Environmental and Resource Economics
https://doi.org/10.1007/s10640-024-00952-0

Individual and Firm Taxation in a CO_2 Emitting Economy

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Accepted: 17 December 2024

Abstract

Typical problems of negative effects of CO_2 emissions are that (i) they are suffered and generated not by the same agent and that (ii) individuals consider them as too small to influence the aggregated effect. Additionally, only little is known about how the behavior depends on the age-composition of a population and individual age-dependent life-cycle effects. We address these issues by an overlapping generations (OLG) structured population and a firm sector producing a homogeneous final consumption good. While firms generate CO_2 emission during the production process, individuals suffer from the aggregated effect. We analyze the difference between the decentralized market and the social welfare solution and study to which extent social optimality can be attained with different taxes on individual consumption and/or production. We find that firm taxation is always sufficient to reach the socially optimal level of CO_2 emissions. A social optimal distribution of consumption across cohorts, however, can only be attained by firm taxes in the steady state. In the general case, i.e., along a dynamic transitional path, additionally age-specific individual taxation is needed.

Keywords OR in environment and climate change · OLG-structured population · Social optimal solution · Tax schemes

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Published online: 02 January 2025

1 Introduction

The taxation of activities causing CO_2-equivalent emissions is one of the tools most frequently used both in the scientific literature as well as in the ongoing public political discussion. The taxes are designed to contain and overcome the negative effects of climate change, which are caused by CO_2 emissions of firms and suffered by the population. Although this clear dichotomy is not completely correct as individuals may also be responsible for emissions by their individual consumption behavior, the consideration in a modeling framework highlights two often neglected issues in the economic literature on climate and environmental economics, which in a combined form changes the established consideration of CO_2 taxation.

First, it is not clear who should be taxed among the different market participants. In that respect we identify the supply side with firms and the demand side with consumers or respectively households. The classic (static and, under certain conditions, also dynamic) economic literature implies that firm taxes carry over to (homogeneous) consumers in a market equilibrium if no market externalities undermine this substantial market mechanism on either side. From the environmental economics literature it is known that environmental pollution (like CO_2 emissions) is indeed an externality. Additionally, another important property of taxation, moreover, is the implementation of the first-best (i.e., socially optimal) solution in a decentralized market setting. This quite critical point is frequently implicitly assumed in the economic analysis of Integrated Assessment Models (IAMs)[1], deriving the Social Cost of Carbon (SCC) without addressing the question of its effectiveness in a decentralized market setup.[2]

Second, the bunch of the economic literature considers the consumer/household side, referred to as individuals in our model, to be homogeneous. While heterogeneity with respect to wealth or education is modeled occasionally, heterogeneity with respect to age and finite life-time implying is assumed only in rare examples and if it is, then in an abstract and simplified way (see e.g., Bassetto (2008) or Yakita (2003)). Important and highly non-trivial in this respect is the asynchronicity of the time-horizon, which makes it difficult to eliminate externalities in a distribution-fair manner, as shown in economic models with population dynamics. While individual controls such as consumption can be smoothed under certain conditions in the individual optimum (as already shown by Yaari (1965)), externalities that are (partly) outside the individual influence are the key issue for a transitional behavior over once life-cycle and for different behavior among cohorts (see Wrzaczek et al. (2014)).

In this paper we address both critical points in regard of the worsening climate crisis by formulating a model that integrates

- firms (supply side): producing a homogeneous product
- age-structured population (demand side): consuming the homogeneous products, born continuously over time, suffering age-specific climate-related effects, living a finite life
- CO_2 emissions: emitted by the production process of the firms, contributing to aggregated emissions which negatively affect the utility of individuals

[1] See Nordhaus (1994, 2008); Nordhaus and Boyer (2001), or Stern (2008) for prominent examples.

[2] For an example we refer to the literature on the established Dynamic Integrated Climate Economy (DICE) model (see e.g., Nordhaus (1994, 2008); Nordhaus and Boyer (2001)). For a counter example see, for instance, Golosov et al. (2014) who explicitly considers also the decentralized economy.

Using this model our paper explores the interplay between a CO_2-emitting production sector and a consumption side represented by a population with overlapping generation (OLG) structure. We are able to derive the first-best solution and test the efficiency of taxes in a decentralized setting. In particular, we contribute to the economic literature by addressing the following questions: (a) Characterizing differences of a decentralized market solution with the socially optimal one. (b) Exploring whether firm or individual taxes are useful to reduce CO_2 emissions to implement the socially optimal solution in the decentralized market setup. The analysis distinguishes between the transitional path and the stationary long-run solution (i.e., steady state), which shows interesting differences among the tax schemes. While firm taxes can achieve social optimality in the steady state but not in general, individual taxes are sufficient in any situation. The implementation of a socially optimal CO_2 level implies cross-cohort inefficiencies.

Our work is situated at the intersection of the economic and rather mathematical (challenging) strands of the existing literature. Our paper fits to the extant literature on the effects of emission taxation and contributes to two debates that overlap each other along this specific dimension.

The first is the analysis of differential oligopoly games in which the regulator adopts an emission tax to induce firms to reduce emissions, either by output contractions, or by adjusting output and investing in abatement technologies. The first approach dates back to Benchekroun and Long (1998, 2002), where it is shown that optimal taxation may indeed drive the industry to replicate the performance of the social planner, taking the brown nature of the technology as given. The second approach, taken by Feichtinger et al. (2016, 2022) includes green R&D investments (and also resource extraction), and provides a full-fledged assessment of the scope of emission taxation on welfare and the preservation of the environment.[3] However, this strand of literature relies on a time-invariant representative consumer populating forever the demand side of the market. In this respect, it is indeed desirable to consider overlapping generation models, which is the alternative route taken by a parallel flow of contributions.

