A METHOD FOR INTEGRATING ACTIVITY ANALYSIS SUBMODELS WITH NEOCLASSICAL GENERAL EQUILIBRIUM MODELS.

Stefan Lundgren

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

PREFACE

Computable general equilibrium models have become a more common tool in economic analysis as progress in computer science has made efficient solution techniques available. These models are often based on the neoclassical economic theory. One example is that the production possibilities are usually represented by neoclassical production functions.

In certain model applications, however, one is interested in a more disaggregated representation of the production possibilities in one or more of the model sectors. Such disaggregated sector models are primarily activity analysis models.

This paper describes a method for integrating activity analysis submodels with a neoclassical general equilibrium model. The starting point is the well-known efficiency properties of a general equilibrium which permit us to reformulate it as a nonlinear optimization problem. We then have a system of optimization models, and the integration is straightforward.

Some preliminary numerical experience is also reported.

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CONTENTS

1. INTRODUCTION

The computable general equilibrium model of Sweden by Bergman and Por (forthcoming) was developed for quantitative analyses of resource allocation issues in the Swedish economy. It has been extensively used to study the sensitivity of the Swedish economy to changes in the cost and availability of energy and to evaluate energy policy. It has also been used in analyses unrelated to energy, for instance a study on possible future changes in Sweden's comparative advantages and how they would affect the domestic economy (Bergman and Ohlsson 1981).

The production system in the Bergman-Por model is represented in a manner introduced by Johansen (1959). There is a one to one correspondence between domestic commodities and production sectors in the model economy. Each sector produces a single, homogeneous output. The production possibilities of a sector are represented by a neoclassical production function, in this case with four flexible inputs: the two primary inputs capital and labour, and the intermediate inputs, electricity and fuel. In addition, the input requirements of the remaining intermediate inputs are given by fixed input-output coefficients.

The number of sectors in the model can be varied, so it is possible to adjust the level of detail adopted in the model between different studies. However, the model view of each individual sector is still quite aggregated. At the same time there exist, for some sectors, more information concerning production possibilities, primarily in the form of activity analysis models. Such information is not relevant in all applications of the general equilibrium model and it would soon become unmanageably large if one tried to plug all available information, which may be relevant for some application, into it. But in applications where the focus is on the impact of developments in the overall economy on a particular sector or conversely on the overall economic impact of developments originating in a particular sector, it would be valuable to be able to integrate available information concerning that sector into the general equilibrium model.

Consider as an example the study of the economic impact of nuclear power discontinuation in Sweden (Bergman 1980). In this study, the ban on nuclear power was basically accounted for by shifting the production function for the electricity sector so as to reflect more expensive methods of electricity generation. The nature of the shift in the production function was calculated on the basis of estimates concerning the likely design of the electricity system in the case where nuclear power would be allowed and in the case where it would not. An existing linear programming model of electricity production was not used in this study. However, ideally one would have liked to be able to delete the aggregate production function for electricity in the model and replace it with the activity analysis model and then solving the integrated model with different constraints on the use of nuclear power.

In this paper, I will describe a method of integrating activity analysis models of individual production sectors into the Bergman-Por model. In principle, the method does not put any limit on the number of model production sectors which could be represented by activity analysis submodels. But in most applications, it would probably only be relevant to integrate such submodels for one or possibly two sectors. Hence, I will concentrate, without loss of generality, on the problem of integrating one activity analysis model of one production sector into the Bergman-Por model.

A solution to the Bergman-Por general equilibrium model is obtained by solving a system of nonlinear equations. The activity analysis model on the other hand is solved by standard linear programming techniques. The method of integration described in this paper makes use of the well-known efficiency properties of an economic general equilibrium to re-formulate the Bergman-Por model as a nonlinear optimization problem. The activity analysis model is then integrated by incorporating it in the constraint system of that nonlinear optimization problem.

Section 2 describes how a solution to the general equilibrium model can be obtained by solving an optimization problem. Then, in Section 3, it is shown how an activity analysis model of one production sector can straightforwardly be integrated with that optimization version of the general equilibrium model. Finally, in Section 4, some preliminary computational experience is reported.

2. THE OPTIMIZATION VERSION OF THE BERGMAN-POR MODEL

A general equilibrium is defined as a set of prices and a set of quantities (output and input levels, consumption levels, etc.) such that three conditions are fulfilled: 1) the total available supply of any commodity, including factors of production, is at least as large as the total demand for it, 2) each producer, taking prices as given, maximises his profits subject to the constraints imposed by the production technology and 3) each consumer, also taking prices as given, maximises his utility subject to the budget restriction. The efficiency properties of such a state of the economy are well-known. An equilibrium in the Bergman-Por model is explicitly defined only by condition 1) above, but since its behavioural equations are derived from profit and utility maximization assumptions, conditions 2) and 3) will also hold. Furthermore, there is only one aggregated consumer in the model so the pareto-optimum corresponding to an equilibrium in the model is defined by one utility function only. However, foreign trade complicates things a bit. The import demand functions of the model can be shown to be consistent with maximising behaviour (see Bergman and Por forthcoming) and, by applying the same reasoning to the behaviour of the rest of the world, so can the export demand functions. But an equilibrium in the Bergman-Por model is then also an international trade general equilibrium and the Pareto-optimum of this is defined by both the domestic utility and that of the rest of the world. The distribution of utilities between the home country and the rest of the world in the model is primarily determined by the exogenous world market prices and the balance of payments requirement.

An equilibrium in the Bergman-Por model thus corresponds to a maximum of the domestic utility function *given* that the utility level of the rest of the world is at a certain level. It is this fact which is exploited in re-formulating the Bergman-Por model as an optimization problem.

To illustrate in a framework as simply as possible, the rationale for the optimization approach, we start by considering a simplified, one-sector version of the general equilibrium model. Although simplified, this one-sector version still contains all the essential properties of the full model. (A complete description of the general equilibrium model can be found in Bergman and Por forthcoming.) After the one sector example the optimization problem corresponding to the n-sector case is presented and briefly discussed.

