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LINEAR PROGRAMMING AND ENTROPY MAXIMIZING MODELS

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For any assignment/interaction problem, let c_{ij} be transportation cost between i and j, T_{ij} the flow, O_i and D_j row and column sums of $\{T_{ij}\}$.

Then, the transportation problem of linear programming is

$$\text{Min} \quad C = \sum_{ij} T_{ij} c_{ij} \tag{1}$$

s.t.

$$\sum_j T_{ij} = O_i \tag{2}$$

$$\sum_i T_{ij} = D_j \quad . \tag{3}$$

In a situation where C takes a sub-optimal value, it can be shown that the probability of $\{T_{ij}\}$ occurring is proportional to

$$W = \frac{T!}{\prod_{ij} T_{ij}!} \tag{4}$$

and for many purposes a useful assignment (e.g. for an urban transport problem) is obtained by maximizing log W subject to (2) and (3) <u>and</u>

$$\sum_{ij} T_{ij} c_{ij} = C \quad , \tag{5}$$

i.e.

$$\text{Max } S = -\sum_{ij} \log T_{ij}! \tag{6}$$

(which is an entropy function) subject to (2), (3), (5). This gives (Wilson [5, 6]).

$$T_{ij} = e^{-\lambda_i^{(1)}} e^{-\lambda_j^{(2)}} e^{-\beta c_{ij}} \tag{7}$$

where $\lambda_i^{(1)}$, $\lambda_j^{(2)}$, and β are the Lagrangian multipliers associated with (2), (3), and (5) respectively. This is written more conveniently as

$$T_{ij} = A_i B_j O_i D_j e^{-\beta c_{ij}} \tag{8}$$

where

$$A_i O_i = e^{-\lambda_i^{(1)}} \tag{9}$$

$$B_j D_j = e^{-\lambda_i^{(2)}} \quad . \tag{10}$$

A_i and B_j are calculated to ensure that (2) and (3) are satisfied:

$$A_i = 1 \Big/ \sum_j B_j D_j e^{-\beta c_{ij}} \tag{11}$$

$$B_j = 1 \Big/ \sum_i A_i O_i e^{-\beta c_{ij}} \quad . \tag{12}$$

These equations are solved iteratively and converge (Evans [2], Bacharach [1]). β can be found by solving (5) numerically.

The linear programming model--(1)-(3)--and the entropy maximizing model--(8), (11), (12)--can be linked as follows:

as $\beta \to \infty$ in (8), (11), (12)

$T_{ij} \to$ the linear programming T_{ij}

(Evans [3])

$$\frac{-\lambda_i^{(1)}}{\beta} \to \alpha_i \quad , \quad \frac{-\lambda_i^{(2)}}{\beta} \to \beta_j$$

(Wilson and Senior [8])

where α_i, β_j are the dual variables associated with (2) and (3) in the linear program. For a residential location model application, see Senior and Wilson [4], and a general review of related models, see Wilson [7].

Note

Equations (11) and (12) can be seen as part of a general matrix adjustment procedure: given $\hat{T}_{ij}$

s.t. $$\sum_j \hat{T}_{ij} \neq O_i$$
$$\sum_i \hat{T}_{ij} \neq D_j$$

form

$$T_{ij} = A_i B_j \hat{T}_{ij}$$

s.t. (2) and (3) are satisfied. Then

$$A_i = O_i \Big/ \sum_j B_j \hat{T}_{ij}$$

$$B_j = D_j \Big/ \sum_i A_i \hat{T}_{ij} \quad .$$

Computationally, proceed as follows:

$$T_{ij}^{(2n+1)} = T_{ij}^{(2n)} \cdot \frac{O_i}{\sum_j T_{ij}^{(2n)}} \quad , \qquad n \geq 0$$

$$T_{ij}^{(2n)} = T_{ij}^{(2n-1)} \cdot \frac{D_j}{\sum_i T_{ij}^{(2n-1)}} \quad , \qquad n \geq 1$$

with $T_{ij}^{(0)} = \hat{T}_{ij}$.

References

[1] Bacharach, M. Biproportional Matrices. C.U.P.

[2] Evans, A. Transportation Research, 4 (1970).

[3] Evans, A. Transportation Research, 7 (1973).

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[5] Wilson, A.G. "A Statistical Theory of Spatial Distribution Models," Transportation Research, 1 (1967).

[6] Wilson, A.G. Entropy in Urban and Regional Modelling. Pion, London, 1970.

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[8] Wilson, A.G. and Senior, M. "Entropy Maximising Models, Linear Programming Models and Their Duals," Journal of Regional Science, to be published, 1974.