Our model is closely related to that of Ono (1996), which employs a two-period OLG framework to model the demand side. However, due to its simplifying assumptions, Ono's analysis does not fully capture the intricate dynamics of intertemporal and intergenerational interactions. First, the social optimum in Ono's model is constrained to a two-period horizon, rather than extending across the entire future. Consequently, the framework cannot analyze both the steady state and the transitional dynamics in depth. Furthermore, this simplification implies that the social optimum can be achieved through uniform, age-independent tax rates on consumption and interest income. In contrast, our generalized model demonstrates that while age-independent tax rates are sufficient in the steady state, age-dependent tax rates are essential along the transitional path towards the steady state. Second, Ono's model simplifies the population structure into two discrete generations (young and old) rather than adopting a continuous age distribution. This abstraction precludes a comprehensive analysis of the individual life-cycle consumption profile relative to the socially optimal allocation-particularly regarding the conditions under which consumption smoothing is optimal. Additionally, the model does not account for externalities related to mortality rates, an issue addressed

[3] For a more detailed discussion of the literature on these matters, including also static multistage games, see Lambertini (2013, 2017, 2018). An exhaustive survey of differential games dealing with pollution control, which goes well beyond the scope of the present paper, is in De Zeeuw (2014).

in Gutiérrez (2008), albeit within a similar two-period OLG framework and thus subject to comparable limitations. In contrast, studies such as Bovenberg and Heijdra (1998), Heijdra and Ligthart (2000), and Heijdra et al. (2006) employ a continuous OLG framework to examine the dynamic allocative effects and intergenerational welfare implications of environmental taxes. However, these approaches focus primarily on the demand side and omit the interplay between individual and firm-level taxation in achieving social optimality. Our contribution extends this literature by integrating the supply side and rigorously analyzing the effectiveness and interaction of taxes at both individual and firm levels in the pursuit of social optimality.

The mathematical literature on continuous time- and age-structured overlapping generations (OLG) models is closely linked to different formulations of Maximum Principles for both decentralized and social welfare (first-best) frameworks. In the decentralized economy, the analysis focuses on the life-cycle of individual agents who regard both supply and environmental damage as exogenous variables. Consequently, individuals do not internalize the impact of their actions on aggregated state variables, which aligns with standard (age-structured) optimal control theory (see, e.g., Grass et al. (2008)) applied along their life-cycle. Aggregating across cohorts involves a continuum of overlapping generations, characterized by asynchronous time horizons, to derive the total demand function. While such approaches are commonly employed in large-scale simulation models, analytical treatments of this class of models remain relatively scarce (see, for instance, Wrzaczek et al. (2014) and Wrzaczek (2021), which explore similar population structures but without explicitly modeling the supply side). The social welfare model, in contrast, is formulated as an age-structured optimal control problem and analyzed using the corresponding Maximum Principle (see, e.g., Brokate (1985) and Feichtinger et al. (2003)). Despite occasional applications, the incorporation of age-structure (also referred to as *vintage-structure* in certain contexts, as discussed in Feichtinger et al. (2006) and Boucekkine et al. (2002)) remains far from standard in theoretical economics. The analytical and numerical challenges posed by such frameworks are considerable, yet they yield insights that cannot be obtained through conventional optimal control theory.

The rest of the paper is organized as follows. In section 2 we introduce the individual and the firm problem and combine it to a decentralized model, for which we derive the solution in section 3. Section 4 presents the social welfare model and its optimal solution. Different tax schemes are considered in section 5. Finally section 6 concludes.

2 The Decentralized Model

Consider an economy with infinite time horizon with demand and supply (i.e., firms) side of the market. The first is represented by the total consumption of an age-structured population consisting of individuals that do not consider the dynamics of aggregated CO_2 emission, but only their own age and age-structured related individual state variables. The supply side is represented by firms producing according to a production function with capital and labor as production inputs. The firms do not anticipate the age-structured composition of the demand side, as well as aggregated CO_2 emissions.

In the following subsections both market sides are introduced in detail.

2.1 Demand Side - Individuals

Individuals are living over a finite time horizon up to maximal age ω according to a survival probability, which depends on aggregated CO_2 emissions. Since time is continuous, the equally continuous birth and death process implies an overlapping generations structure of the population with time t and age a evolving at the same pace as independent variables (see e.g., Wrzaczek et al. (2014), followed by Wrzaczek (2021)).

An a-year old individual at t enjoys utility from (non-negative) consumption $c(t, a)$ diminished by negative effects (i.e., environmental disutility) of aggregated CO_2 emissions at t, denoted by $E(t)$. We may define the instantaneous individual net utility of an individual born at $t-a$ at $t \geq a \geq 0$ as

$$U\left(a, c(t,a), E(t)\right) = u(c(t,a))\left(1-\kappa+\kappa d(a,E(t))\right) - (1-\kappa)\, d(a,E(t)), \tag{1}$$

where the (age-independent) utility from consumption utility is denoted by $u(c(t, a))$ and disutility of $E(t)$ by $d(a, E(t))$. Note that (i) $d(a, E(t))$ potentially (and also realistically) depends on the age of the individual, and (ii) we do not limit the disutility to decrease consumption utility additively or multiplicatively at this stage but specify it by the exogenous parameter κ. An additive effect is captured by $\kappa = 0$ and a multiplicative one by $\kappa = 1$ (obviously implying that the form of $d(a, E(t))$ differs according to the choice of κ). Individual consumption is financed by the life-cycle income. As in many individual choice models[4], we assume a balanced budget with a perfect annuity market as suggested in the seminal paper by Yaari (1965). Thus, denoting assets of an a-year old individual at t by $A(t, a)$, the asset's kinematic equation is

$$\dot{A}(t,a) = (r(t) + \mu(a, E(t)))A(t,a) + w(t)p(a,E(t)) - \bar{p}c(t,a) - \tau_I(t,a)c(t,a). \tag{2}$$