2.1. A One-Sector Example

There is one good produced within the economy. The production possibilities are given by a concave and linearly homogeneous production function. For illustration we use the Cobb-Douglas parameterization

$$X = A\ N^{1-a}K^{a} \tag{1}$$

where X is the production of the good and N and K are the inputs of labour and capital, respectively. Given the output price P, the wage rate W, and the cost of capital services Q, producers are assumed to maximize profits defined by

$$P_{\gamma} = P \cdot X - W \cdot N - Q \cdot K \tag{2}$$

The first order conditions for profit maximum implicitly define the input demand for labour and capital services

$$P \cdot X(1 - a) = w \cdot N \tag{3}$$

$$P \cdot X \cdot a = Q \cdot K \tag{4}$$

The demand for the domestically produced good comes from two sources: consumption demand by an aggregated household sector and export demand. The household sector is assumed to consume a composite good consisting of the domestic good and imports. The domestic good and imports are not perfect substitutes and the substitution possibilities between them are given by a constant elasticity of substitution function. The household sector chooses the mix of the domestic and the import good which minimizes the cost of consumption. The minimum cost p^D, of the composite good is then given by

$$p^D = \{h^{\frac{1}{1-u}} \cdot P^{\frac{-u}{1-u}} + m^{\frac{1}{1-u}} (vP^M)^{\frac{-u}{1-u}}\}^{\frac{u-1}{u}} \tag{5}$$

where p^M is the price of the import good and v the exchange rate. The cost function of the composite good is concave and homogenous of degree one. By Shephard's lemma the domestic demand, X^0, for domestically produced goods is given by

$$X^0 = \frac{\partial p_d}{\partial p} \cdot C = h^{\frac{1}{1-u}} \left(\frac{p^D}{P}\right)^{\frac{1}{1-u}} \cdot C \qquad (6)$$

and the demand for imports by

$$M^D = \frac{\partial p_d}{\partial (vP^M)} \cdot C = m^{\frac{1}{1-u}} \left(\frac{p_d}{vP^M}\right)^{\frac{1}{1-u}} \cdot C \qquad (7)$$

The consumption demand, C, is determined by the demand function

$$C = E/P \qquad (8)$$

where E is total consumer expenditure[1/].

Exports are determined by

$$Z = Z^0 \left(\frac{P}{vP^E}\right)^{\varepsilon}$$

where P^E is the price of foreign goods with which the home country's exports compete.

The model is closed by a set of equilibrium conditions. The markets for the two primary inputs as well as the markets for the domestically produced good and the import good shall clear. Furthermore, an exogenously specified target for the balance of payments must be met. The supplies of the primary resources are fixed so the equilibrium conditions for the input markets are simply

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1) It should be observed that equation (8) only is a definition of E. For a given set of prices, the level of consumption, C, is actually determined by the balance of payments requirement (eq.(14)). Thus equation (8) could be deleted without affecting the economic content of the model, but it has been incorporated to make our one-sector example as similar to the n-sector general equilibrium model as possible.

$$N = \bar{N} \tag{10}$$

$$K = \bar{K} \tag{11}$$

where $\bar{N}$ and $\bar{K}$ are the fixed supplies of labour and capital respectively.

Total demand for the domestically produced good consists of domestic demand, X^0, and export demand Z. The equilibrium condition thus is

$$X = X^0 + Z \tag{12}$$

The model economy is assumed to be a small economy in the sense that it faces a perfectly elastic supply of import goods. Hence, the equilibrium condition for the import good market is simply

$$M = M^D \tag{13}$$

Denoting the exogenously specified target for the balance of payments by D, the condition for external equilibrium is

$$\left(\frac{P}{v}\right) \cdot Z - P^M \cdot M = D \tag{14}$$

The profit maximization conditions (3) and (4) and the assumption of constant returns to scale in production, implies that profits must be zero in equilibrium. Consequently, there are thirteen equations in the fourteen unknowns: P, P^D, W, Q, v, E, X, C, M, Z, N, K, X^0 and M^0. As usual in general equilibrium models only relative prices matter, that is the price level is indeterminiate, so by choosing a numeraire, say $v = 1$, the number of unknowns reduces to thirteen and we have a determinate system.

Consider now the following optimization problem

$$\max\ u(C) = \text{Ln}C \quad \text{1/} \tag{15}$$

subject to the constraints (the variables in brackets denote the Langrange multipliers associated with the constraints).

$$X - AN^{1-\alpha} K^{\alpha} \leq 0 \quad (P) \quad 0 \leq \alpha \leq 1 \tag{16}$$

$$C - \{h(X-Z)^{u} + mM^{u}\}^{\frac{1}{u}} \leq 0 \quad (P^{D}) \quad u < 1 \tag{17}$$

$$\left(\frac{\varepsilon}{1+\varepsilon}\right) \cdot P^{E} \cdot Z0^{-\frac{1}{\varepsilon}} \cdot Z^{\frac{1+\varepsilon}{\varepsilon}} - P^{M} \cdot M \geq k \quad (v) \quad \varepsilon < 0 \text{ and } \varepsilon \neq -1 \tag{18}$$

$$N \leq \overline{N} \quad (w) \tag{19}$$

$$K \leq \overline{K} \quad (Q) \tag{20}$$

and the nonnegativety constraints $C \geq 0$, $X \geq 0$, $N \geq 0$, $K \geq 0$, $Z \geq 0$, $M \geq 0$.

u(C) is a utility function which generates the demand function (8) of the general equilibrium model. We recognize constraint (16) as the production function and the constraints (19) and (20) are obvious. The remaining two constraints may not be so obvious however.

