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Innovative Applications of O.R.

The digital economy and advertising diffusion models: Critical mass and the Stalling equilibrium

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ABSTRACT

In the digital economy it is frequently observed that products become more valuable the larger is the number of people that use it. To account for such network effects, we introduce a new diffusion equation in a dynamic model of the firm with the aim to obtain the advertising policy that maximizes firm profits. Also an advertising budget is introduced.

First, we find that introduction of the network effect leads to a critical mass of the sales level. Only if the sales are greater or equal to this level, is it optimal for the firm to have a substantial sales level in the long run. Second, introduction of an advertising budget could result in the existence of a Stalling equilibrium. This new type of equilibrium, at which the advertising amount is at its upper bound, serves as the critical mass threshold separating growth and decline. Third, the advertising policy is continuous in the sales level either when the discount rate is large and the rate of forgetfulness is low, when the Stalling equilibrium exists, or when the rate of forgetfulness is that high that the firm is always in decline. Furthermore, existence of the Stalling equilibrium involves an important message to the management of the firm in that a slight increase of the advertising budget would open up an avenue of growth in sales.

1. Introduction

We consider a firm that sells products, which become more valuable if they attract a larger number of customers. The purpose of this paper is to determine advertising strategies for these products where the objective is to maximize the firm's discounted profit stream.

The research is motivated by the fact that in the digital economy it is frequently observed that products attracting a larger customer base are more valuable. Here we think of online games, communication platforms like Whatsapp, Zoom and Microsoft Teams, social media platforms like YouTube, Facebook, and Instagram, and online platforms for selling products like Amazon and Zalando. Also in the case of smart products like automated vehicles this phenomenon can be observed. For instance, as noted by Dawid et al. (2017), the ability of an automated vehicle to avoid a traffic jam will increase with the number of other automated vehicles it can communicate and coordinate with.

Evans and Schmalensee (2010) state that firms with substantial network effects may be able to grow rapidly because customers attract more customers. As an example, Evans and Schmalensee (2010) mention MySpace, which grew to more than 2 million registered users in its first year. Another example is Facebook. Two weeks after its launch in February 2004, two-thirds of the students of Harvard College were using it. After achieving substantial participation from other colleges as well, it opened its network in 2006. Within two years it reached 110 million users (Evans & Schmalensee, 2010).

To achieve our purpose of designing advertising strategies for such products, we develop a dynamic model of the firm in which we introduce a diffusion equation that includes this network effect. In particular, we impose that consumer utility of using the product increases in the number of consumers. In the diffusion equation this implies that when the current sales level is larger, it is easier for a consumer to convince a currently non-user to buy the product. Advertising can amplify

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[1] See https://www.bloomberg.com/graphics/2019-amazon-reach-across-markets/ (last accessed February 27th, 2023)
[2] See https://www.cnbc.com/2018/03/29/why-microsoft-failed-in-phones.html (last accessed February 27th, 2023)

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this effect, where it increases the likelihood of product adoption as a result of encounters between users and non-users.

The analysis of our model results in the identification of a critical mass level for the sales, such that only when this level is reached, it is optimal for the firm to have a substantial sales level in the long run. The network effect is at work here, in that a sufficient number of customers should have been attracted in order for the firm to be able to grow without advertising expenses being too high. This is exemplified in the start of Facebook that opened its network only after it had reached substantial participation from Harvard College and other colleges. Another example is Amazon when it launched its business as online bookstore. Once it acquired a large enough customer base it opened its platform Amazon Marketplace for other sellers. Because of the large number of users, it attracted a substantial number of sellers. These sellers, in turn, extended Amazon's product variety[1] and made the platform more attractive for customers. An example for product failure due to network effects is Windows Phone. This operating system entered the market too late and could never attract a sufficient amount of users. Then, it did not pay off for important app developers to make their product available on Windows Phone[2]. The lack of apps in turn made the product less attractive for users in comparison with other firms, which resulted in the failure of Windows Phone.

We analyze the problem with and without a budget constraint. Imposing a budget constraint implies that the rate of advertising is bounded from above. As the impact of advertising is not tangible, it is often first at risk to be cut when a firm faces financial difficulties. We obtain, however, that a low budget can be very problematic for a firm's success. In particular, we show that the resulting model solution could exhibit a Stalling equilibrium (Feichtinger et al., 2021; Grass et al., 2021), which is a steady state where the advertising rate is at its upper bound. Due to this upper bound and the network effect, it is impossible for the firm to increase the sales level from there on. In fact, the firm has the choice between keeping the advertising rate at its maximum with the sales level being constant, or choosing a lower advertising rate with the consequence that sales decline and approach zero in the long run. Right at the Stalling equilibrium it is optimal to choose for maximal advertising so that the firm admits the steady state sales level throughout. It follows that the Stalling equilibrium serves as the critical mass threshold separating growth and decline of the sales.

Interesting from a mathematical point of view is that at the Stalling equilibrium the solution is abnormal when the discount rate is small. In this case the marginal value of the sales level, and thus the costate variable, is very large, where it is infinite at the Stalling equilibrium itself. This is understandable in the sense that an infinitesimal increase of the sales level would make a growth pattern possible for the firm. Without this infinitesimal increase, at best the firm admits the sales level at the Stalling equilibrium forever with maximal advertising effort. A solution with an infinite value of the costate is not an admissible solution for the maximum principle. Therefore one has to resort to the abnormal problem. Still this mathematical feature contains also an important message for the firm's management. When the Stalling equilibrium exists and the firm's sales level is slightly below or equal to the level at this equilibrium, a slight increase of the advertising budget would enable the firm to increase sales, which in turn would open up an avenue of sales growth.

Also relevant for the advertising strategy is that we obtain that the advertising rate is a continuous function of the sales level, except when the discount rate is small and the rate of forgetfulness in the diffusion equation has an intermediate level. In the latter case the solution is history-dependent with a Skiba point (Forster, 1975; Sethi, 1977, 1979; Skiba, 1978) serving as the critical mass threshold for the sales. Right at the Skiba point the firm is indifferent between choosing for growth or decline. The advertising policy function is discontinuous in the sense that the growth trajectory begins with a considerably higher advertising rate than the decline trajectory. If the rate of forgetfulness is smaller, it is easier for the firm to force sales growth, implying that advertising effort does not jump there. Also, if the discount rate is larger, future profits have less value, and therefore the firm does not want to have large advertising costs at the beginning of a growth trajectory, which again implies that the advertising policy is continuous at the Skiba point. A Skiba point where the control variable is continuous is denoted as a weak Skiba point (Grass et al., 2008), where the term weak is used to describe that at the Skiba point itself the firm will neither grow nor decline, so the sales level stays constant.

The paper is organized as follows. Section 2 gives an overview of the relevant literature in this area, whereas in Section 3 we present the model. In Section 4 we analyze the model variant without budget constraint. Section 5 analyzes the full model, thus with the advertising upper bound included. We conclude our paper in Section 6.

2. Literature review

The advertising diffusion literature starts out with Vidale and Wolfe (1957). Advertising appears linearly in their diffusion equation, where advertising is directed to the potential consumers not currently buying from the firm. Sethi (1973) incorporates the Vidale and Wolfe diffusion dynamics into an optimal control model and provides a full analysis. Ozga (1960) takes advertising to be a word-of-mouth phenomenon, which is analyzed within the diffusion context by Gould (1970) and Sethi (1979). For an overview of the early literature in this area we refer to Feichtinger et al. (1994). Feinberg (2001) builds in a convex–concave effect of advertising and shows that this allows for periodic solutions. Bass (1969) introduces a diffusion dynamics departing from the fact that there are different reasons for adopting a product. One can either have an eye for new developments or imitate other customers. This concept is used in the models of Fruchter and Van den Bulte (2011), Van den Bulte and Joshi (2007). Fruchter et al. (2022) propose a setting where consumers are divided in two segments, elite and followers, where the elite users exercise a positive peer effect on the followers. The firm has the ability to advertise in each of the elite and follower segments. Similar to most of these papers, our paper also analyzes an optimal control model with an advertising diffusion function. The difference is that our diffusion function is designed for products that become more valuable the larger is the number of customers.

This network effect is an essential element in our model. The literature on this topic starts with Rohlfs (1974), in which it is noted that a subscriber's utility derived from a communications service, increases as others join the system. As we have derived from our dynamic optimization model, Rohlfs (1974) also obtains a critical mass constraint that must be satisfied for a firm to be viable. Evans and Schmalensee (2010) build on Rohlf's work to show for two-sided platforms how the size of the critical mass depends on the nature of network effects, the dynamics of consumer behavior, and the distribution of consumer tastes. Up until now this literature refrains from deriving profit-maximizing strategies, and this is what we do in the present paper.