We assume zero assets at the time of birth and an initial asset distribution $A_0(a)$ (for $a \in [0, \omega]$). A balanced budget also implies zero assets at the maximal age ω. Thus, the initial and end conditions for the individual can be formulated as

$$\text{initial condition}: \quad \begin{cases} A(t-a,0) = 0 & \text{if } t \geq a \\ A(0,a-t) = A_0(a-t) & \text{if } t < a \end{cases}, \tag{3}$$

$$\text{end condition}: \quad A(t-a+\omega,\omega) = 0. \tag{4}$$

Equation (3) distinguishes whether the individual is born after the beginning of the planning period ($t \geq a$) or already before ($t < a$). In equation (2), $r(t) > 0$ denotes the market interest rate, $w(t)p(a, E(t))$ the wage rate that subdivides into a time dependent market component $w(t)$ and an age-dependent productivity component $p(a, E(t))$, and $\bar{p}$ the constant unit price of consumption which will be used as numéraire for the rest of the paper, i.e., $\bar{p} = 1$. In general, the age-dependent productivity also depends on $E(t)$ as it may suffer from difficult environmental conditions. The market interest rate as well as the market component of the wage rate are endogenously determined in the market equilibrium, but exogenous for a single individual. The tax rate on individual consumption, denoted by $\tau_I(t,a)$, depends both

[4] See, for instance, the rational addiction model of Becker and Murphy (1988).

on time and age, which is possible only by the OLG structure of the population. Different assumptions on the individual tax rate and its implications will be addressed later on, in section 5. The presence of a perfect annuity market, where $\mu(a, E(t))$ equals the age-specific mortality rate (which also negatively depends on $E(t)$), ensures that the model is closed and that assets from deceased people are distributed among the cohort.

The generic individual chooses the consumption path over the life-cycle in order to optimize the expected life-time utility, i.e.,

$$\mathcal{V}(t_0, a_0) := \max_{c(t,a)\geq 0} \int_{a_0}^{\omega} e^{-\rho(a-a_0)} S(t, a) U(a, c(t, a), E(t)) \mathrm{d}a, \tag{5}$$

where (t_0, a_0) is either $(t_0, 0)$ for individuals born after the beginning of the planning period (corresponding to first line of (3)), or $(0, a_0)$ for individuals that are born earlier (corresponding to second line of (3)). Parameter $\rho > 0$ denotes the individual time discount rate, while $S(t, a)$ is the survival probability up to age a, which depends on aggregated emissions during the life-time of the individual.[5] Individuals are atomistic and behave as such, that is, they consider themselves *small* to exert no influence either on the aggregated emissions $E(t)$ or on the supply supply of the market (see also Daube and Ulph (2016)). Therefore, from consumers' individual and collective standpoint, both are treated as exogenous magnitudes. Consequently, the mortality rate/survival probability, the market interest rate, the wage rate, and the environmental disutility are treated as exogenous variables (i.e., market and environmental externalities) and not included in the individual optimization.

2.2 Supply Side - Firms

The supply side consists of a market with perfect competition with n identical firms, producing the same undifferentiated product based on the same common technology. The firms produce according to a Cobb-Douglas production function with capital and labor as production input, i.e., using K_i units of capital and L_i units of labor firm i produces

$$F_i(K_i, L_i) = \bar{F} K_i^{\beta} L_i^{1-\beta}, \qquad \beta \in (0, 1), \tag{6}$$

where $\bar{F}$ denotes total factor productivity, and β and $1-\beta$ the output elasticities of capital and labor, respectively. In the following we assume w.l.o.g. $n = 1$ (subindex i in (6) becoming irrelevant). The production process causes CO_2 emissions, which are linear in the output $F(K, L)$ with parameter $\bar{a}$. Starting from an initial stock of emissions E_0, the stock of CO_2 emissions therefore evolves according to

$$\dot{E}(t) = \bar{a} F(K(t), L(t)) - \delta E(t), \qquad E(0) = E_0, \tag{7}$$

$$F(K(t), L(t)) := \bar{F} K^{\beta}(t) L^{1-\beta}(t), \tag{8}$$

[5] In correspondence with the mortality rate $\mu(a, E(t))$ (used in the asset dynamics (2)) the survival probability evolves according to $\dot{S}(t, a) = -\mu(a, E(t)) S(t, a)$ (with initial condition $S(t-a, 0) = 1$) which gives $S(t, a) = e^{-\int_0^a \mu(s, E(t-a+s)) \mathrm{d}s}$.

where $F(K(t), L(t))$ denotes the total production of the production sector at t, and $\delta \geq 0$ measures the exogenous absorption rate of the natural carbon sinks. Note that w.l.o.g. we can choose $\bar{a} = 1$.
The firm maximizes the profit $\pi(K, L)$ at every t defined as (recall $\bar{p} = 1$)

$$\pi(K(t), L(t)) = F(K(t), L(t)) - r(t)K(t) - w(t)L(t) - \tau_F(t)F(K(t), L(t)), \tag{9}$$

with respect to K and L. The first term measures revenues according to the constant market price $\bar{p}$. The second and third term denote the production costs, i.e., $r(t)$ and $w(t)$ denote the costs of capital rent and labor. Profits are reduced by output taxation at exogenous rate $\tau_F(t)$. The rate is assumed to be non-negative and smaller than 1, i.e., $\tau_F(t) \in [0, 1)$, implying that the tax rate cannot be turned into a subvention.
Note that, emissions can alternatively be expressed as a function of current aggregated consumption, as modeled, for instance, by Ono (1996). Under this assumption, the emission dynamics (7) would take the form:

$$E(t) = \bar{a}C(t) - \delta E(t), \tag{10}$$

represents aggregated consumption (see the definition provided for the first-best solution in (55)). Adopting this alternative formulation does not alter the core results of the paper, as the core assumptions regarding the individual behavior and anticipation remains the same, i.e., individuals consider themselves as too small to have an influence on the aggregate emission dynamics. The result may slightly differ on the transitional path towards a steady state, where the solutions corresponding to the different solution dynamics eventually coincide (here production equals aggregated consumption). Importantly, all qualitative results regarding the structure of the solution remain intact and can be derived analogously under this alternative specification.