Let us first look at constraint (17). In the general equilibrium model, C is a composite good with a price index defined by the cost function (5). This cost function is arrived at by assuming the aggregate household to minimize the cost, given domestic and import prices, of domestic goods and imports used

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1/ Again, this particular objective function is used to make the one-sector example similar to the n-sector case, where a logarithmic utility function is used. Obviously, in the one-sector case, we could just as well maximize the consumption level C and obtain the same solution (but with a different normalization of prices).

for consumption purposes, provided that a certain consumption level should be attained. The CES-function in equation (17) shows the feasible combinations of domestic goods and imports which result in the prescribed consumption level. It may be viewed as a "production function" showing how the consumption good is "produced" by various combinations of domestic and import goods.

Equation (18) can be viewed as a pseudo-utility function for the rest of the world. It is increasing in Z and decreasing in M. Furthermore, it is concave provided $\varepsilon < 0$, which is quite a natural assumption, since ε is the price elasticity of the export function. The meaning of this constraint will become clear when we discuss the first order conditions of the maximization problem.

The maximization problem has the following economic interpretation. We choose levels of domestic production, consumption, exports and imports, so as to maximize domestic utility subject to the production technology, the definition of the composite consumption good, the availability of primary productive resources and given that the utility level of the rest of the world should be at least k.

The objective function is concave and all the constraints are convex, so we have a concave programming problem. Suppose all decision variables are non-zero at the optimum. Then the necessary, as well as sufficient, conditions for maximum are that there exist a set of nonnegative Langrange multipliers such that the constraints (16) - (20) are satisfied, as well as the following equalities:

$$\frac{\partial L}{\partial X} = 0 \quad \text{or} \quad P^D\{h(x - Z)^u + mM^u\}^{\frac{1}{u}-1} \; h(X - Z)^{u-1} = P \tag{21}$$

$$\frac{\partial L}{\partial N} = 0 \quad \text{or} \quad P \cdot X(1 - a) = W \cdot N \tag{22}$$

$$\frac{\partial L}{\partial K} = 0 \quad \text{or} \quad P \cdot X \cdot a = Q \cdot N \tag{23}$$

$$\frac{\partial L}{\partial C} = 0 \quad \text{or} \quad P^D \cdot C = 1 \tag{24}$$

$$\frac{\partial L}{\partial M} = 0 \quad \text{or} \quad P^D\{h(x - Z)^u + mM^u\}^{\frac{1}{u}-1} = vP^M \tag{25}$$

$$\frac{\partial L}{\partial Z} = 0 \quad \text{or} \quad P^D\{h(x - Z)^u + mM^u\}^{\frac{1}{u}-1} h(x - Z)^{u-1} - vP^E Z_0^{\frac{1}{\varepsilon}} \cdot Z^{\frac{1}{\varepsilon}} \tag{26}$$

where L is the Lagrangean to the maximization problem.

Interpreting the Lagrange multipliers as prices in the conventional manner, it is obvious that (22) and (23) together with (16) are similar to the equations (3), (4) and (1) of the general equilibrium model.

Equation (24) can be rewritten as

$$C = 1/P^D \tag{24'}$$

which corresponds to equation (8) of the general equilibrium model, assuming that prices are normalized so as to make $E = 1$. As we have assumed C to be strictly positive in the optimum, so will P^D, and hence the constraint (17) must hold as an equality in the optimum. Taking account of (17) as an equality, equations (21) and (25) may be rewritten as

$$X - Z = h^{\frac{1}{1-u}} \left(\frac{P^D}{P}\right)^{\frac{1}{1-u}} \cdot C \tag{21'}$$

$$M = m^{\frac{1}{1-u}} \left(\frac{P^D}{vP^M} \right)^{\frac{1}{1-u}} \cdot C \tag{25'}$$

Obviously (25') corresponds to (7) and (21') to the combination of (6) and (12).

Substituting (21) into (26) and rearranging we obtain[1/]

$$Z = Z^0 \left(\frac{P}{vP^E} \right)^{\varepsilon} \tag{26'}$$

Equation (26') corresponds to the export function (9) of the general equilibrium model.

The optimization formulation implicitly assumes import supply to be always equal to the demand for import goods, so equation (3) is automatically accounted for. The constraints (19) and (20) obviously correspond to (10) and (11), the only difference being that the optimization version allows for the (rather uninteresting) case of a zero shadow price on any of the two primary resources.

So far we have shown the correspondence between the necessary and sufficient conditions for a maximum and all but one of the equations of the general equilibrium model. The remaining equation is the balance of payments requirement[2/]. From (26) we have that

$$\frac{P}{v} = P^E \, Z^{0^{-\frac{1}{\varepsilon}}} \cdot Z^{1/\varepsilon} \tag{27}$$

1/ Note that (18) is not defined for $\varepsilon = -1$. If one wants to work with a unitary price elasticity the first term in (18) should be replaced with $Z^0 \cdot P^E \cdot \text{Ln}Z$.

2/ Equation (5) has also passed unmentioned. This function is defined as the solution to $\min\{P*(X-Z) + vP^M*M: (h*(X-Z)^u + M*M^u)^{1/u} > 1.\}$. From (21) and (25) it is obvious that a solution to the maximization problem also implies a solution to this minimization problem with the Lagrange multiplier P^D being the minimum value of the cost of the composite good.

and hence

$$D = \frac{P}{V}\cdot Z - P^M\cdot M = P^E Z^{\circ\frac{1}{\varepsilon}}\cdot Z^{\frac{1+\varepsilon}{\varepsilon}} - P^M\cdot M$$

D will deviate from k in a manner determined by the sign of $(\varepsilon/1+\varepsilon)$, and hence the realized surplus in the balance of payments inter alia depends on the exogenously specified k. However, it is not *a priori* possible to determine what value should be specified for k in order to obtain a certain balance of payments surplus D. Consequently, we have to solve a sequence of the optimization problem, with different k-values, until we get the desired D. It is obvious from (28) that an increase (decrease) in k will imply an increase (decrease) in D. So, given a solution to the optimization problem, the direction in which k should be changed in order to approach the desired D will always be known.

