall x \in L_i \;,\quad p \to d_i(x,p) \text{ is affine} \quad .$$

Let us assume moreover

$$(7)\qquad M = M_0 - R^{\ell}_{+} \text{ is closed and convex, where } M_0 \text{ is compact}$$

that

(8) $\forall\ i = 1,\ldots,n$, L_i is closed, convex and bounded below,

and that

(9) $$0 \in \mathrm{Int}\left(\sum_{i=1}^{n} L_i - M\right)$$

If the instantaneous Walras laws hold true, then

a) there exists an equilibrium $(\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_n,\overline{p})$.

b) For every initial allocation $x^0 \in K$, there exist n absolutely continuous functions $x_i(\cdot) : [0,\infty[\to R^\ell$ and a measurable function $p(\cdot) : [0,\infty[\to S^\ell$ solutions to the differential system which satisfy the viability conditions

(10) $$\begin{cases} \text{i)} & \forall t \geq 0 \ , \ \forall i = 1,\ldots,n \ , \ x_i(t) \in L_i \\ \text{ii)} & \forall t \geq 0 \ , \ \sum_{i=1}^{n} x_i(t) \in M \end{cases}$$

and the budget constraint

(11) For almost all $t \geq 0$, $\langle p(t) , \sum_{i=1}^{n} x_i'(t)\rangle \leq 0$

c) The price p(t) plays the role of a feedback control:

(12) for almost all $t \geq 0$, $p(t) \in C(x_1(t),\ldots,x_n(t))$. ▲

7. LIAPUNOV FUNCTIONS

We shall investigate whether differential inclusions

(1) $$x'(t) \in F(x(t)) \ , \quad x(0) = x_0$$

do have trajectories satisfying the property

(2) $$\forall t \geq s \ , \ V(x(t)) - V(x(s)) + \int_s^t W(x(\tau),x'(\tau))dx \leq 0$$

where

$$(3)\quad \begin{cases} \text{i)} & V \text{ is a function from } K := \text{Dom } F \text{ to } R_+ \\ \text{ii)} & W \text{ is a function from Graph } (F) \text{ to } R_+ \end{cases}$$

Trajectories $x(\cdot)$ of differential inclusion (1) satisfying (2) will be called "monotone trajectories" (with respect to V and W).

We shall answer the following questions:

1. What are the *necessary and sufficient* conditions linking F, V and W for the differential inclusion (1) to have monotone trajectories with respect to V and W?

2. Do these necessary and sufficient conditions imply the existence of pairs $(x_*,v_*) \in \text{graph } (F)$ satisfying $W(x_*,v_*) = 0$? Observe that if the values $W(x,v)$ are strictly positive whenever v is different from 0, then such an x_* is an *equilibrium*.

3. Are the cluster points x_* and v_* of the functions $t \to x(t)$ and $t \to x'(t)$, when $t \to \infty$, solutions to the equation $W(x_*,v_*) = 0$?

4. The set-valued map F and the function W from graph (F) to R_+ being given, can we construct a function V such that these necessary and sufficient conditions are satisfied?

For answering these questions positively, we have to introduce the concept of *upper contingent derivative* of a proper function V from X to $R \cup \{+\infty\}$ at a point x_0 in a direction u_0:

$$(4)\qquad D_+V(x_0)(u_0) := \liminf_{\substack{h \to 0_+ \\ u \to u_0}} \frac{V(x_0 + hu) - V(x_0)}{h}$$

We remark the following facts:

a) When V is Gâteaux-differentiable, $D_+V(x)$ coincides with the gradient $\nabla V(x)$:

$$(5)\qquad D_+V(x)(u) = \langle \nabla V(x), u \rangle \quad \text{for all } u \in X \quad .$$

b) When V is convex, the upper contingent derivative is related to the derivative from the right by the formula

$$(6)\qquad D_+V(x)(u_0) = \liminf_{u \to u_0} DV(x)(u) \quad .$$

They coincide when the latter is lower semicontinuous.

c) When V is locally Lipschitz, the upper contingent derivative coincides with a Dini derivative:

$$(7)\qquad D_+V(x)(u_0) \;=\; \liminf_{h \to 0_+} \frac{V(x+hu_0) - V(x)}{h} \quad .$$

The same is true when V is defined on a right open interval of R containing x_0.

d) When V is Gâteaux-differentiable on a neighborhood of a subset K, then

$$(8)\qquad D_+(V|_K)(x)(u) \;=\; \begin{cases} \langle \nabla V(x), u \rangle & \text{when } u \in T_K(x) \\ +\infty & \text{when } u \notin T_K(x) \end{cases}$$

This means that the upper contingent derivative of the restriction of a function to a subset K is the restriction of its gradient to the contingent cone.