Advertising strategies where advertising is restricted by some budget have also been studied. For instance, Yang et al. (2022) investigate optimal advertising strategies, where the budget is such that within some time frame the firm cannot spend more on advertising than the available budget. This is different from what we have, because in our case advertising is bounded from above at each specific point in time. To model the effect of advertising on the firm's sales level, Yang et al. (2022) employ the Vidale–Wolfe framework. Our paper is different in that our optimal control model replaces the Vidale Wolfe framework

[1] See https://www.bloomberg.com/graphics/2019-amazon-reach-across-markets/ (last accessed February 27th, 2023)

[2] See https://www.cnbc.com/2018/03/29/why-microsoft-failed-in-phones.html (last accessed February 27th, 2023)

by a diffusion function build on the idea that products become more valuable the larger is the number of customers.

Further recent contributions that consider the impact of advertising in an optimal control framework include Chenavaz and Eynan (2021), which analyzes the impact of advertising and goodwill on the Veblen effect and Caulkins et al. (2017), Chenavaz et al. (2020) which study the interaction of product quality, advertising and pricing. We share with these papers the optimal control approach, but differ in that we focus on deriving the optimal advertising strategy for products that become more valuable when more people are using it.

3. Model formulation

We consider a monopolist launching a new product at time $t = 0$ in a market with a network effect as a driving factor for product adoption. The total market size is normalized to one. From the time of the launch on, the firm can control the advertising denoted by $u(t)$. Let $x(t)$ be the fraction of customers, who use the product at time t. The inverse demand function is assumed to be linear and given by

$$p(x(t)) = d - \varphi x(t),$$

in which p is the price, d is a constant representing the reservation price, and φ a constant representing the sensitivity of price with respect to quantity. The kind of examples we have in mind belong to the so-called digital economy. In many of these platforms price plays no role (think of, for instance, Facebook) so then the parameter φ will be zero. In such a case price should be read as revenue per customer. This is different, however, in online platforms designed to sell products like Amazon and Zalando, in which case φ will be positive. In our base example we choose a rather flat inverse demand function to cover both cases.

It follows that the firm's instantaneous profit, π, is given by

$$\pi(x(t)) = (d - \varphi x(t))\, x(t) - u(t).$$

Note that the revenue maximizing level equals

$$\tilde{x} = \frac{d}{2\varphi}. \tag{1}$$

Since advertising is costly, it can be expected that the fraction of customers will never be higher than $\tilde{x}$, provided that its initial level does not exceed $\tilde{x}$.

The diffusion dynamics is given by

$$\dot{x}(t) = g(x(t))\, f(u(t)) - \delta x(t), \quad x(0) = x_0. \tag{2}$$

in which the positive parameter δ represents the forgetting rate. The diffusion function $g(x(t))$ captures the network effect, as will be explained below, and describes how diffusion depends on the number of customers. The advertising effect is given by the function $f(u(t))$. This effect will increase in the advertising rate, i.e. more advertising will generate more customers, so that the function f is increasing in u, i.e. $f'(u(t)) > 0$. A little advertising will capture the more eager customers, and, as advertising increases, it will become more and more difficult to convince additional customers to use the product. Therefore, we impose that $f(u(t))$ is a concavely increasing function, i.e $f''(u(t)) < 0$.

The diffusion function $g(x(t))$ reflects that the utility of using the product increases with the number of users, and thus with the fraction of customers using the product, $x(t)$. *For small values of* $x(t)$, the product is not that valuable and it is difficult to attract new customers. Therefore, although increasing, the diffusion function $g(x)$ has a rather flat shape for small values of $x(t)$. *For larger values of* $x(t)$ the product value goes up, leading to a more steep increase of the diffusion function. Since the total market size is fixed, *for values of* $x(t)$*even larger*, the number of non-users, which are the potential new customers, is low. Consequently, the steepness of the diffusion function first declines and eventually diffusion decreases in the fraction of current customers, $x(t)$, which reflects the market saturation effect.

All in all a diffusion function $g(x(t))$ that satisfies these characteristics must increase in a convex–concave way for $x(t)$ sufficiently small, and decrease for larger values of $x(t)$ to capture the saturation effect. Essentially it has an inflection point, $x(t) = x_1$, and a maximal value being reached for $x(t) = x_2$, for which it holds that $x_1 < x_2$. Then, in general the diffusion function satisfies

$$\begin{aligned}
&g'(x(t)) = 0, \quad g''(x(t)) = 0 \quad \text{for } x = 0,\\
&g'(x(t)) > 0, \quad g''(x(t)) > 0 \quad \text{for } x \in (0, x_1),\\
&g'(x(t)) > 0, \quad g''(x(t)) = 0 \quad \text{for } x = x_1,\\
&g'(x(t)) > 0, \quad g''(x(t)) < 0 \quad \text{for } x \in (x_1, x_2),\\
&g'(x(t)) = 0, \quad g''(x(t)) < 0 \quad \text{for } x = x_2,\\
&g'(x(t)) < 0, \quad g''(x(t)) < 0 \quad \text{for } x \in (x_2, \infty).
\end{aligned}$$

In order to carry out the calculations for the optimal control model we need to specify the functional forms. The concave increasing function $f(u(t))$ will be represented by

$$f(u(t)) = a\sqrt{u(t)},$$

in which a is a positive parameter representing the efficiency of advertising. For our diffusion function we choose the following functional form:

$$g(x(t)) = x^2(t)(1 - x(t)). \tag{3}$$

Note that since the market size is normalized to one and $x(t)$ represents the fraction of customers using the product, the fraction of non-users is given by $1 - x(t)$. Thus, the rate of encounters by users and non-users is $x(t)(1 - x(t))$. The rate at which users convince non-users to use a product depends positively on the number of people using it, as due to network effects the utility is higher when many people use the product. Assuming that rate depends linearly on $x(t)$, we arrive at the functional form specified in (3).

In the remainder of the paper we omit time argument t unless necessary for understanding.

This paper considers two different models, both being centered around the diffusion function with the network effect. The firm's objective function is the net present value (NPV) of its profit stream over an infinite time horizon, where the discount rate is denoted by $r > 0$. Consequently, our first model is given by

$$\max_u \int_0^\infty e^{-rt}\,((d - \varphi x)\,x - u)\,dt, \tag{4}$$

subject to

$$\dot{x} = x^2(1 - x)\,a\sqrt{u} - \delta x, \quad x(0) = x_0, \tag{5}$$

$$x, u \geq 0. \tag{6}$$

This model will be analyzed in Section 4.

The second model extends the first model by taking into account that within a firm one has to deal with advertising budgets. To model this, we simply employ the first model (4)–(6) and add the constraint

$$u \leq B, \tag{7}$$

in which B, the advertising budget, is a fixed constant. So, the second model is represented by (4)–(7), and will be analyzed in Section 5. Note that in Section 5.1 we will also consider a variant where the budget is a fraction of sales.

4. Optimal advertising under network effects

Here we analyze the dynamic optimization problem (4)–(6), which involves forming the current-value Hamiltonian

$$H = \lambda_0 (x(d - \varphi x) - u) + \lambda \left(a x^2 (1-x) \sqrt{u} - \delta x \right),$$

where λ_0 is the Lagrange multiplier corresponding to the objective function in the Hamiltonian and $\lambda(t)$ is the current-value co-state variable, see e.g. Sethi (2019). The following proposition presents some characterizations of the optimal solution.

Proposition 1. *The dynamic optimization problem (4)–(6) is normal, i.e.*

$\lambda_0 = 1.$

The optimal advertising rate satisfies

$$u = \left(\frac{\lambda a x^2 (1-x)}{2} \right)^2. \tag{8}$$

Furthermore, there exists a steady state $(\hat{x}_Z, \hat{u}_Z) = (0,0)$, which is a saddle point, and there are steady states with a positive value of x, satisfying

$$u = \left(\frac{\delta}{a x (1-x)} \right)^2.$$

These steady states with positive $x-$ value can only be numerically determined.

Proof. The optimal advertising rate is determined by setting the first order derivative of the Hamiltonian equal to zero, i.e.

$$H_u = -\lambda_0 + 0.5 \lambda a x^2 (1-x) \frac{1}{\sqrt{u}} = 0. \tag{9}$$

The costate equation is

$$\dot{\lambda} = (r+\delta)\lambda - \lambda_0 (d - 2\varphi x) - a\lambda \sqrt{u} \left(2x - 3x^2 \right) \tag{10}$$

The canonical system in the interior of the feasible region can be written as

$$\dot{x} = a^2 x^4 (1-x)^2 \frac{\lambda}{2\lambda_0} - \delta x \tag{11}$$

$$\dot{\lambda} = (r+\delta)\lambda - \lambda_0 (d - 2\varphi x) - \frac{\lambda^2}{2\lambda_0} a^2 x^3 (1-x)(2-3x) \tag{12}$$

Setting $\dot{x}$ and $\dot{\lambda}$ to zero, we can analytically find a steady state of the canonical system at $(\hat{x}_Z, \hat{\lambda}_Z) = (0, \frac{\lambda_0 d}{r+\delta})$, which implies that the optimal advertising rate $\hat{u_Z}$ is also zero in this steady state. In order to determine a steady state with positive market share, x, the roots of a higher order polynomial have to be found, which is not possible analytically.