2.3 The Market Equilibrium and Full Model

In the market equilibrium the firm produces with the total capital stock and labor available on the market. Both are defined by the (age-structured) population. Total capital stock equals the sum of all assets owned by the individuals and the available labor is defined by all individuals weighted by $p(a, E(t))$:

$$K(t) = \int_0^\omega A(t, a)\mathrm{d}a, \tag{11}$$

$$L(t) = \int_0^\omega N_0 S(t, a) p(a, E(t))\mathrm{d}a, \tag{12}$$

where N_0 denotes the (exogenous, constant) number of newborns at every t.[6] Hence, putting the demand side (individual problem, subsection 2.1), the supply side (firm problem,

[6] Note, that we abstract from a time dependent rate of newborns or even an endogenous birth process, as this effect is not at the core interest of this paper. However, the analysis would be analogous, but extended by and additional term including future effects of newborns, see e.g., Wrzaczek et al. (2010).

subsection 2.2) and the market equilibrium together, the full model of the decentralized market reads

$$\text{Demand side:} \quad \max_{c(t,a)\geq 0} \int_{a_0}^{\omega} e^{-\rho(a-a_0)} S(t,a) U(a, c(t,a), E(t)) \mathrm{d}a, \tag{13}$$

$$\begin{array}{ll} \forall a_0 \in (0,\omega] & \text{individuals born before } t=0 \\ \forall\,(t-a)\geq 0 & \text{individuals born after } t=0 \end{array} \tag{14}$$

$$\text{Supply side:} \quad \max_{K(t),L(t)\geq 0} F(K(t),L(t)) - r(t)K(t) - w(t)L(t) - \tau_F(t)F(K(t),L(t)), \qquad \forall t, \tag{15}$$

subject to

$$\begin{aligned} \text{Ind. assets: } A_t(t,a) + A_a(t,a) &= (r(t) + \mu(a, E(t)))A(t,a) \\ &+ w(t)p(a,E(t)) - c(t,a) - \tau_I(t,a)c(t,a), \end{aligned} \tag{16}$$

$$A(t-a,0) = 0 \qquad \text{if } t \geq a \tag{17}$$

$$A(0,a-t) = A_0(a-t) \qquad \text{if } t < a \tag{18}$$

$$A(t-a+\omega,\omega) = 0, \tag{19}$$

$$\text{Agg. emissions: } \dot{E}(t) = \quad F(K(t),L(t)) - \delta E(t), \qquad E(0) = E_0, \tag{20}$$

$$C(t) := \int_0^{\omega} N_0 S(t,a) c(t,a) \mathrm{d}a, \tag{21}$$

$$\text{Aggregation: } K(t) := \int_0^{\omega} A(t,a) \mathrm{d}a, \tag{22}$$

$$L(t) := \int_0^{\omega} N_0 S(t,a) p(a, E(t)) \mathrm{d}a. \tag{23}$$

Remark: The asset dynamics of individuals originally has been formulated from the viewpoint of the focal individual. As time and age evolve at the same pace the time derivative in (2) is denoted by a dot, i.e., $\dot{A}(t,a) := \frac{\mathrm{d}A(t_0+a,a)}{\mathrm{d}a}$ where $t_0 := t-a$ denotes the individual's time of birth. From the viewpoint of the entire market, however, time and age are different independent variables and the same derivative (16) is denoted by the partial derivative as $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) A(t,a) := A_t(t,a) + A_a(t,a)$. Both notations are standard in the relevant literature on OLG models with continuous cohorts and age-structured optimal control theory (see e.g., Aniţa (2000)) and are excellent to distinguish different viewpoints of the problem.

Note that formulating the full structure of the model reveals the problematic property in a decentralized market setup. CO_2 is emitted during the production process of the firms, who do not suffer from the emission stock (except a reduced labor which is not anticipated by firms) and therefore optimize their production profit. Consumers, on the other hand, believe to be *small* so that they cannot directly influence the size of the output, which is responsible for the emission level at any *t*. Hence, causation and effect of CO_2 emissions enter at different sides. Both types of agents are connected by the market equilibrium but are both lacking any chance of steering the other side of the market. This is a classical situation, where the decentralized solution does not achieve social optimality without market intervention by a regulator (e.g., a policy maker or a government). Formulating a tax scheme to overcome this market failure is standard in economic theory. However, the realistic OLG structure of the demand side makes the analysis more involved but addresses the implication to social optimality under different conditions.

To facilitate certain parts of the analysis in the subsequent sections, we choose specific forms of the functions mentioned above. For disutility from the CO_2 stock, individual net utility and the consumption utility, we assume

$$d(a, E) = \left(\bar{d}(a)E\right)^{\alpha}, \tag{24}$$

$$U(c, E) = u(c)\left(1 - \kappa + \kappa d(a, E)\right) - (1 - \kappa)\, d(a, E), \tag{25}$$

$$u(c) = c^{\sigma}, \qquad 0 < \sigma < 1. \tag{26}$$

The exponent α is assumed to be greater than one if the effect of aggregated emissions is additive, and negative if it is multiplicative, i.e.,

$$\alpha \geq 1 \text{ for } \quad \kappa = 0, \tag{27}$$

$$\alpha < 0 \text{ for } \quad \kappa = 1. \tag{28}$$

The disutility of $E(t)$ is assumed to be non-decreasing over age, i.e., $\frac{\partial \bar{d}(a)}{\partial a} = \bar{d}_a(a) \geq 0$ $(\forall a)$. Although it is realistic that the disutility increases in age, we will discuss both cases $\bar{d}_a(a) > 0$ and $\bar{d}_a(a) = 0$ in the analysis of the following section separately. This allows to distinguish the *ageing*-effect of the individuals, possibly implying a different behavior over the life-cycle, from the market failure effect which is age-independent. For $u(c)$ we assume a concave function that fulfills the usual Inada conditions, i.e., $\lim_{c \to 0+} u_c(c) = \infty$, $\lim_{c \to \infty} u_c(c) = 0$. The functional forms of $d(a, E)$ and $u(c)$ together imply that the individual net utility function is concave, thereby being compatible with sufficiency conditions from optimal control theory.
The following table 1 summarizes the control and state variables as well as the parameters of the model.