2.2. The Complete Optimization Version. The n-Sector Case.

We now turn to the optimization problem corresponding to the n-sector version of the Bergman-Por model. First, in section 2.2.1, the maximization problem is stated. Then, in section 2.2.2, we use the necessary and sufficient conditions for a maximum to derive a set of equations which correspond to the equations of the Bergman-Por model.

2.2.1. The Maximization Problem

There are n production sectors, each producing one homogenous good. There are nt traded goods and consequently n-nt nontraded goods. There is furthermore one bookkeeping sector, n+1, in which different goods are combined into a single capital good, and ns other bookkeeping sectors combining goods into ns consumption commodities, by means of fixed coefficients. In each production sector capital, labour, fuels and electricity are substitutable factors of production, whereas the use of nonenergy intermediate inputs are proportional to output. Final demand consists of an exogenously determined public demand for the nontraded good produced in the public sector, an exogenous net investment demand for the capital good, and finally an endogenously determined demand for consumption by an aggregated household sector and export demand.

The problem is to choose the output and input quantities in each production sector, the quantities of consumption goods so as to maximize the utility of the aggregated household sector. The maximization problem is constrained by the production technology, the availability of the primary inputs labour and capital and by the requirement that the exogenous demand components must be met. Exports and imports must be determined so as to make the "utility" level of the rest of the world at least reach a certain level.

The Objective Function

$$\max_{\{C_s X_j K_j N_j X_{ij} Z_j M_j\}} \sum_{s=1}^{ns} \beta_s \, \ell n(C_s - q_s) \quad , \quad \sum_{s=1}^{ns} \beta_s = 1 \qquad (29)$$

1/ A list of variables and parameters is given in the Appendix.

The Production Technology Constraints

$$X_j - A_j \Big\{ a_j (K_j^{\alpha_j} N_j^{(1-\alpha_j)} C^{d_j t})^{P_j} + b_j (C_j X_{1j}^{\gamma_j} + d_j X_{2j}^{\gamma_j})^{P_j/\gamma_j} e^{\gamma_j^* P_{it}} \Big\}^{1/P_j} \leq 0$$

$$j = 1,2,\ldots,n. \qquad (30)$$

The Market Clearing Constraints

Energy

$$\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=1}^{n} X_{ji} - \{h_j (X_j - Z_j)^{\mu_j} + m_j M_j^{\mu_j}\}^{\frac{1}{\mu_i}} \leq 0$$

$$j = 1,2,\ldots \qquad (31)$$

Nonenergy, trading sectors

$$\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=1}^{n+1} a_{ji} X_i - \{h_i (X_j - Z_j)^{\mu_j} + m_j M_j^{\mu_j}\}^{\frac{1}{\mu_j}} \leq 0$$

$$j = 3,4,\ldots,nt. \qquad (32)$$

Nontraded sectors, except the public sector

$$\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=1}^{n+1} a_{ji} X_i - X_j \leq 0$$

$$j = nt+1,\ldots,n-1. \qquad (33)$$

The public sector

$$X_n \geq G \qquad (34)$$

Investment

$$X_{n+1} - \sum_{j=1}^{n} \delta_j K_j \geq I \qquad (35)$$

Capital and Labor

$$\sum_{j=1}^{n} K_j \leq K \qquad (36)$$

$$\sum_{j=1}^{n} N_j \leq N$$

"Utility" constraint of the rest of the world

$$\sum_{j=1}^{nt} \left(\left(\frac{\varepsilon_j}{1+\varepsilon_j}\right) P_j^E Z_j^0{}^{-\frac{1}{\varepsilon_j}} \cdot Z_j^{\frac{1+\varepsilon_j}{\varepsilon_j}} - P_j^M \cdot M_j \right) - \sum_{j=1}^{2} \bar{P}_j \bar{b}_j X_j \geq k \qquad (37)$$

$$\varepsilon_j \neq -1 \qquad j=1,2,\ldots,nt \qquad (38)$$

All decision variables must be nonnegative and also $X_j \geq Z_j$, $j = 1,2,\ldots,nt$, and $C_i > q_i$, $i = 1,2,\ldots,ns$.

The Lagrangian for this problem is

$$L = \sum_{s=1}^{ns} \beta_s \ln(C_s - q_s) - \sum_{j=1}^{n} P_j^* \Big[X_j - A_j \{ a_j (K_j^{\alpha_j} N_j^{1-\alpha_j} e^{d_j})^{\rho_j}$$

$$+ b_j \left(c_j X_{1j}^{\gamma_j} + d_j X_{2j}^{\gamma_j} \right)^{\rho_i/\gamma_j} e^{\lambda_j^* \rho_j t} \}^{\frac{1}{\rho_j}} \Big]$$

$$- \sum_{j=1}^{2} P_j^D \left(\sum_{i=1}^{ns} k_{ji} C_i + \sum_{i=1}^{n} X_{ij} - \{ h_j (X_j - Z_j)^{\mu_j} + m_j M_j^{\mu_j} \}^{1/\mu_j} \right)$$

$$- \sum_{j=3}^{nt} P_j^D \left(\sum_{i=1}^{ns} k_{ji} C_i + \sum_{i=1}^{n+1} a_{ji} X_i - \{h_j (X_j - 2_j)^{\mu_j} + m_j M_j^{\mu_j}\}^{1/\mu_j} \right.$$

$$- \sum_{j=nt+1}^{n-1} P_j^D \left(\sum_{i=1}^{ns} k_{ji} C_i + \sum_{i=1}^{n+1} a_{ji} X_i - X_j \right) - P_n^D (G - X_n)$$

$$- P_{n+1}^D \left(I + \sum_{j=1}^{n} \delta_j K_j - X_{n+1}\right) - Q\left(\sum_{j=1}^{n} K_j - K\right) - W\left(\sum_{j=1}^{n} N_j - N\right) \qquad (39)$$

$$+ v\left(\sum_{j=1}^{nt} \left(\left(\frac{\varepsilon_j}{1+\varepsilon_j}\right) P_j^E \cdot Z_j^{0^{-\frac{1}{\varepsilon_j}}} \cdot Z_j^{\frac{1+\varepsilon_j}{\varepsilon_j}} - P_j^M \cdot M_j \right) - \sum_{j=1}^{2} \bar{P}_j b_j X_j \right)$$