In order to analyze the stability of $(\hat{x}_Z, \hat{\lambda}_Z)$, we evaluate the Jacobian, which is given by

$$J = \begin{pmatrix} a^2 \frac{\lambda}{\lambda_0} x^3 (3x-2)(x-1) - \delta & \frac{a^2}{2\lambda_0} x^4 (1-x)^2 \\ -\frac{1}{2\lambda_0} a^2 x^2 \lambda^2 (-20x + 15x^2 + 6) + 2\lambda_0 \varphi & (r+\delta) - \frac{\lambda}{\lambda_0} a^2 x^3 (1-x)(2-3x) \end{pmatrix},$$

at this point. This implies that the steady state is a saddle point with the eigenvalues $\xi_1 = r + \delta$ and $\xi_2 = -\delta$. The corresponding eigenvectors are $\chi_1 = (0,1)'$ and $\chi_2 = \left(-\frac{2\delta + r}{2\lambda_0 \varphi}, 1 \right)'$.

For an abnormal solution, it has to hold that $\lambda_0 = 0$. Condition (9) implies that for $\lambda_0 = 0$, it has to hold that $\lambda = 0$ (except for $x = 0$ and $x = 1$). Then, however, all multipliers become zero simultaneously which contradicts Pontryagin's maximum principle. Concerning $x = 1$, it is not possible to choose the control in a way such that one stays at this point (see (5)) and since for lower state values than one we know that the solution must not be abnormal, we can also exclude abnormality for $x = 1$. At $x = 0$, one can easily show that the advertising rate that maximizes the objective is zero, which we obtain by solving the normal problem. Thus, the problem is normal.

Table 1
Parameter values in the base case scenario.

r	d	φ	a	δ
0.05	10	0.1	1	0.2

From (11) we find that at an interior steady state it has to hold that

$$\hat{\lambda}_I = \frac{2\delta}{a^2 x^3 (1-x)^2}.$$

which implies when inserting this impression into (8) that at an interior steady state it has to hold that $u = \left(\frac{\delta}{ax(1-x)} \right)^2$. □

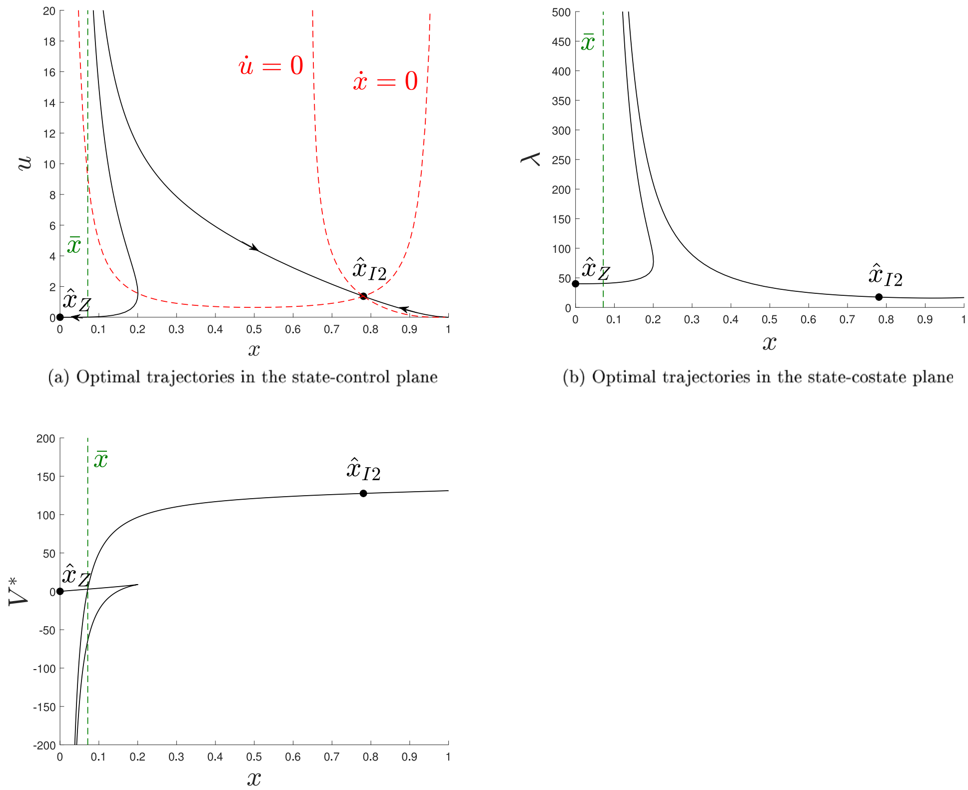
Based on the results of this proposition, we develop our numerical results. As base case scenario we use the parameter values depicted in Table 1.

Fig. 1 depicts the optimal solution for this situation. Fig. 1(a) shows that in the long run the firm is either still active with a positive number of customers, or the firm will have no customers anymore and therefore has to cease business. The solution is history-dependent in the sense that only if the initial fraction of customers is large enough, it is optimal to employ an advertising policy with the aim to let the fraction of customers approach the long run steady state where this fraction is positive. The intuition behind this result is that the utility of consuming the product increases with the number of customers. This implies that for a larger fraction of customers less advertising is needed to attract additional customers and let sales increase. So, only if the initial amount of customers is large enough, advertising expenses need not be too high to expand the firm's operations and converge to the long run steady state with a positive amount of customers. We conclude that the network effect creates the history dependence aspect of the solution, or, in other words, a critical mass level of firm activity, given by $\bar{x}$ in Fig. 1(a), is needed to remain in business in the long run. For x very large, i.e. the fraction of customers using the product is close to one, the saturation effect kicks in, resulting in a decline of sales and convergence towards the positive steady state.

Figs. 1(b) and 1(c) help to understand this history-dependent solution. Fig. 1(b) shows the development of the co-state variable, being the shadow price associated with an infinitesimal increase in the fraction of customers, along the two trajectories. The co-state has a low value either for small or large values of x. For small x, the value is low, because the firm decides to leave business anyhow. For large x, the fraction of customers using the product is already substantial, which implies that the network effect is stimulating demand to a large extent. A marginal increase in the fraction of customers will therefore have not much influence, which explains why the value of the costate is low when x is large. Some very large values of the co-state, which can be interpreted as shadow price of x (see, e.g., Grass et al. (2008)), can be observed in the intermediate segment. There, the firm wants to increase sales, but at the same time a lot of advertising is needed to do so. Then a marginal increase in the fraction of customers is very valuable because it raises customer utility, implying that the firm needs to undertake considerably less advertising expenses to pursue firm growth.

Fig. 1(c) depicts the firm value, which we denote by V^*, as a function of the initial customer fraction. Not surprising it is to observe that this value is increasing in x. Further, it confirms Fig. 1(b) in that the firm value is steeply increasing in the intermediate segment of $x-$ values. The two trajectories generate their own curves with the intersection at the critical mass level $\bar{x}$. In mathematical terms $\bar{x}$ is called a Skiba point, see, e.g., Grass et al. (2008), which separates the regions of attractions of the zero and the positive steady state.

Fig. 2 shows the development of the fraction of customers and the advertising rate over time for the two optimal trajectories converging to the zero and the positive steady state, where in both cases the starting point is the Skiba point $\bar{x}$. This means that at the initial point in time the firm is indifferent between decline, i.e. choosing the trajectory

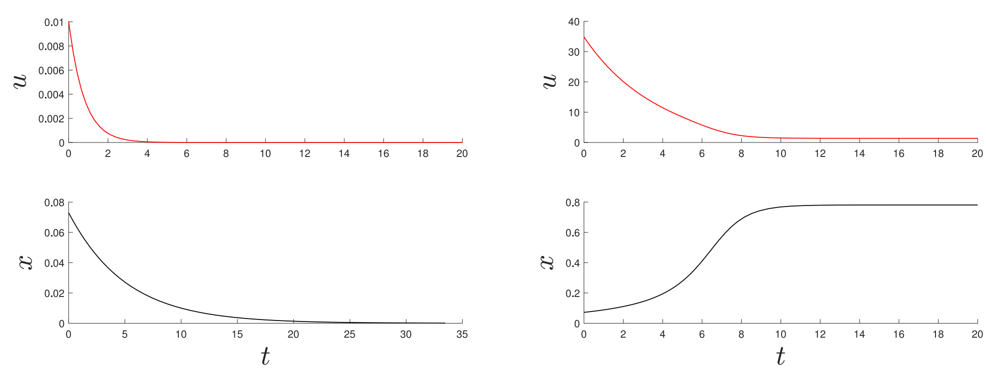


(a) Optimal trajectories in the state-control plane

(b) Optimal trajectories in the state-costate plane

(c) Value of the firm as a function of x, where the value for each x is the maximum of the depicted curves

Fig. 1. The optimal solution for the base case parameter values.