The main justification for the introduction of the upper contingent derivatives is the following characterization:

Theorem 2. Assume that F is a bounded upper semicontinuous map from a locally compact subset K of X to the convex compact subsets of X, V is a continuous function from K to R_+ and W is a lower semicontinuous function from Graph (F) to R_+, convex with respect to the second argument. A necessary and sufficient condition for the differential inclusion (1) to have monotone trajectories with respect to V and W is that:

(9) $\forall x \in K \quad , \quad \exists v \in F(x)$ such that $D_+V(x)(v) + W(x,v) \leq 0$.

We shall say that a function V from K to R_+ satisfying the above condition is a *Liapunov function for* F *with respect to* W. Indeed, we recognize that when K is open, V is differentiable and F is single-valued, this condition is nothing other than the usual property

(10) $\forall x \in K \quad , \quad \langle \nabla V(x), F(x) \rangle + W(x,F(x)) \leq 0$

used in Liapunov's method for studying the stability of solutions to differential equations. We also point out that condition (9) implies the existence of a pair $(x_*,v_*) \in \text{graph}\ (F)$ satisfying $W(x_*,v_*) = 0$.

The next problem we investigate is the construction of Liapunov functions. Let $\mathcal{T}(x)$ denote the set of trajectories of the differential inclusion (1) starting at x.

We define the function V_F by

(11) $$V_F(x) := \inf_{x(\cdot) \in \mathcal{T}(x)} \int_0^\infty W(x(\tau),x'(\tau))d\tau \quad .$$

We begin by pointing out the following remark.

Proposition 1

Let $V : \text{Dom}\ (F) \to R_+$ and $W : \text{graph}\ (F) \to R_+$ be nonnegative functions.

(12) If there exists a monotone trajectory $x(\cdot) \in \mathcal{T}(x_0)$ with respect to V and W, then

$$0 \leq V_F(x_0) \leq V(x_0) \quad .$$

(13) If $\bar{x} \in \mathcal{T}(x_0)$ is a monotone trajectory with respect to V_F and W and if $V_F(x_0)$ is finite, it achieves the minimum of

$$x \to \int_0^\infty W(x(\tau),x'(\tau))d\tau \quad \text{on} \quad \mathcal{T}(x_0) \quad .$$

(14) Conversely, if $\bar{x} \in T(x_0)$ achieves the minimum of $x \to \int_0^\infty W(x(\tau), x'(\tau))d\tau$ on $T(x_0)$, then it is a monotone trajectory with respect to V_F and W and furthermore

(15) $$\forall t \geq 0 \quad , \quad V_F(\bar{x}(t)) = \int_t^\infty W(\bar{x}(\tau), \bar{x}'(\tau))d\tau \quad .$$ ▲

Remark

Equality (15) is the "*principle of optimality*". It states that if $\bar{x}$ is a solution to the differential inclusion $x' \in F(x)$, $x(0) = x_0$ that minimizes on $T(x_0)$ the functional $x \to \int_0^\infty W(x(\tau), x'(\tau))d\tau$, then its restriction to $[t,\infty[$ minimizes the functional $x \to \int_0^\infty W(x(\tau), x'(\tau))d\tau$ over the set of solutions to the differential inclusion $x' \in F(x)$, $x(t) = \bar{x}(t)$.

We then state a result whose origin can be traced back to Carathéodory, Jacobi and Hamilton: If for all initial state x there exists a trajectory $x(\cdot) \in T(x)$ that minimizes the above functional, then V_F is a Liapunov function for F with respect to W.