3 The Optimal (Decentralized) Solution

We derive the optimal solution of the individual consumer's and firm's problem as stated by (2-5) and (9). The derivation of the necessary optimality conditions are possible for general functions. However, whenever helpful in the analysis and the derivation of specific results, we will resort to the forms specified in (24).

Table 1 Summary of notation

Control variables	$c(t, a)$	Individual consumption
State variables	$E(t)$	Aggregated CO_2 emissions
	$A(t, a)$	Individual assets
Aggregation	$C(t)$	Total consumption
	$K(t)$	Capital stock/used for production
	$L(t)$	Labor/used for production
Functions	$u(c)$	(Consumption) utility function
	$U(a, c, E)$	Individual net utility
	$d(a, E)$	Disutility of aggregated CO_2 emissions
Parametric functions	$S(t, a)$	survival probability
	$\mu(a, E)$	(Age-specific) mortality rate
	$p(a, E)$	(Age-specific) productivity
	$\tau_I(t, a)$	(Age- and time-specific) tax rate on consumption
	$\tau_F(t)$	(Time-specific) tax rate on production
Parameters	ρ	Discount rate
	ω	Maximal individual age
	κ	Utility function parameter
	σ	Utility function parameter
	N_0	Number of newborns
	$\bar{d}(a)$	(Individual) disutility of aggregated CO_2 emissions
	α	Exponent of (individual) emission disutility
	β	Output elasticity
	$\bar{F}$	Total factor productivity
	δ	Natural decay rate of total emissions
	$r(t)$	Market interest rate
	$w(t)$	Wage rate

3.1 The Consumer's Problem

The individual consumer's problem (2-5) is a standard finite time horizon optimal control problem, which can be solved with the Maximum Principle (see e.g., Grass et al. (2008)). Maximizing the Hamiltonian (and suppressing independent variables t and a) yields the first order condition for consumption:[7]

$$Su_c\left(1-\kappa+\kappa d(a,E)\right)-(1+\tau_I)\lambda^A=0, \tag{29}$$

where the adjoint variables $\lambda^E(t,a)$ and $\lambda^A(t,a)$ for aggregated emissions and assets respectively are defined by

$$\lambda_t^E+\lambda_a^E=(\rho+\delta)\,\lambda^E-\alpha S\left(u(c)\kappa-1+\kappa\right)\left(\bar{d}(a)E\right)^{\alpha-1}-\lambda^A w_E, \tag{30}$$

[7] Due to the Inada conditions on the consumption utility function boundary values for $c(t, a)$ can be excluded.

$$\lambda_t^A + \lambda_a^A = (\rho - r - \mu)\,\lambda^A. \tag{31}$$

Though the individual problem is a standard optimal control problem in one independent variable (recall that an individual does not distinguish between time and age) with ω as end of the planning period, all individual variables depend on time and age to distinguish the corresponding variables w.r.t. the focal individual. While for λ^E the standard transversality condition applies, the exogenous initial and terminal values of A implies that the condition does not exist for λ^A (replaced by initial and end condition for A). Thus,

$$\lambda^E(t-a+\omega,\omega) = 0, \tag{32}$$

$$\lambda^A(t-a+\omega,\omega) \quad \text{free.} \tag{33}$$

The individual considers itself as *small*, i.e., without any influence on the behavior of others and on the market equilibrium. Though the decentralized model is embedded in the full market equilibrium, what potentially implies several non-linear effects, it is possible to characterize the individual consumption dynamics along the life-cycle by considering the corresponding consumption Euler equation. This is summarized by the following lemma.

Lemma 1 Consider an individual entering the economic system at t_0 at age a_0 (see (3)), whose optimal consumption profile along the life-cycle is determined by (29-33). Then the consumption along the life-cycle of an individual develops according to the consumption Euler equation:

$$c_t + c_a = \frac{u_c}{u_{cc}}\left(\frac{\dot{\tau}_I}{p+\tau_I} + \rho - r - \frac{\kappa\alpha d}{1-\kappa+\kappa d}\left(\frac{\bar{d}_a}{\bar{d}} + \frac{\dot{E}}{E}\right)\right). \tag{34}$$

Let's further assume that no taxes are imposed to individual consumption ($\tau_I = 0$ for $\forall(t,a)$). If the disutility of $E(t)$ diminishes the net utility additively ($\kappa = 0$), the consumption path is independent of $E(t)$ and follows the usual relation between the discount and the market interest rate:

$$\kappa = 0: \qquad c_t(a,t) + c_a(a,t) \gtreqless 0 \quad \text{if} \quad r(t) \gtreqless \rho, \qquad\qquad 0 \le a \le \omega, \quad t \ge 0. \tag{35}$$

In case of a multiplicative effect of $E(t)$ the consumption path also adapts for the change of the disutility:

$$\kappa = 1: \qquad c_t(a,t) + c_a(a,t) \gtreqless 0 \quad \text{if} \quad r(t) + \alpha\left(\frac{\bar{d}_a}{\bar{d}} + \frac{\dot{E}}{E}\right) \gtreqless \rho, \qquad\qquad 0 \le a \le \omega, \quad t \ge 0. \tag{36}$$

Proof The general expression (34) can be directly derived from (29). Assuming $\tau_I = 0$ yields

$$c_t + c_a = \frac{u_c}{u_{cc}}\left(\rho - r - \frac{\kappa\alpha d}{1-\kappa+\kappa d}\left(\frac{\bar{d}_a}{\bar{d}} + \frac{\dot{E}}{E}\right)\right). \tag{37}$$

From the assumption on the utility function we have $\frac{u_c}{u_{cc}} < 0$, which proves (35) and (36). □
The lemma characterizes how the consumption path adapts to changes in the conditions of the market and the environment. It follows the standard economic result that individuals aim at consumption smoothing along the life-cycle (under the condition of a perfect annuity market) if the time preference equals the market interest rate. The deviation in case of a multiplicative disutility of $E(t)$ is due to the change of the marginal utility from consumption along the life-cycle. In this case, ageing as well as the development of total pollution are considered. In contrast to that, an additive effect only shifts the utility without additional effect on the path. The following corollary clarifies this effect by assuming a steady state of the market and total emissions.