The objective function is a concave function in the consumption levels C_s . Furthermore, of the nonlinear constraints (30) - (32), (31) and (32) are convex for $\mu_j < 1$, $j = 1,2,\ldots,nt$. Also, for $\rho_j < 1$ and $\gamma_j < 1$, $j = 1,2,\ldots,n$, the constraints (30) are convex. Finally, the constraint (38) is the sum of nt concave functions so it is concave. The rest of the constraints are linear and consequently the constraints (30) - (38) define a convex feasible set. We thus have a concave programming problem.

2.2.2. The necessary and sufficient conditions for a maximum.

A vector of decision variables values ($C_1,\ldots,C_{ns}$; $X_i,\ldots,X_n$; $K_1,\ldots,K_n$; $N_1,\ldots,N_{ns}$; $X_1,\ldots,X_{1n}$; $X_{21},\ldots,X_{2n}$; $Z_1,\ldots,Z_{nt}$; $M_i,\ldots,M_{nt}$) solves the maximization problem if and only if there exist a set of nonnegative Lagrange multipliers such that if any constraint holds as a strict inequality the corresponding muliplier is zero and such that the Lagrangean is maximized. For the Lagrangean to be maximized the decision variables and the Lagrange mulipliers must satisfy the following equations (assuming all decision variables are nonzero in optimum).

$$\frac{\partial L}{\partial C_s} = \beta_s (C_s - q_s)^{-1} - \sum_{j=1}^{n-1} k_{is} P_j^D = 0 \qquad s=1,2,\ldots,ns \tag{40}$$

$$\frac{\partial L}{\partial X_j} = P_j - \sum_{i=3}^{n-1} a_{ij} P_i^D - v\bar{P}_j \bar{b}_j$$

$$+ P_j^D \{h_j (X_j - Z_j)^{\mu_j} + m_j M_j^{\mu_j}\}^{\frac{1}{\mu_j} - 1} h(x_j - Z_j)^{\mu_j - 1} = 0$$

$$j=1,2. \tag{41}$$

$$\frac{\partial L}{\partial X_j} = - P_j^* - \sum_{i=3}^{n-1} a_{ij} P_i^D$$

$$+ P_j^D \{h_j (X_j - Z_j)^{\mu_j} + m_j M_j^{\mu_j}\}^{\frac{1}{\mu_j} - 1} h(X_j - Z_j)^{\mu_j - 1} = 0$$

$$j=3,4,\ldots,nt \tag{42}$$

$$\frac{\partial L}{\partial X_j} = - P_j^* - \sum_{i=3}^{n-1} a_{ij} P_i^D + P_j^D = 0 \qquad j= nt+1,\ldots,n \tag{43}$$

$$\frac{\partial L}{\partial X_{n+1}} = - \sum_{i=3}^{n-1} a_{ij} P_i^D + P_{n+1}^D = 0 \tag{44}$$

$$\frac{\partial L}{\partial N_j} = P_j^* A_j \{\cdot\}^{\frac{1}{\rho_j} - 1} (\cdot)^{\rho_j - 1} K_j^{\alpha_j} N_j^{-\alpha_j} (1 - \alpha_j) a_j - W = 0$$

$j=1,2,...,n$ (45)

$$\frac{\partial L}{\partial K_j} = P_j^* A_j \{\cdot\}^{\frac{1}{\rho_j} - 1} (\cdot)^{\rho_j - 1} K_j^{\alpha_j - 1} N_j^{1 - \alpha_j} \alpha_j \cdot a_j - Q - P_{n+1}^D \cdot \delta_j = 0$$

$j=1,2,...,n$ (46)

$$\frac{\partial L}{\partial X_{ij}} = P_j^* A_j \{\cdot\}^{\frac{1}{\rho_j} - 1} (\cdot)^{\frac{\rho_j - 1}{\gamma_j}} X_{ij}^{\gamma_j - 1} c_j b_j - P_i^D = 0 \quad (47)$$

$j=1,2,...,n$ (47)

$$\frac{\partial L}{\partial X_{2j}} = P_j^* A_j \{\cdot\}^{\frac{1}{\rho_j} - 1} (\cdot)^{\frac{\rho_j - 1}{\gamma_j}} X_{2j}^{\gamma_j - 1} d_j b_j - P_2^D = 0$$

$j=1,2,...,n$ (48)

$$\frac{\partial L}{\partial Z_j} = - P_j^D \{h_j (x_j - Z_j)^{\mu_j}$$

$$+ m_j M_j^{\mu_j}\}^{\frac{1}{\mu_j} - 1} h_j (X_j - Z_j)^{\mu_j - 1} t\, v\, P_j^E Z_j^{0^{-\frac{1}{\varepsilon_j}}} Z_j^{\frac{1}{\varepsilon_j}} = 0$$

$j=1,2,...,nt$ (49)

$$\frac{\partial L}{\partial M_j} = P_j^D \{h_j (X_j - Z_j)^{\mu_j} + m_j M_j^{\mu_j}\}^{\frac{1}{\mu_j} - 1} m_j M_j - v\, P_j^M = 0$$

$j=1,2,...,nt$ (50)

Let

$$P^C_s \equiv \sum_{j=1}^{n-1} k_{js} P^D_j \qquad (51)$$

and rearranging (40) we obtain

$$c_s = q_s + \frac{\beta_s}{P^C_s} \qquad s=1,2,\ldots,ns \quad (40')$$

Summing (40) over the ns consumer goods we get

$$\sum_{s=1}^{ns} \beta_s = 1 = \sum_{s=1}^{ns} P^C_s C_s - \sum_{s=1}^{ns} P^C_s q_s = E - \sum_{s=1}^{ns} P^C_s q_s$$

where

$$E \equiv \sum_{s=1}^{ns} P^C_s \cdot C_s$$

Equation (40') gives the demand for consumer goods, with prices normalized so as to make $E - \sum_{s=1}^{ns} P^C_s \cdot q_s$ equal to unity. These consumption demand equations are the same as those in the general equilibrium model.