Proposition 2

Let F be a bounded upper semicontinuous map with compact convex images and $W : \text{graph}\,(F) \to R_+$ be a nonnegative lower semicontinuous function that is convex with respect to v. If the minimum in $V_F(x_0)$ is achieved for $x_0 \in K$, V_F satisfies not only the Liapunov condition, but the following generalization of Hamilton-Jacobi-Carathéodory equation:

(16) $$\exists v_0 \in F(x_0) \quad \text{such that} \quad D_+V_F(x_0)(v_0) + W(x_0, v_0) = 0 \quad .$$ ▲

We recognize this fact when V_F is a smooth function, since equation (16) can be written

(17) $$\inf_{v \in F(x_0)} \left(\sum_{i=1}^{n} \frac{\partial V_F}{\partial x_i} v_i + W(x_0, v) \right) = 0 \quad .$$

We translate these results into the time dependent case.

Let F be a set-valued map from $R_+ \times X$ to X, the domain of which is the graph of a set-valued map $t \to K(t)$ from R_+ to X. We introduce a nonnegative function W defined on the graph of F.

We denote by $\mathcal{T}(t_0,x_0)$ the set of solutions $x(\cdot) \in \mathcal{C}(t_0,\infty;x)$ of the differential inclusions

$$(18) \qquad x'(t) \in F(t,x(t)) \quad , \quad x(t_0) = x_0 \quad .$$

We introduce

$$(19) \qquad V_F(t_0,x_0) := \inf_{x(\cdot) \in \mathcal{T}(t_0,x_0)} \int_{t_0}^{\infty} W(\tau,x(\tau),x'(\tau))d\tau \quad .$$

Theorem 2. Let F be a bounded upper semicontinuous map from the closed graph of a set-valued map $K(\cdot) : R_+ \to X$ to the compact convex subsets of X, satisfying

$$\forall t \geq 0 \quad , \quad \forall x \in K(t) \quad , \quad F(t,x) \cap DK(t,x)(1) \neq \emptyset \quad .$$

Let $W : \text{graph}\ (F) \to R_+$ be a nonnegative lower semicontinuous function which is convex with respect to the last argument. If for all $(t_0,x_0) \in \text{graph}\ (K)$ the function $V_F(t_0,x_0)$ is finite, it is the smallest nonnegative lower semicontinuous Liapunov function for F with respect to W: it satisfies

$$\exists v_0 \in F(t_0,x_0) \text{ such that } D_+V_F(t_0,x_0)(v_0) + W(t_0,x_0,v_0) = 0 \quad .$$

The optimal trajectories $\bar{x}(\cdot)$ satisfy

$$\forall t \geq t_0 \quad , \quad V_F(t,\bar{x}(t)) = \int_t^{\infty} W(\tau,\bar{x}(\tau),\bar{x}'(\tau))d\tau \quad .$$

▲

We list now some properties of monotone trajectories with respect to functions V and W.

$$(20) \qquad \begin{cases} a) & t \to V(x(t)) \text{ is non increasing} \\ b) & \int_0^{\infty} W(x(\tau),x'(\tau))d\tau < +\infty \end{cases}$$

We show also that the cluster points x_* and v_* of the functions $x(\cdot)$ and $x'(\cdot)$ when $t \to \infty$ solve the equation

$$(x_*, v_*) \in \text{Graph}\ (F) \quad \text{and} \quad W(x_*, v_*) = 0 \quad .$$

But we have to be careful, because $x'(\cdot)$ is not defined everywhere. So, we have to make precise the notion of "almost cluster point" of a measurable function.

We single out two important instances:

a) $W(x,v) := \|v\|$

Condition (20) states that the *length* $\int_0^\infty \|x'(\tau)\| d\tau$ of the trajectory is finite and that $x(t)$ *has a limit when* $t \to \infty$ *which is an equilibrium of* F.

b) $W(x,v) := \phi(V(x))$ where $\phi: [0,\infty[\to R$ is a bounded continuous function. Let w be a solution to the differential equation:

$$w'(t) + \phi(w(t)) = 0 \quad , \quad w(0) = V(x_0) \quad .$$

Then monotone trajectories do enjoy the estimate

$$V(x(t)) \leq w(t) \quad \text{for all} \quad t \geq 0 \quad .$$

8. DIFFERENTIAL INCLUSIONS WITH MEMORY

Differential inclusions express that at every instant the velocity of the system depends upon its state at this very instant. *Differential inclusions with memory*, or, as they are also called, *functional differential inclusions*, express that the velocity depends not only on the state of the system at this instant, but depends upon the history of the trajectory until this instant. To formalize this concept, we introduce the Fréchet space $C(-\infty,0;X)$ of continuous functions from $]-\infty,0[$ to X supplied with the topology of uniform convergence on compact intervals.