Corollary 2 Consider an individual entering the economic system at t_0 at age a_0 (see (3)), whose optimal consumption profile along the life-cycle is determined by (29-33), and assume that no taxes are imposed to individual consumption ($\tau_I = 0$ for $\forall(t, a)$). Consider that the economy has reached a steady state, i.e., $\dot{E}(t) = 0$, and $A_t(t, a) = 0$. Then individuals choose a smooth consumption path along the life-cycle if

- $\kappa = 0$: $\rho = r(t)$
- $\kappa = 1$: $\rho = r(t)$, and $\bar{d}_a = 0$.

According to the corollary consumption smoothing in a steady state needs (i) the full market to be in a steady state, (ii) a market interest rate that is equal to the individual time preference rate, and (iii) an additive disutility from $E(t)$ or, alternatively a multiplicative disutility that affects all ages equally. Point (iii) corresponds to the individual optimality condition for consumption (29) that balances marginal costs with the marginal consumption utility. While assets can be shifted freely across the individual life-cycle due to the perfect annuity market, the marginal utility is independent of age only in case of $\kappa = 0$. For $\kappa = 1$ the marginal utility is weighted by disutility from aggregated emissions (see left hand side of (29)) implying that consumption will possibly increase along age.

The steady state per definition means that individual consumption of an a-year old individual is constant over time, i.e., $c_t(t, a) = 0$. Across age, however, the conclusion depends on the form of the disutility from aggregated emissions. While $\kappa = 0$ implies additionally $c_a(t, a) = 0$, $\kappa = 1$ implies only $c_t(t, a) >$ (if $\bar{d}_a > 0$). Therefore, older individuals consume more than younger ones though the market has eventually reached a steady state.

Lemma 1 and Corollary 2 show two relevant problems. First, every single individual only considers its own life-cycle. Anything happening after one's own life does not influence the consumption decision. Wrzaczek et al. (2014) come to a similar conclusion in a simpler setup and even show that a bequest motive can only diminish, but not prevent this problem from the finiteness of life. Second, the effect of consumption on the disutility of the emission stock (via production) is not internalized but assumed to be exogenous from the individual point of view. This is an example of the problem known as the tragedy of commons, in this case strengthened by decoupling production and consumption choices. Both problems do not arise in a socially optimal (first best) solution addressed in section 4. The corresponding objective function considers the net utilities of all cohorts during the entire time horizon solving the finiteness of life-issue, and the intertemporal cross-cohort effect of individual behavior on emissions is included by attaching adjoint variables to the stock of

CO_2 emissions and total consumption. The resulting outcome serves as a desirable solution benchmark that policy makers aim to achieve by setting proper policy instruments such as taxes and subsidies.

We close the analysis of the individual problem by stating an analytic expression for the specific choice of the consumption utility function (24).

Lemma 3 Consider an individual entering the economic system at t_0 at age a_0, whose optimal consumption profile (along the life-cycle) is described by (29-33). If no taxes are imposed to individual consumption ($\tau_I = 0$ for $\forall(t, a)$) and the consumption utility function (24), then individual consumption at t_0 at age a_0 is given by

$$c(t_0, a_0) = \frac{\Psi_I(t_0, a_0)}{\Psi_{LE}(t_0, a_0)}, \tag{38}$$

with discounted aggregated life-time income $\Psi_I(t_0, a_0)$ and discounted weighted rest-life-time expectancy $\Psi_{LE}(t_0, a_0)$, i.e.,

$$\Psi_I(t_0, a_0) := \int_{a_0}^{\omega} e^{-(a-a_0)r} \frac{S(t, a)}{S(t_0, a_0)} w(t) p(a, E(t)) \mathrm{d}a, \tag{39}$$

$$\Psi_{LE}(t_0, a_0) := \int_{a_0}^{\omega} e^{-(a-a_0)r} \frac{S(t, a)}{S(t_0, a_0)} e^{\int_0^a \Delta(t-a+s, s) \mathrm{d}s} \mathrm{d}a, \tag{40}$$

and

$$\Delta(t, s) := \frac{1}{\sigma - 1} \left(\rho - r(t) - \frac{\kappa \alpha d(s, E(t))}{1 - \kappa + \kappa d(s, E(t))} \left(\frac{\bar{d}_a(s, E(t))}{\bar{d}(s, E(t))} + \frac{\dot{E}(t)}{E(t)} \right) \right). \tag{41}$$

Consumption at *t* and *a* is then implied by application of the consumption Euler equation (34).

Proof (Sketch) Integration of the asset dynamics along the life-cycle has to equal zero due to the terminal condition. From the consumption Euler equation (34) we can deduce $c(t, a)$ depending on $c(t - a, 0)$. Plugging into the asset equation and isolating $c(t - a, 0)$ implies the assertion. □

Thus (38) reveals that individual consumption equals the remaining life-time income divided by weighted life-time expectancy. All terms are discounted with respect to the individual discount rate and the conditional survival probability.

It is worth emphasizing that the analysis in this subsection generalizes the findings of the two-period OLG model by Ono (1996), which established that older individuals tend to consume more than their younger counterparts. By leveraging the consumption Euler equation, our framework not only delineates the optimal consumption trajectory over an individual's lifetime but also explicitly identifies the underlying mechanisms driving this behavior. Furthermore, the adoption of a continuous OLG structure facilitates a deeper examination of the conditions under which consumption smoothing becomes optimal - an aspect not addressed in Ono (1996). This broader perspective provides richer insights into the intertemporal allocation decisions of agents in the model.