Multiplying equations (17) - (20) with N_j, K_j, X_{1j}, X_{2j}, respectively and summing we obtain

$$P^*_j \cdot X_j = W \cdot N_j + (P_{n+1}\delta_j + Q)K_j + P^D_1 X_{1j} + P^D_2 X_{2j}$$

$$j=1,2,\ldots,n \qquad (53)$$

where W is the shadow wage rate; P^D_1 and P^D_2 the shadow prices of the two energy commodities and Q the shadow value of capital services. P^*_j apparently is the value added price of commodity j, the returns to the energy inputs being inclued in the concept of value added. Thus, in optimum the production function constraints are satisfied, there are zero profits in all sectors and equations (17)-(20) are the familiar conditions for profit maximum, which, together with the production functions determines the sectoral demands for the four flexible inputs. Therefore, as far as production technology and implied producer behavior are concerned there is a complete correspondence between the general equilibrium model and the maximum conditions in the optimization version.

Define the producer costs of the n commodities as

$$\begin{cases} P_j \equiv P_j^* + \sum_{i=3}^{n-1} a_{ij} P_i^D + v\bar{P}_j \bar{b}_j & j=1,2 \\ P_j \equiv P_j^* + \sum_{i=3}^{n-1} a_{ij} P_i^D & j=3,4,\ldots, \end{cases}$$

Substituting (54) into (41) and (42) and supposing constraints (31) and (32) hold as equalities in optimum, we obtain

$$X_j - Z_j = h_j^{\frac{1}{1-\mu_j}} \left(\frac{P_j^D}{P_j}\right)^{\frac{1}{1-\mu_j}} \cdot \left[\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=1}^{n} X_{ji}\right] \qquad j=1,2 \quad (41')$$

and

$$X_j - Z_j = h_j^{\frac{1}{1-\mu_j}} \left(\frac{P_j^D}{P_j}\right)^{\frac{1}{1-\mu_j}} \left[\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=1}^{n+1} a_{ji} X_i\right] \qquad j=3,4,\ldots nt \quad (42')$$

The right-hand sides of (41') and (42') are the domestic demand for domestically produced goods while the left-hand sides give the supplies of domestically produced goods available for domestic use. These equations correspond to the market clearing conditions for domestically produced commodities in the general equilibrium model.

Next, substituting (54), (31) and (32) (the latter two supposed to hold as equalities) into (50) we obtain the import demand functions:

$$\begin{cases} M_j = m_j^{\frac{1}{1-\mu_j}} \left(\dfrac{P_j^D}{vP_j^M}\right)^{\frac{1}{1-\mu_j}} \cdot \left[\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=1}^{n} X_{ji}\right] & j=1,2 \\ M_j = m_j^{\frac{1}{1-\mu_j}} \left(\dfrac{P_j^D}{vP_j^M}\right)^{\frac{1}{1-\mu_j}} \cdot \left[\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=1}^{n+1} a_{ji} X_i\right] & j=3,4,\ldots,nt \end{cases} \qquad (50')$$

Since the optimization version implicitly equates import supply and import demand, (50') corresponds to the market clearing conditions for import goods in the general equilibrium model.

Substituting (41), (42) and (54) into (49) and rearranging, we obtain the export demand functions

$$Z_j = Z_j^0 \left(\frac{P_j}{vP_j^E}\right)^{\varepsilon_j} \qquad j=1,2,\ldots,nt \qquad (21')$$

which corresponds to those of the general equilibrium model. Obviously, with a unitary price elasticity ($\varepsilon = -1$) for some good, the export demand for that good is not defined in the optimization version. For such commodities the corresponding term in the constraint (10) should be replaced by the logarithmic term $P_j^E \cdot Z_j^0 \cdot \ln Z_j$.

Finally we note that the equation (44) defines the price of the capital good in exactly the same way as in the general equilibrium model. By dividing Ω with this price we get a measure of the real rate of interest.

Exactly as in the one sector case we have now established the similarity between the equations of the general equilibrium model and the necessary and sufficient conditions for maximum in the optimization version for all but one relation; the balance of payments requirement.

In the same way as in the one sector example the realized balance of payments surplus will depend on the chosen foreign utility level k. In general, the higher the value of k the higher will be the balance of payments surplus. However, *à priori* we do not know what value we should assign to k in order to get a certain surplus (or deficit), even though in many cases we will be able to make fairly good estimates. Thus, we have to solve a sequence of the maximization problem with different values for k until we get the desired balance of payments. Then the solution of the optimization version will correspond to a solution of the general equilibrium model.

The correspondence shown above between the necessary and sufficient conditions for the maximization problem and the equations of the general equilibrium model rests on the assumption that the various tax parameters and the parameters imposing a sectoral structure on wages and rates of return on capital, are not used in the latter. If they are, wedges will be created between input and output prices of the goods and wages, and costs of capital services will vary between sectors.

In its present formulation the optimization version cannot account for such wedges and intersectoral price differences. There are no limits on the sectoral distribution of primary and intermediate inputs, and hence an optimum solution requires inputs to be distributed so that the marginal value products, in terms of the objective function, are equalized in all cases. Their shadow prices will then be the same in all cases.