(a)

(b)

Fig. 2. The two optimal trajectories starting out from the Skiba point $\bar{x}$. *The decline pattern is depicted in Fig. 2(a), whereas the growth pattern is shown in Fig. 2(b).*

converging to the origin, or growth, thus start advertising a lot and eventually converge to the positive steady state. If the firm chooses the decline pattern, initially the customer fraction, and thus also the utility of consuming the product, has still a reasonable value. This implies that advertising has some effect. However, over time the customer fraction, and therefore also the advertising rate, decreases, until no customer is left and the firm exits the market.

Regarding the growth trajectory, the firm initially has to spend a lot on advertising to accomplish an increase in the fraction of customers using the product. As this fraction keeps on increasing, the utility of consuming the product goes up. Over time this first results in a steeper increase of x, but later on the steepness declines due to the saturation effect. In this way we get the typical S-shaped development of the sales level (compare Rogers (2003)). As the fraction of customers x increases, the same holds for the utility of using the product, and therefore advertising is less needed to accomplish an increase of the sales. This explains why advertising decreases over time along this trajectory.

Regarding a numerical solution it is important to investigate the effect of changes in the value of the different parameters. In Fig. 3 we show the qualitative solution patterns that can occur for varying levels of the forgetting rate and the discount rate. The base parameter value case from Table 1, the corresponding solution of which we extensively discussed already, belongs to region S, where S denotes *Skiba region*. For a lower forgetfulness rate, or a higher discount rate, we end up in region W, which is the *Weak Skiba region*. The resulting solution pattern, as depicted in Fig. 4, has a weak Skiba point (Grass et al., 2008), which distinguishes itself from a "normal" Skiba point in that the policy function, in this case the advertising policy, is a continuous function of the state variable, in this case the consumer fraction x. The weak Skiba point is a steady state of the canonical system (11)–(12)

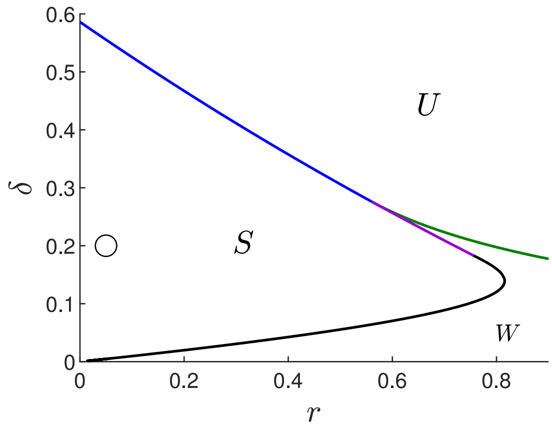


Fig. 3. Bifurcation diagram, o depicts the base case parameter values, S denotes the Skiba region, W the Weak Skiba region, and U the region with a unique steady state.

(which a "normal" Skiba point obviously is not). Thus, starting exactly on this point one will stay there, starting with a higher or lower initial state value, one will either approach the higher or the zero steady state.

A low forgetfulness rate implies that advertising has a long-lasting effect. Therefore, close to the Skiba point, being the critical mass of the sales level, the amount of advertising needed to accomplish growth is not substantially higher than the amount of advertising used to slow down decline at the pattern leading to the zero steady state. If the discount rate is higher, the firm is more myopic. Then it does not assign a high value to the long term effect of advertising. The implication is that the firm does not want to advertise too much at the beginning of the growth pattern, and therefore the policy function is continuous at the level of x corresponding to the critical mass of the sales level.

Advertising has a less long lasting effect if the rate of forgetfulness is large, whereas the long term effect of advertising is not appreciated by a myopic firm, thus if the discount rate is large. Fig. 3 shows that in such a situation we are in Region U, which denotes that the solution is *unique* in the sense that only the decline pattern is optimal. In other words, the optimal solution, as depicted in Fig. 5, is that the firm ceases business in finite time. The larger the discount rate, the lower is the forgetfulness rate that is needed for this unique solution to prevail, which makes sense.

5. Impact of the budget constraint

Here we analyze the problem (4)–(6) to which we add the budget constraint (7). To do so we start with presenting the current-value Hamiltonian

$$H = \lambda_0 (x (d - \varphi x) - cu) + \lambda \left(a x^2 (1 - x) \sqrt{u} - \delta x \right),$$

and the Lagrangian to take the constraint (7) into account:

$$L = H + \nu (B - u).$$

The following proposition presents some properties of the optimal solution.

Proposition 2. *The optimal advertising rate satisfies*

$$u = \begin{cases} \left(\frac{a\lambda x^2 (1-x)}{2c\lambda_0} \right)^2 & \text{if} \quad a\lambda x^2 (1-x) \le 2c\lambda_0 \sqrt{B} \\ B & \text{otherwise} \end{cases}.$$

Next to the steady states of Proposition 1, the two following steady states arise for $u = B$:

$$\hat{x}_{B1} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}, \tag{13}$$

$$\hat{x}_{B2} = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}, \tag{14}$$

provided that

$$a\sqrt{B} \ge 4\delta. \tag{15}$$

Proof. See the Appendix in Appendix A.1. □

To investigate the role of the boundary steady states (13) and (14) in the optimal solution, it turns out to be crucial how the $\dot{x} = 0$ isocline relates to the budget constraint $u \le B$. Let $\bar{u}$ be the advertising level associated with the $\dot{x} = 0$ isocline, i.e. $\bar{u}$ is the advertising rate that keeps the sales level constant. From (5) follows that

$$\bar{u} = \frac{\delta^2}{a^2 x^2 (1 - x)^2},$$

which admits its minimum for $x = 1/2$, corresponding to $\bar{u} = \frac{16\delta^2}{a^2}$. Fig. 6 shows the relationship between the $\dot{x} = 0$ isocline and the budget constraint. If the minimum $\bar{u} = \frac{16\delta^2}{a^2}$ exceeds the available budget B, no feasible advertising rate can prevent the sales level from declining so that in the long run the firm ceases business. Otherwise, the inequality (15) is satisfied implying that the boundary steady states (13) and (14) exist, as Fig. 6 confirms.

Fig. 6 also shows that for values of the sales level x lower than $\hat{x}_{B1}$, sales are decreasing for any level of the advertising rate u satisfying the budget constraint. This implies that $\hat{x}_{B1}$ is the critical mass threshold the sales level need to exceed in order for sales growth to be possible. Below that level, the product has too few consumers to be sufficiently attractive for surviving in the long run. Here it is the network effect that prevails, saying that product utility increases with the number of users.

In line with Feichtinger et al. (2021), $\hat{x}_{B1}$ can be denoted as *Stalling equilibrium*, which means that at the steady state $\hat{x}_{B1}$ the control u is at its upper bound, and for lower values of x the state dynamics is negative for all feasible values of u. The Stalling equilibrium has the property that maximum advertising has to be applied to keep the sales at the same level and it is not possible to find a feasible control such that the state value can be increased above this steady state. The next proposition describes the role of the boundary steady states $\hat{x}_{B1}$ and $\hat{x}_{B2}$.

Proposition 3. *In the scenario*

$$r \le \delta, \tag{16}$$

and

$$B \ge \frac{(r - 2\delta)^4}{a^2 (\delta - r)^2} \tag{17}$$

the Stalling equilibrium $\hat{x}_{B1}$ is abnormal, i.e. $\lambda_0 = 0$.

The larger boundary steady state $\hat{x}_{B2}$ is normal, $\lambda_0 = 1$, and a saddle point.

Proof. See the Appendix in Appendix A.2. □

Abnormality of the Stalling equilibrium $\hat{x}_{B1}$ goes along with the costate variable λ approaching infinity in its vicinity on the stable manifolds approaching the high and the low steady state, see Fig. 7. Note that the costate value has a shadow price interpretation, see, e.g., Grass et al. (2008), in that it equals the contribution to the objective of an infinitesimal increase of the corresponding state variable x. At the critical mass threshold $\hat{x}_{B1}$, it is not possible for the sales level to grow any further for any feasible advertising rate u. To accomplish growth an infinitesimal increase of x is needed, the value of which is measured by λ. Growth is particularly beneficial in terms of payoff, when, first, future benefits are appreciated, and, second, when it is easy to quickly increase sales. The first can happen when the discount rate r

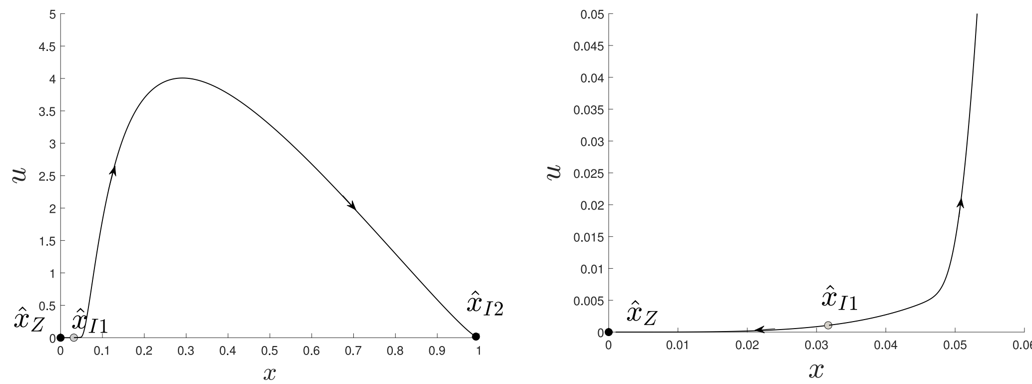


Fig. 4. Phase portrait (with a zooming in the right panel) for the weak Skiba case ($r = 0.15$ and $\delta = 0.001$).