We "embed" the "past history" of a trajectory $x(\cdot)$ of $C(-\infty,+\infty;X)$ in this space $C(-\infty,0;X)$ by associating with it the function $T(t)x$ of $C(-\infty,0;X)$ defined by

(1) $\quad \forall\tau \in]-\infty,0] \quad , \quad T(t)x(\tau) := x(t+\tau) \quad .$

Hence a differential inclusion with memory describes the dependence of the velocity $x'(t)$ upon the history $T(t)x$ of $x(\cdot)$ up to time t through a set-valued map F from a subset $\Omega \subset R \times C(-\infty,T;X)$ to X.

Solving a differential inclusion with memory is the problem of finding an absolutely continuous function $x(\cdot) \in C(-\infty,T;X)$ saitsfying

(2) $\quad \forall t \geq 0 \quad , \quad x'(t) \in F(t,T(t)x) \quad .$

This class of problems covers many examples:

a) differential-difference inclusions, associated to a set-valued map G from a subset of $R \times X^p$ to X, defined by

(3) $\quad x'(t) \in G(t,x(t-r_1(t)),\dots,x(t-r_p(t)))$

belong to this class since we can define the set-valued map F by

$$F(t,\varphi) := G(t,\varphi(-r_1(t)),\dots,\varphi(-r_p(t))) \quad .$$

The functions $r_i(t)$ $(1 \leq i \leq p)$ are called the *delay functions.*

b) Volterra inclusions, which are inclusions of the form

(4) $\quad x'(t) \in G\left(t , \int_{-\infty}^{t} k(t,s,x(s))ds\right)$

where k maps $R \times R \times X$ to X and where G is a set-valued map from $R \times X$ to X are also differential inclusions with memory. Indeed, we define F from $R \times C(-\infty,0;R)$ by

$$F(t,\varphi) := G\left(t , \int_{-\infty}^{0} k(t,t+\tau,\varphi(\tau))d\tau\right) \quad .$$

c) *Differential Trajectory processing inclusions.* A "trajectory-processor" is a family of maps $P(t)$ from $C(-\infty,+\infty;X)$ to a Hilbert space Y satisfying the property

(5) $\varphi(s) = \psi(s)$ for all $s \leq t$, then $P(t)\varphi = P(t)\psi$.

Differential Trajectory processing inclusions are problems of the form

(6) $x'(t) \in G(t,P(t)x)$

where G maps $R \times Y$ to X.

Initial-value problems for differential inclusions with memory are problems of the form

(7) $$\begin{cases} \text{i)} & \text{for almost all } t \geq 0 \;,\; x'(t) \in F(t,T(t)x) \\ \text{ii)} & T(0)x = \varphi_0 \text{ where } \varphi_0 \text{ is given in } C(-\infty,0;X) \;. \end{cases}$$

Theorems about differential inclusions whose right-hand side is upper semicontinuous with compact convex images can be extended to differential inclusions with memory.

We choose, for instance, to state and prove the time dependent Viability Theorem.

Theorem 1. Let K be a set-valued map with closed graph from $[0,\infty[$ to X. We set

(8) $$\forall t \geq 0 \;,\; \mathcal{K}(t) := \{\varphi \in C(-\infty,0;X) \mid \varphi(0) \in K(t)\} \;.$$

Let F be a bounded semicontinuous map from graph $\mathcal{K}$ to the compact convex subsets of X.

We assume that

(9) $$\forall t \geq 0 \;,\; \forall \varphi \text{ such that } \varphi(t) \in K(t),\; F(t,\varphi) \cap DK(t,\varphi(t))(1) \neq \emptyset \;.$$

Then, for all $\varphi_0 \in \mathcal{K}(0)$, there exists a solution to the differential inclusion with memory

$$(10)\quad \begin{cases} \text{for almost all } t \geq 0 , \quad x'(t) \in F(t,T(t)x) \\ T(0)x = \varphi_0 \end{cases}$$

which is viable in the sense that

$$(11)\quad \forall t \geq 0 , \quad x(t) \in K(t) .$$

▲

Remark

As in the case of differential inclusion, we can prove that condition (9) is necessary.