However, intersectoral input price differences can be introduced in the optimization version by imposing lower and upper bounds on the use of different inputs in different sectors. (see, for instance, Zalai (1980)). The problem is to know what bounds to impose in order to get a certain price structure for some input. On the other hand it may be argued that when the model is used for projections into the future, often it is more reasonable to impose bounds on the allocation of primary and intermediate inputs than to impose certain intersectoral price structures of these inputs.

3. THE INCORPORATION OF THE ACTIVITY ANALYSIS SUBMODEL

Suppose that for one sector (say Sector 1) we have a detailed activity analysis model which can be formulated in the following way:

$$\min \sum_{i=1}^{m1} \omega_i U_i \tag{55}$$

subject to:

$$\sum_{j=1}^{na} g_{ij}Y_j + U_i \geq 0 \qquad i=1,2,\ldots,m1 \qquad (56)$$

$$\sum_{j=1}^{na} g_{ij}Y_j \geq b_i \qquad i=m1+1,\ldots,m. \qquad (57)$$

$$\sum_{j=1}^{na} h_{ij}Y_j \leq S_i \qquad i=1,2,\ldots,s \qquad (58)$$

There are m commodities and s capacities in the model. Of the m commodities, m1 can be supplied from the rest of the economy. The remaining m-m1 goods are only produced within sector 1 and may be final output as well as internal intermediate goods. The vector U denotes the amounts of the external inputs delivered to sector 1. The vector Y gives the levels of the production activities and, if negative, the g_{ij} coefficients denote the use of good i in production activity j when it is run at unit level, while if they are positive, they denote the output of good i from production activity j. The h_{ij} coefficients give the amount of capacity i utilized by production activity j at unit level. The vector S is the total availabilities of the s capacities and the vector b is a set of net output requirements on the m-m1 goods produced within sector 1.

Thus the problem is to minimize the cost (55) of external supplies to sector 1, subject to the restrictions that the supplies of the m1 external commodities shall be at least as large as the net use of them in sector 1 (constraint (56)), that for nonexternal commodities net supply shall be at least b (constraint (57)) and that available capacities must not be exceeded (constraint (58)).

In general the commodity classification is more detailed in the activity analysis submodel than in the Bergman-Por model. Therefore, we must introduce an aggregation interface.

We will assume a fixed coefficient aggregation. Let X_1 be the level of sector 1 output in the general equilibrium model. We then assume there is a vector r which disaggregates X_1 into the m-m1 net outputs of the activity analysis model[1/]. Consequently, $b = r\,X_1$. Thus, we can rewrite the constraint (57) as

$$r\,X_1 - G_2Y \leq 0 \tag{59}$$

where G_2 is the matrix of g_{ij} coefficients from rows m1+1 to m.

Let R be a [(n+1)*m1] matrix which aggregates the m1 external commodities of the activity analysis model into the n-1 intermediate and two primary inputs of the general equilibrium model. For a given vector U of external supplies to the activity analysis model, the use of goods according to the general equilibrium model classification are:

$$a1 = RU \tag{60}$$

where a1 is the input vector of sector 1, including inputs of primary factors of production.

To integrate the activity analysis submodel of sector 1 into the optimization version of the general equilibrium model, we first delete the production function of sector 1, as well as the first column of the matrix of nonenergy input-coefficients. Instead we add the constraint system (56)-(58) of the activity

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1/ If some of the m-m1 goods are pure internal intermediate goods, so that the element of b corresponding to that good is zero, then the corresponding element in r is also zero.

analysis model to the remaining constraints of the optimization version in the following way.

The production function constraint (30) for sector 1 is replaced by:

$$r\,X_i - G_2Y \leq 0 \qquad (61)$$

$$\sum_{j=1}^{na} h_{ij}Y_j \leq S_i \qquad i=1,2,\ldots,s \qquad (62)$$

In each of the market equilibrium constraints the terms representing the inputs into sector 1 are replaced by the corresponding elements of the input vector RG_1Y, where G_1 is the matrix of g_{ij} coefficients of the first m_1 rows.

With these changes the activity analysis submodel has been integrated into the optimization version of the general equilibrium model. For the sake of completeness we now state the whole integrated optimization version.

The objective function

$$\max_{\{C_sX_jK_jN_jX_{2j}Z_jM_j\}} \sum_{s=1}^{ns} \beta_s \ln(C_s - q_s) \; , \; \sum_{s=1}^{ns} \beta_s = 1$$

subject to

production technology constraints

$$r_i\,X_1 - \sum_{j=1}^{n\alpha} g_{ij}Y_j \leq 0 \qquad i=m1+1,\ldots,m$$

$$\sum_{j=1}^{n\alpha} h_{ij}\,Y_j \leq S_i \qquad i=1,2,\ldots,s$$

$$X_j - A_j \left\{ a_j \left(K_j^{\alpha_j} N_j^{1-\alpha_j} C^{d_j t} \right)^{\rho_j} \right.$$

$$\left. + b_j \left(c_j X_{1j}^{\gamma_j} + d_j X_{2j}^{\gamma_j} \right)^{\rho_j/\gamma_j} e^{\lambda_j^* \rho_j t} \right\}^{1/\rho_j} \leq 0 \qquad j=2,3,\ldots,n$$

market clearing constraints

energy

$$\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=2}^{n} X_{1i} - \sum_{h=1}^{m1} r_{jh} \sum_{k=1}^{na} g_{hk} Y_k$$

$$- \left\{ h_j (X_j - Z_j)^{\mu_j} + m_j M_j^{\mu_j} \right\}^{\frac{1}{\mu_j}} \leq 0 \qquad j=1,2,.$$

non-energy trading sectors

$$\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=2}^{n+1} a_{ji} X_i - \sum_{h=1}^{m2} r_{jh} \sum_{k=1}^{na} g_{hk} Y_k$$