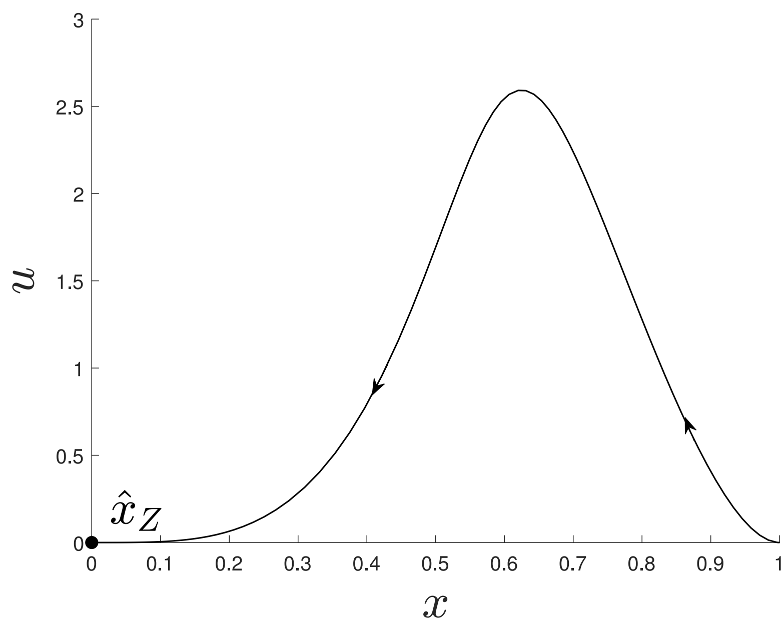


Fig. 5. Phase portrait for $\delta = 0.6$.

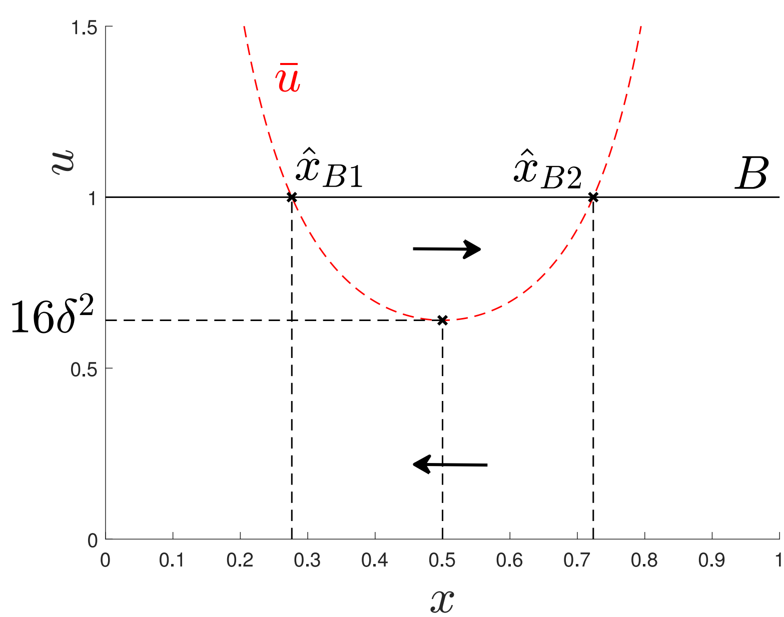


Fig. 6. The $\dot{x} = 0$ isocline $\bar{u}(x)$ versus the control upper bound, $u = B$, making clear that a critical mass of $\hat{x}_{B1}$ is needed to survive in the long run.

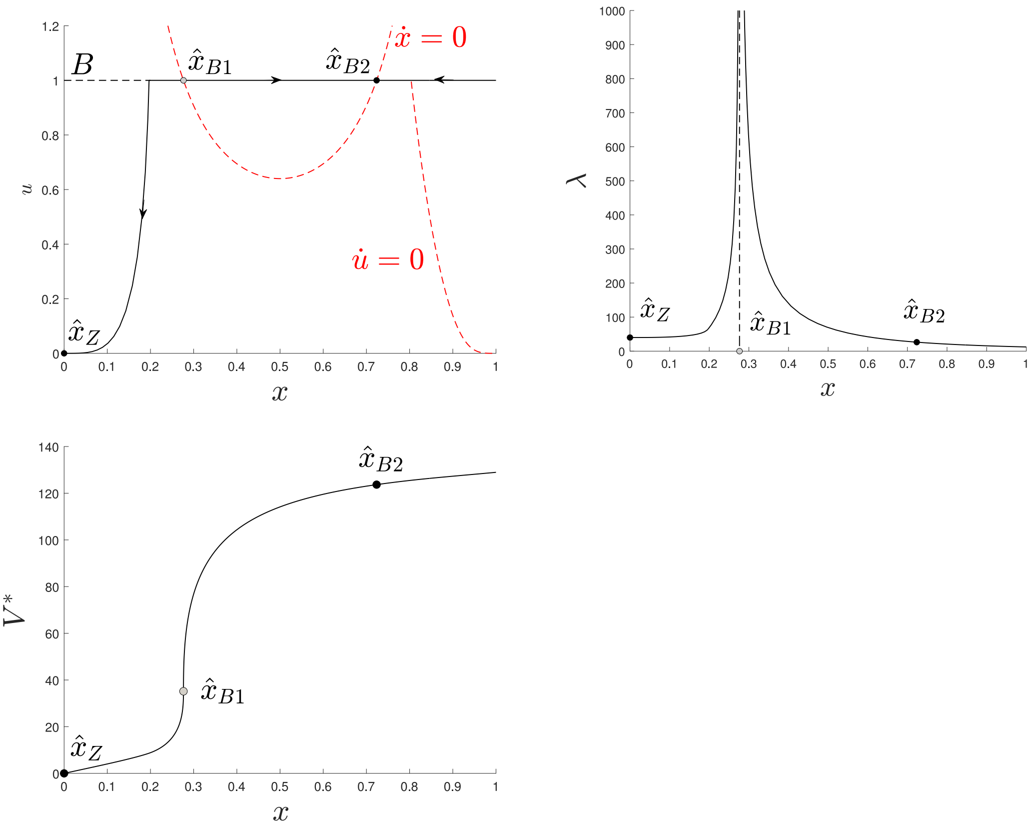


Fig. 7. Base case: phase portrait (state-costate space) left panel; phase portrait (state-costate space) right panel, corresponding objective value lower left panel.

is small, the second when the budget B is large and the diffusion effect, resulting from a high value of a, is strong. We conclude that satisfaction of conditions (16) and (17) create a scenario in which increasing the sales level will result in a high payoff. An infinitesimal increase of x at the sales level $\hat{x}_{B1}$ implies that growth of sales is a possibility for the firm. This explains why the costate variable is especially large, even infinite in this case, under the two conditions of Proposition 3. However, the Maximum principle does not allow λ to be infinite, and therefore we have to resort to the abnormal problem. We see that the firm value V is steeply increasing in x for $x = \hat{x}_{B1}$, which confirms the infinite value of the costate in its shadow price interpretation.