3.2 The Firm's Problem

The firm maximizes (9) with respect to the production input factors $K(t)$ and $L(t)$ at every t. The first order conditions for all t read

$$\pi_K(K, L) = (1 - \tau_F)F_K(K, L) - r = 0, \tag{42}$$

$$\pi_L(K, L) = (1 - \tau_F)F_L(K, L) - w = 0, \tag{43}$$

which, for the Cobb-Douglas form as assumed in (8), imply the following interest and wage rates:

$$r = \beta K^{-1}(1 - \tau_F)F(K, L), \tag{44}$$

$$w = (1 - \beta)L^{-1}(1 - \tau_F)F(K, L). \tag{45}$$

Hence, the wage and the interest rate are defined by the market equilibrium as it is standard in economics. We assume that the market is in perfect competition such that, the firm does not make any profit at any t, i.e.,

$$0 = (1 - \tau_F(t))F(K(t), L(t)) - r(t)K(t) - w(t)L(t), \tag{46}$$

for which the explicit values for r and w can be used to obtain

$$(1 - \tau_F(t))C(t) = (1 - \tau_F(t))F_K(K(t), L(t))K(t) + (1 - \tau_F(t))F_L(K(t), L(t))L(t). \tag{47}$$

Taxes diminish the wage and the tax rates by the same proportion, so are completely transferred to the individuals/consumer sector. The firm, moreover, does not internalize any cross-cohort (re-)distributional effect neither of the tax rate nor of total pollution. From the regulators' point of view, it cannot be followed immediately from the results of this section to which extent firm taxes can be used to fix a desired emission level at every t, as taxes just transit from the firms to individuals. To investigate the latter, we first define and solve the first best problem in the next section followed by a thorough analysis of individual and firm taxes.

4 The First Best Under Social Planning

A benevolent social planner does not consider individual assets along the life-cycle, but the capital stock as a whole, allowing to share wealth across generations. Hence, from (22) it is straightforward to derive the classical dynamics of the capital stock as follows:

$$\begin{aligned}\dot{K}(t) = &\int_0^\infty A_t(t,a) + A_a(t,a)\mathrm{d}a + \underbrace{A(t,0) - A(t,\omega)}_{=:(a)} - \underbrace{\int_0^\infty N_0 S(t,a)\mu(a,E(t))A(t,a)\mathrm{d}a}_{=:(b)} \\ =&r(t)K(t) + w(t)L(t) - C(t),\end{aligned} \tag{48}$$

where (a) correspond to assets from individuals at (minimal or maximal) age $a = 0$ or $a = \omega$, and (b) comes from individuals that die at *t*. Defining the social welfare (in Benthamite form) as the sum of all individual utilities (therefore considering the same time discount rate ρ), the social planning problem reads

$$\mathcal{SW}(E_0, K_0) := \int_0^\omega N_0 \mathcal{V}(0,a;E_0)\mathrm{d}a + \int_0^\infty e^{-\rho t} N_0 \mathcal{V}(t,0;E(t))\mathrm{d}t \tag{49}$$

$$= \max_{c(t,a)\geq 0} \int_0^\infty \int_{a_0}^\omega e^{-\rho t} N_0 S(t,a) U(a, c(t,a), E(t))\mathrm{d}a\mathrm{d}t \tag{50}$$

$$\text{s.t.} \quad \dot{E}(t) = F(K(t), L(t)) - \delta E(t), \qquad E(0) = E_0 \tag{51}$$

$$\dot{K}(t) = F(K(t), L(t)) - C(t), \qquad K(0) = K_0 = \int_0^\infty A_0(a)\mathrm{d}a \tag{52}$$

$$S_t(t,a) + S_a(t,a) = -\mu(a, E(t))S(t,a) \tag{53}$$

$$S(0,a) = S_0(a), S(t,0) = 1 \tag{54}$$

$$C(t) = \int_0^\omega N_0 S(t,a)c(t,a)\mathrm{d}a \tag{55}$$

$$L(t) = \int_0^\omega N_0 S(t,a)p(a,E(t))\mathrm{d}a. \tag{56}$$

The firms and its optimal behavior need not be modeled explicitly. The market equilibrium conditions on the interest and the wage rate are implicitly included in the problem, as capital is considered in aggregated form. That is, differently to the decentralized solution the effect of production on the individual problem and on the dynamics of total emissions, which in turn is entering individual utilities and the social welfare function, is automatically included in the optimality conditions. The social planner also considers the endogenous survival probability as it endogenously depends on *E*(*t*) (and, through this channel, on labor, total demand and the social welfare). It is not necessary to consider balanced budgets of all cohort. However, we assume the capital stock to be non-negative for all *t*. This assumption remains to be implicit, as it does not get active for our specific choice of a Cobb-Douglas production function.

4.1 Optimality Conditions of the Social Optimum

The social welfare maximization problem (49) is an age-structured optimal control problem with additional concentrated state variables (i.e., depending only on time), which can be solved with the age-specific Maximum Principle (see e.g., Feichtinger and Wrzaczek (2024) for the age-structured Maximum Principle extended by a concentrated state variable including a sketch of the proof). Maximizing the Hamiltonian with respect to $c(t, a)$ gives the following first order condition (suppressing t and a):[8]

$$N_0 S u_c(c)\left(1-\kappa+\kappa d(a, E)\right)+\zeta^D N_0 S = 0, \tag{57}$$

where $\zeta^D(t)$ is the adjoint variable for aggregated consumption. Using the specific form of the individual net utility (see (24)), we get

$$c = \left(-\frac{\zeta^D}{\sigma\left(1-\kappa+\kappa d(a, E)\right)}\right)^{\frac{1}{\sigma-1}}, \tag{58}$$

which directly implies that also in the social optimum consumption differs along the life-cycle of individuals in case $\kappa = 1$. For the adjoint equations, we obtain

$$\dot{\xi}^E = (\rho+\delta)\,\xi^E - \int_0^\omega N_0 S\left(u(c)\kappa d_E(a, E) - \xi^S \mu_E(a, E) + \zeta^L N_0 p_E(a, E)\right) \mathrm{d}a \tag{59}$$