$$- \left\{ h_j (X_j - Z_j)^{\mu_j} + m_j M_j^{\mu_j} \right\}^{\frac{1}{\mu_j}} \leq 0 \qquad j=3,4,\ldots,nt$$

non-trade sectors, except the public sector

$$\sum_{s=1}^{ns} k_{js} C_s + \sum_{i=2}^{n+1} a_{ji} X_i - \sum_{h=1}^{m1} r_{jh} \sum_{k=1}^{na} g_{hk} Y_k - X_j \leq 0$$

$$j=nt+1,\ldots,n-1$$

public sector

$$X_n \geq G$$

investment

$$X_{n+1} - \sum_{j=2}^{n} \delta_j K_j - \delta_1 \left(\sum_{h=1}^{m1} r_{nh} \sum_{k=1}^{na} g_{nk} Y_k \right) \geq I$$

capital and labour

$$\sum_{j=2}^{n} K_j - \sum_{h=1}^{m1} r_{nh} \sum_{k=1}^{na} g_{nk} Y_k \leq K$$

$$\sum_{j=2}^{n} N_j - \sum_{h=1}^{m1} r_{n+1;h} \sum_{k=1}^{na} g_{n+1;k} Y_k \leq N$$

utility constraint for the rest of the world

$$\sum_{j=1}^{nt} \left(\left(\frac{\varepsilon_j}{1+\varepsilon_j} \right) P_j^E \, Z_j^{0^{-\frac{1}{\varepsilon_j}}} \, Z_j^{\left(\frac{1+\varepsilon_j}{\varepsilon_j}\right)} - P_j^M M_j \right) - \sum_{j=1}^{2} \bar{P}_j \bar{b}_j X_j \geq k$$

The incorporation of the activity analysis submodel does not alter the structure of the optimization problem in any fundamental way. One nonlinear constraint (the production function of sector 1) is replaced by m-m1+s linear constraints. The four nonlinear variables N_1, K_1, X_{11}, and X_{12} are replaced by the na linear variables Y.

4. PRELIMINARY COMPUTATIONAL EXPERIENCE

The nonlinear maximization problem described in section 2.2 has been implemented and solved for two cases: n=3 and n=8. The model is solved by the MINOS software system developed at Stanford's

Systems Optimization Laboratory by B. Murtagh and M. Saunders (1981).

In the first testcase (n=3), nt equals 2 and ns equals 5. This problem has 23 nonlinear variables (that is, variables entering the objective and/or the constraints in a nonlinear way) and 3 linear ones. Of the 23 nonlinear variables, 5 enter the objective function. There are 12 constraints, 6 of which are nonlinear.

In the second testcase (n=8), nt equals 6 and ns equals 7. There are 57 nonlinear variables and 9 linear ones. 7 of the nonlinear variables appear in the objective function. The number of constraints are 26 and 15 of them are nonlinear.

In table 1 some preliminary computational experience is recorded. In both cases the algorithm started from scratch (that is, no initial basis was provided). However, in both cases, the initial valuation of the objective and the constraint functions and their gradients were made at the known optimum point. The results of table 1 are not necessarily the most efficient ones. Time has not yet allowed a thorough testing, experimenting with the various options in the MINOS software system.

Table 1. Summary of results from testcases

	1. n=3	2. n=8
Number of major iterations	5	25
Number of total iterations	61	490
Number of evaluations of objective and constraint functions and their gradients	180	1160
Time, seconds	15	170

The results of the test cases are satisfactory. Experience from the general equilibrium model shows that disaggregation beyond the 8 sector level rarely contributes much additional information, so the practical feasibility of the optimization version should be clear. However, how well it works with further sectoral disaggregation and with the addition of a considerable number of more linear constraints and variables, which will be the case, when an activity analysis submodel is incorporated, has yet to be studied.

APPENDIX

The following is an explanation of the variables and parameters used in section 2.2.

Endogenous variables

X_j gross output in sector j=1,2,...,n

X_{n+1} output of investment goods

X_{ij} use of commodity i=1,2 in sector j=1,2,...,n

K_j use of capital services in sector j=1,2,...,n

N_j use of labour in sector j=1,2,...,n

Z_j export of production sector output j=1,2,...,nt

M_j imports of goods competing with production sector j=1,2,...,nt

C_s household consumption of consumption good s=1,2,...,ns

E total household consumption expenditures

P_j^* value added per unit of gross output in sector j=1,2,...,n

P_j price of production sector output j=1,2,...,n

P_i^D user price of commodity i=1,2,...,n+1

P_i^C price of consumption good i=1,2,...,ns

v the exchange rate

W the wage rate

Q the cost of capital services net of depreciation.

Exogenous variables

P_j^M world market prices of import goods in foreign currency j=1,2,...,nt

P_j^E world market prices in foreign currency of goods with which domestically produced goods are competing j=1,2,...,nt

$\bar{P}_j$ prices in foreign currency on complementary imports used as inputs in sector j=0,1

G public expenditures on goods and services

I net investment requirement

N total supply of labor

K total supply of capital

k required 'utility' level for the rest of the world

Parameters

k_{js} use of commodity j=1,2,.n-1 in consumption good s=1,2,...,ns

a_{ji} use of commodity j=3,4,...,n-1 in production of good i=1,2,...,n+1

δ_j annual rate of depreciation of capital in sector j=1,2,...,n

A_j, α_j, a_j, b_j, c_j, d_j, λ_j, λ_j^* production function parameters

ρ_j, γ_j parameters determining the elasticities of substitution on the production functions

h_j, m_j, μ_j parameters of the composite good aggregation functions

z_j^0 constant term of the export function $j=1,2,\ldots,nt$

ε_j the price elasticity of exports $j=1,2,\ldots,nt$

$\bar{b}_j$ complementary imports used on sector $j=1,2,$ per unit of output

β_s distribution parameter in the domestic utility function $s=1,2,\ldots,ns$

q_s minimum consumption requirements of the consumption goods $s=1,2,\ldots,ns$.

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