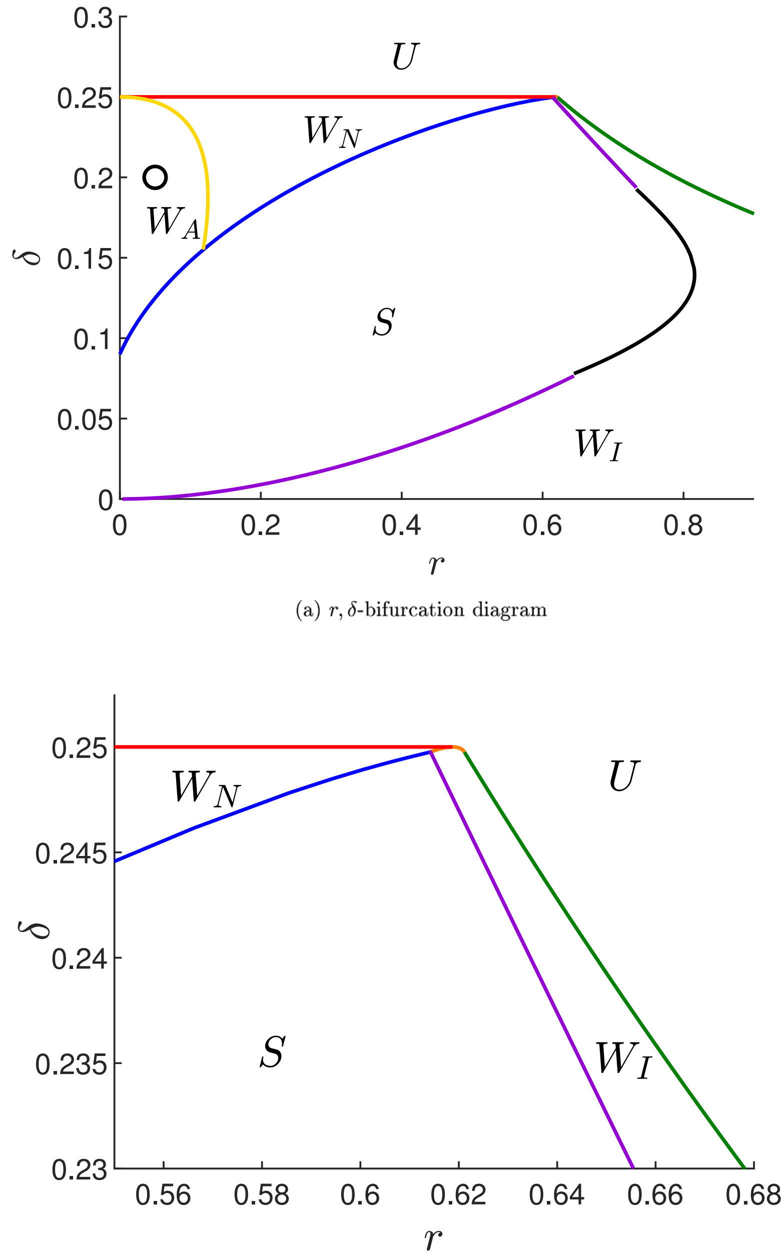
Fig. 7 presents the optimal solution for the base case parameter values of Table 1. This scenario satisfies the two parameter conditions in Proposition 3, so that conditions for growth are good. As we just said $\hat{x}_{B1}$ is the Stalling equilibrium where the solution is abnormal, and this sales level serves as the critical mass threshold separating growth and decline. In particular, three long run outcomes could prevail. First, if the initial sales level falls below the critical mass threshold $\hat{x}_{B1}$, the sales rate declines and the firm ceases business in finite time. Conditions for growth are good, but the budget is too tight for growth to let it happen. Second, if the initial sales level is larger than $\hat{x}_{B1}$, the firm spends maximally on advertising and the sales level converges to the larger boundary steady state $\hat{x}_{B2}$. Third, if the initial sales level exactly equals the critical mass threshold $\hat{x}_{B1}$, the optimal strategy is to also spend the entire budget on advertising, but this time it will only accomplish that the firm's sales level remains constant at the level $\hat{x}_{B1}$ forever.

As in the previous section, also here it holds that with numerically obtained solutions it is important to investigate the effect of changes in value of the different parameters. Compared to Fig. 3, Fig. 8 shows the effect of the budget constraint in the (r, δ)-plane. First of all, where in Fig. 3 we have W denoting a Weak Skiba solution, in Fig. 8 we have three different denotations of W, all relating to qualitatively different Weak Skiba solutions. The one that mostly corresponds to W in Fig. 3 is W_I, where the advertising rate at the critical mass threshold lies in the interior of the control region, meaning that the firm does not use the complete budget for advertising here. Contrastingly, the whole budget is spent on advertising at the critical mass thresholds denoted by W_A and W_N, where the distinction between the two lies in the fact that W_A means that the solution is abnormal at the critical mass threshold, whereas we have a normal solution in case of W_N.

Compared with Fig. 3, we can conclude that configurations are similar for small values of the forgetfulness rate δ and a large discount rate r. If the forgetfulness rate is small, it is easier for the firm to stimulate sales with advertising. For this reason it is less needed to use the whole budget for that, and therefore the constraint model solution of Fig. 8 matches the unconstrained one in Fig. 3 for small δ. If the discount rate is large, future profits are less appreciated, and therefore the firm is less eager to advertise a lot to increase future sales levels. This results in the firm being less inclined to spend the whole budget on advertising when r is large, which explains why Figs. 3 and 8 also match for large r.

Hence, the effect of introducing an advertising budget is especially noticeable in the north-west corner of the (r, δ)-bifurcation diagram. For a very large forgetfulness rate, i.e. $\delta > 0.25$, the budget level is too low to let the advertising rate be high enough to accomplish an increase in sales. Therefore, Region U applies there, which denotes a trajectory of declining sales where the firm ceases business in finite time. If the forgetfulness rate is a little bit lower and the discount rate is small, we are in the range of the conditions of Proposition 3, which corresponds to Region W_A. These conditions are good for growth, which will be pursued whenever possible. As we know, the latter means that the sales level should be beyond the critical mass level $\hat{x}_{B1}$ for sales growth to take place.

In the Region W_N, the discount rate is larger compared to the region W_S, implying that the conditions for growth are still good but not so



(a) r, δ-bifurcation diagram

(b) magnification of the upper right edge

Fig. 8. Bifurcation diagram, o depicts the base case parameter values.

good that the shadow price of an infinitesimal increase of the sales level is infinite at the critical mass threshold $\hat{x}_{B1}$ (see Fig. 9 for a typical solution pattern). Therefore, the problem is normal, but still it holds that in between the sales levels $\hat{x}_{B1}$ and $\hat{x}_{B2}$ the firm uses its whole budget for advertising in order for the sales level to converge to the larger boundary steady state $\hat{x}_{B2}$. At $\hat{x}_{B1}$ we still have the Stalling property: the full budget is used for advertising and the firm stays put at this sales level.

The solutions occurring in the regions W_I, W_A, and W_N have in common that the advertising policy is continuous in the sales level at the critical mass threshold $\hat{x}_{B1}$. However, there is a difference. In region W_I the firm's advertising policy does not use up the complete budget at the critical mass threshold $\hat{x}_{B1}$. The forgetfulness rate being low implies that advertising has a long-lasting effect and therefore the firm does not need to advertise too much to accomplish growth. This keeps the policy function continuous. In Regions W_A and W_N the discount rate is low, creating good conditions for growth. Therefore, right at $\hat{x}_{B1}$ the firm wants to choose an advertising rate that let the sales grow. However, the presence of a limited advertising budget prevents that, so that the firm has to choose a lower advertising rate that keeps the sales level constant. Consequently, the advertising policy is continuous in x, where the advertising rate equals the budget level B around the critical mass threshold $\hat{x}_{B1}$.

Fig. 10 depicts the critical mass threshold and the long run sales level, in case the firm stays in business, as a function of the budget B. For very low budget levels the firm can never advertise enough to let the sales grow. In this case sales decline until x reaches zero and the firm ceases business in finite time (the Region U). If the budget B gets larger it will eventually result in existence of a critical mass threshold $\hat{x}_{B1}$ and a long run steady state sales level $\hat{x}_{B2}$. We observe that the critical mass threshold decreases with B, indicating that it becomes easier for the firm to survive in the long run, if the budget level is larger.

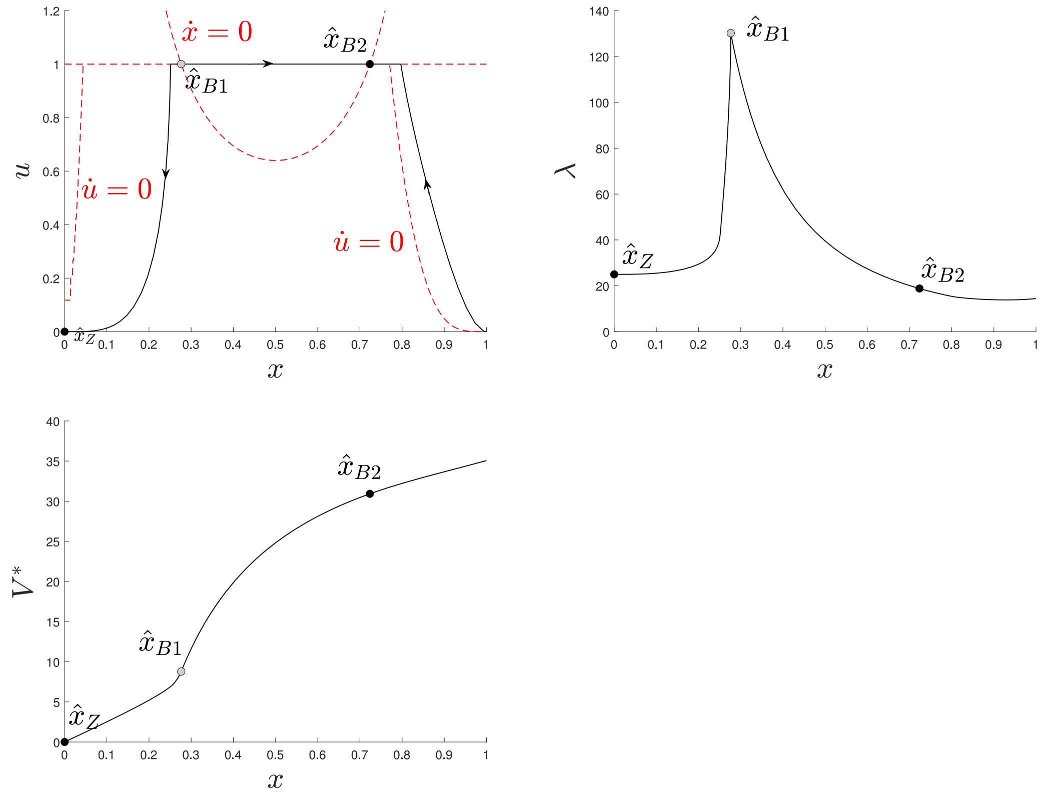


Fig. 9. The optimal solution for the base case parameter values with the exception of r being equal to 0.2.

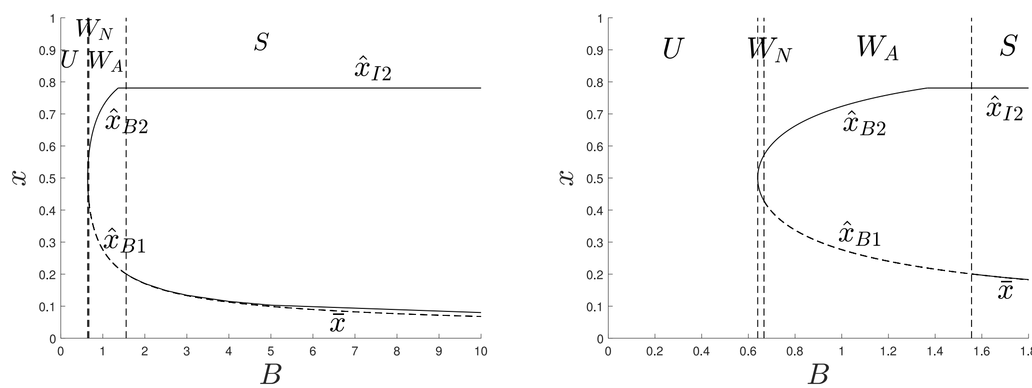


Fig. 10. Bifurcation diagram B.

With a larger B the firm gets access to higher advertising levels, so that sales can increase more, which, due to the network effect, increases consumer utility of using this product.

The long run steady states level $\hat{x}_{B2}$ first increases with B (in the regions W_N and W_A). Then, when sales reach this steady state level, the firm uses the maximum budget for advertising to keep sales at this level. The larger B is, the larger the corresponding steady state level of x will be. For even larger levels of the budget B, i.e. in region S, the unconstrained long run steady state level of the sales of the previous section is reached, denoted by $\hat{x}_{I2}$. Then the budget is so large that not all of it is needed for advertising and the optimal strategy corresponds to the case when no budget constraint is considered (compare Fig. 1).

5.1. Robustness

It makes sense that an advertising department of a firm faces a certain budget, and the implications of this we dealt with in this section. However, where up until now we supposed that the size of this budget is fixed, it may very well be that the budget depends on the current performance of the firm, see, e.g. Tellis (1998). To see whether the main conclusions of our analysis carry over to such a situation, we consider now an advertising budget that is equal to a certain fraction f of the revenue. The problem reads as

$$\max_u \int_0^\infty e^{-rt}\left((d-\varphi x)\,x-u\right)dt$$

$$\text{s.t. } \dot{x} = x^2(1-x)\,a\sqrt{u}-\delta x, \quad x(0)=x_0,$$

$$u \le f\,x\,(d-\varphi x), \quad x,u \ge 0.$$

Again we apply Pontryagin's maximum principle. Due to the dependency of the budget constraint on the state variable, the problem is more involved compared to the problem with a constant budget and less analytical insights are possible. We therefore directly resort to numerical calculations. Fig. 11 depicts the solution for the scenario where parameter values correspond to the base case of Table 1. The upper left panel shows the optimal solution paths (black solid lines) in the state-control space. At the dashed blue line the control constraint is

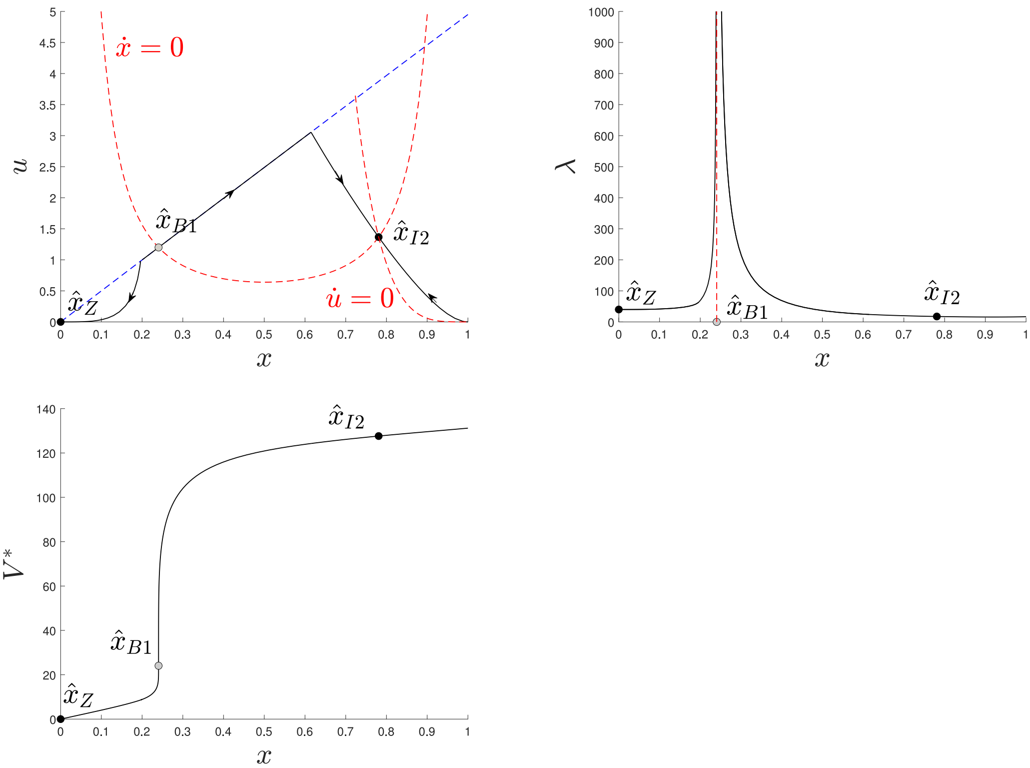


Fig. 11. The optimal solution for the base case parameter value of Table 1, where the constraint $u \le B$ is replaced by $u \le fx(d - \varphi x)$.

active, the red dashed lines depict the isoclines. The upper right panel depicts the solution paths in the state-costate space and the lower left panel the corresponding objective value.

We conclude that the critical mass threshold is still there. The costate is infinite at $\hat{x}_{B1}$, which can be seen in the upper right panel of Fig. 11, and abnormality of the solution at $\hat{x}_{B1}$ is preserved. Indeed the value of the firm V increases steeply in x at $\hat{x}_{B1}$. For initial values of the sales level exceeding the threshold $\hat{x}_{B1}$, sales behave monotonically and converge to the steady state level $\hat{x}_{I2}$ in the long run.

6. Conclusion

Digital products gain momentum in todays' economy. Many such products have in common that consumer utility increases if a larger number of consumers are using it. Then a network effect arises where customers attract more customers. The present paper designs an optimization model to determine the firm's optimal advertising policy in such a situation. We find that for the firm to be viable in the long run, the current sales level should exceed a critical mass threshold. This applies to Facebook that first introduced its product among colleges in the US, and only opened its network for everyone after substantial participation was achieved among students.

Furthermore, we find that in a scenario where the discount rate is low, and the budget and the diffusion effect is large enough, the critical mass threshold is directly affected by this budget. At the threshold the whole budget is spent on advertising, but this is just enough to keep sales at the same level. The shadow price of sales being infinite, signals that an extension of the budget, which enables the firm to increase advertising, which in turn raises the sales level, could bring additional profits to the firm that are substantial. From a more technical point of view it is interesting to remark that the corresponding solution of the optimal control problem is abnormal when the sales level equals the critical mass threshold. This usually occurs in degenerate problems (Halkin, 1974) and not in economically meaningful problems like we consider here.

Limitations of this work are that the framework is deterministic, i.e. uncertainties in for instance the future demand level are not considered. Also, this dynamic model of the firm is a one-decision maker model, excluding competition. It could therefore be interesting to either study such a network effect in a stochastic dynamic framework, or in a duopoly where firms are competing for a market share.

CRediT authorship contribution statement

Gustav Feichtinger: Conceptualization, Supervision, Methodology. **Dieter Grass:** Formal analysis, Investigation, Methodology, Software, Visualization. **Richard F. Hartl:** Conceptualization, Formal analysis, Writing – review & editing, Writing – original draft. **Peter M. Kort:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Andrea Seidl:** Conceptualization, Formal analysis, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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Appendix A

A.1. Proof of Proposition 2

By means of Pontryagin's maximum principle we find that

$$L_u = -\lambda_0 + 0.5\lambda a x^2 (1-x) \frac{1}{\sqrt{u}} - \nu = 0. \tag{18}$$

The complementary slackness condition and the non-negativity of the Lagrange multiplier have to be fulfilled, i.e.

$$\nu (B-u) = 0, \quad \nu \geq 0. \tag{19}$$

Concerning advertising efforts, there are basically three possibilities: either we have minimal advertising, $u = 0$, or maximal advertising, $u = B$, or it has some intermediate level. The first case can be excluded and the third case was already analyzed in Section 4. From (18) and the condition that $u \leq B$ it follows that the optimal control can be expressed in the normal case where $\lambda_0 = 1$ as

$$u = \begin{cases} \frac{\lambda^2 a^2 x^4 (1-x)^2}{4} & \text{if } a\lambda x^2 (1-x) \leq 2\sqrt{B} \\ B & \text{otherwise.} \end{cases} \tag{20}$$

For u to be at its upper bound it has to hold that

$$\nu = -\lambda_0 + 0.5\lambda a x^2 (1-x) \frac{1}{\sqrt{B}} \geq 0 \tag{21}$$

Costate Eq. (10) is not altered by the introduction of the budget constraint when it does not depend on the state variable.

In case of the budget constraint being active, the canonical system reads as

$$\dot{x} = a x^2 (1-x) \sqrt{B} - \delta x \tag{22}$$

$$\dot{\lambda} = (r+\delta)\lambda - \lambda_0 (d - 2\varphi x) - a\lambda\sqrt{B}\left(2x - 3x^2\right) \tag{23}$$

Setting the $\dot{x}$-isocline equal to zero, we obtain as steady state values

$$\hat{x}_{B1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}. \tag{24}$$

The corresponding costate values are From (23) we obtain that at a steady state of the canonical system, where the control is at its upper bound, it has to hold that

$$\hat{\lambda}_{B1,2} = \frac{\lambda_0 \left(d - 2\varphi \hat{x}_{B1,2}\right)}{r + \delta - a\sqrt{B}\left(2\hat{x}_{B1,2} - 3\hat{x}^2_{B1,2}\right)} \tag{25}$$

These steady states exist iff

$$a\sqrt{B} \geq 4\delta \tag{26}$$

and they are both in the interval (0,1).

A.2. Proof of Proposition 3

A steady state with the control at its upper bound can due to condition (21) only be feasible for the normal case with $\lambda_0 = 1$ if the steady state value of the costate $\hat{\lambda}_{B1}$ (see (25)) is positive. Because of (1), the numerator of (25) is always positive. Hence, a solution where the denominator of (25) is non-positive, i.e.

$$r + \delta - a\sqrt{B}\left(2\hat{x}_{B1} - 3\hat{x}^2_{B1}\right) \leq 0, \tag{27}$$

is not feasible and implies that a steady state solution at $\hat{x}_{B1}$ which is normal is not possible.

Concerning the costate, (23) implies that for $\lambda_0 = 0$ the steady state value of $\hat{\lambda}_{B1}$ is zero, which means that all multipliers vanish simultaneously, which contradicts Pontryagin's maximum principle. Thus, we can also exclude an abnormal solution where the costate assumes a steady state value equal to zero.

Note, however, that the costate value at the Stalling equilibrium has no impact on the value of the state variable (which is basically determined by the budget constraint). Thus, when we consider costate Eq. (23) at $\hat{x}_{B1}$ with $\lambda_0 = 0$, we find for any $\lambda(0) > 0$ due to condition (27) it holds that $\dot{\lambda}(t) \leq 0$ for any t with $\lim_{t\to\infty} \lambda(t) = 0$ (or $\lambda(t) = \lambda(0)$ for all t in case of the left side of (27) being equal to zero). This in turn implies that (a) the necessary transversality conditions (see e.g. Grass et al. (2008)) are satisfied and (b) $\lambda(t) > 0$ for any t. This means that an abnormal solution at the Stalling equilibrium with any $\lambda(0) > 0$ fulfills the necessary optimality conditions when condition (27) is fulfilled and is therefore a candidate for the optimal solution. Since a normal solution can be excluded under conditions (27), a solution where one stays at the Stalling equilibrium is only possible if the solution is abnormal.

To investigate when this occurs, we insert the smaller steady state value, $\hat{x}_{B1} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}$ from (24) into (27) and get

$$\frac{r+\delta}{a\sqrt{B}} - 2\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}\right) + 3\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}\right)^2 \leq 0$$

which simplifies to

$$\frac{2(r-2\delta)}{a\sqrt{B}} + 1 \leq \sqrt{1 - \frac{4\delta}{a\sqrt{B}}}.$$

Because of (26), both the left hand side of the inequality as well as the square root on the right hand side are positive. We can take squares and finally get

$$(r-2\delta)^2 + a\sqrt{B}(r-\delta) \leq 0.$$

The left hand side can only be negative if $\delta > r$. Reformulating the inequality above leads to (17).

Concerning the larger steady state $\hat{x}_{B2}$, just like before the abnormal case can only occur if

$$r + \delta - a\sqrt{B}\left(2\hat{x}_{B2} - 3\hat{x}^2_{B2}\right) \leq 0.$$

Inserting $\hat{x}_{B2} = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}$, we finally obtain that this is equivalent to

$$\frac{2(r-2\delta)}{a\sqrt{B}} + 1 \leq -\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}$$

which can be reformulated as

$$\frac{2r - 4\delta + B}{a\sqrt{B}} \leq -\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}. \tag{28}$$

The existence of a boundary steady state requires that $a\sqrt{B} \geq 4\delta$ (see (26)), which implies that the left hand side of (28) is positive and therefore (28) cannot hold.

To learn more about the stability properties of the boundary steady states, we consider the Jacobian evaluated at $\hat{x}_B$, which is

$$J = \begin{pmatrix} a\left(2\hat{x}_B - 3\hat{x}^2_B\right)\sqrt{B} - \delta & 0 \\ 2\lambda_0\varphi - 2a\hat{\lambda}_B\left(1 - 3\hat{x}_B\right)\sqrt{B} & r + \delta - a\hat{x}_B\left(2 - 3\hat{x}_B\right)\sqrt{B} \end{pmatrix}$$

The determinant is

$$\det J_B = \left(a\left(2\hat{x}_B - 3\hat{x}^2_B\right)\sqrt{B} - \delta\right)\left(r + \delta - a\hat{x}_B\left(2 - 3\hat{x}_B\right)\sqrt{B}\right) \tag{29}$$

Upper steady state $\hat{x}_{B2}$:

Inserting $\hat{x}_{B2} = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}$ into (29) we get

$$\det J_{B2} = -\frac{1}{2}\left(a\sqrt{B} - 4\delta\right)\left(a\sqrt{B} - 2\delta + r\right) - \frac{1}{2}a\sqrt{B}\left(a\sqrt{B} - 4\delta + r\right)\sqrt{1 - \frac{4\delta}{a\sqrt{B}}} < 0$$

where the sign follows from (26) and implies that the steady state is a saddle point.

Lower steady state $\hat{x}_{B1}$:

Inserting $\hat{x}_{B1} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\delta}{a\sqrt{B}}}$ into (29) we want to verify that

$$\det J_{B1} = -\frac{1}{2}\left(a\sqrt{B} - 4\delta\right)\left(a\sqrt{B} - 2\delta + r\right) + \frac{1}{2}a\sqrt{B}\left(a\sqrt{B} - 4\delta + r\right)\sqrt{1 - \frac{4\delta}{a\sqrt{B}}} > 0.$$

The inequality can be simplified into

$$4\delta\left(a\sqrt{B} - 4\delta\right)\left((r - 2\delta)^2 + a\sqrt{B}(r - \delta)\right) > 0 \tag{30}$$

which is true for $r > \delta$. Thus, we can conclude that the steady state is unstable.

Concerning $r < \delta$, for the normal case it has to hold that

$$a\sqrt{B} < \frac{(r - 2\delta)^2}{\delta - r},$$

see (17). Hence,

$$a\sqrt{B}(r - \delta) < (r - 2\delta)^2,$$

which means that (30) is satisfied, meaning the steady state is unstable.

Concerning $r < \delta$, for the abnormal case it has to hold that

$$a\sqrt{B} > \frac{(r - 2\delta)^2}{\delta - r}$$

which reverses the sign in (28), implying that the steady state is a saddle point.

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