$$\dot{\xi}^K = \left(\rho - F_K(K, L)\right)\xi^K - \xi_E F_K(K, L) \tag{60}$$

$$\xi_t^S + \xi_a^S = \left(\rho+\mu(a, E)\right)\xi^S - N_0 u(c)\left(1-\kappa+\kappa d(a, E)\right) - \zeta^D N_0 c - \zeta^L N_0 p(a, E) \tag{61}$$

$$\zeta^D = -\,\xi^K \tag{62}$$

$$\zeta^L = \left(\xi^E + \xi^K\right) F_L(K, L), \tag{63}$$

where $\xi^E(t)$, $\xi^K(t)$, and $\xi^S(t, a)$ denote the adjoint variables of $E(t)$, $K(t)$, and $S(t, a)$, respectively. $\zeta^L(t)$ depicts that adjoint variable to the aggregated labor. Before deriving an explicit expression for the dynamic development of the consumption path, i.e., the socially optimal Euler equation along the life-cycle of one individual, let us have a closer look on $\xi^E(t)$ by disentangling different effects and attaching a specific interpretation to them. By solving dynamics of $\xi^E(t)$ we obtain (for any $T > t$)

$$\xi^E(t) = e^{-(\rho+\delta)(T-t)}\xi^E(T) + \int_t^T e^{-(\rho+\delta)(t'-t)} \int_0^\omega \left[S\left(u\kappa d_E - \xi^S \mu_E + \zeta^L N_0 p_E\right)\right] \mathrm{d}a\mathrm{d}t'. \tag{64}$$

[8] Analogously to the individual optimum the Inada conditions for the consumption utility function exclude boundary solutions for $c(t, a)$.

Applying the limiting transversality condition (see Michel (1982)) $\lim_{t\to\infty} e^{-\rho t}\xi^E(t) = 0$, we find that the first term of the above three expressions is nil for $t \to \infty$ (since$\delta > 0$) and, thus,

$$\xi^E(t) = \int_t^\infty e^{-(\rho+\delta)(t'-t)} \int_0^\omega S\left[N_0 u\kappa d_E - \xi^S \mu_E + \zeta^L N_0 p_E\right] \mathrm{d}a\mathrm{d}t', \tag{65}$$

where all three terms in the inner integral account for marginal effects of $E(t)$ to the expected individual life-cycle utility: The first term for the loss of net utility, the second one for the life-cycle effect of a decrease of the survival probability, and the third one for the effect on the productivity of the individuals. The three parts are aggregated over all individuals and over the remaining time horizon. None of these effects is not considered in the decentralized solution (13) but are at the core of the social optimum.
From the first order condition, it is possible to derive the dynamics of consumption to observe its development along the life-cycle and compare it with the decentralized optimum as discussed in Lemma 1. It turns out, as already observed above, that the consumption path of an individual is, in general, not smooth over the life-cycle and that socially optimal individual consumption also considers the dynamic effects of the environment. This is summarized in the following lemma.

Lemma 4 Consider an individual entering the economic system at t_0 at age a_0, whose socially optimal consumption profile (along the life-cycle) is yielded by (57). The social optimal consumption path follows the (social) consumption Euler equation:

$$c_t + c_a = \frac{u_c}{u_{cc}}\left(-\frac{\xi^K}{\zeta^D}\left(\rho - F_K(K,L)\right) - \frac{\kappa\alpha d}{1-\kappa+\kappa d(a,E)}\left(\frac{\bar{d}_a}{\bar{d}} + \frac{\dot{E}}{E}\right) + \frac{\xi_E F_K}{\zeta^D}\right). \tag{66}$$

Proof The Euler equation (66) is obtained from differentiating the first order condition (57) along the life-cycle of an individual analogously to the individual problem. □

The dynamics of the consumption of an *a*-year old individual at *t* at the social optimum consists of the consumption dynamics of the decentralized problem without individual taxes (see (34)), but differs in two points. First, the difference between the discount and the market interest rate (in the social welfare solution represented by $F_K(K, L)$) is weighted by the exchange rate of aggregated capital and consumption. Second, an additional part takes the change in the damage of aggregated emissions into account.

Evaluating (66) for different κ gives

$$\kappa = 0: \quad c_t + c_a = \frac{u_c}{u_{cc}}\left(-\frac{\xi^K}{\zeta^D}\left(\rho - F_K(K,L)\right) + \frac{\xi_E F_K}{\zeta^D}\right), \tag{67}$$

$$\kappa = 1: \quad c_t + c_a = \frac{u_c}{u_{cc}}\left(-\frac{\xi^K}{\zeta^D}\left(\rho - F_K(K,L)\right) - \alpha\left(\frac{\bar{d}_a}{\bar{d}} + \frac{\dot{E}}{E}\right) + \frac{\xi_E F_K}{\zeta^D}\right), \tag{68}$$

which shows that the result on an increasing or decreasing consumption path is, in general, more complicated as compared to the decentralized solution. However, the terms reduce if the social welfare solution has reached a steady state, where the first and third effect cancel each other out by (60). Therefore, the conditions for consumption smoothing are similar to that of the decentralized solution (see Lemma 1):

Corollary 5 Consider an individual entering the economic system at t_0 at age a_0, whose optimal consumption profile along the life-cycle is determined by (57-63). Consider that the economy has reached a steady state, i.e., $\dot{E}(t) = 0$, and $\dot{K}(t) = 0$. Then individuals choose a smooth consumption path along the life-cycle if

- $\kappa = 0$
- $\kappa = 1$ and $\bar{d}_a = 0$.

On the transitional path the consumption path is adjusted (i) for the difference $\rho - F_K(K, L)$ weighted by the marginal exchange rate between *K* and *D* (using the envelop theorem), and (ii) for the effect of total emissions. Both effects (i) and (ii) account for spillover effects of individuals across the cohorts, which are missing in the individual optimization.

Note, that the result for $\kappa = 1$ and $\bar{d}_a = 0$ contrasts the standard economic literature of intergenerational differences, as this implies increasing consumption along age instead of a smooth consumption. Obviously, this result relies on the assumption of age-dependent environmental disutility (increasing in *a*), which can be doubted. However, it is empirically proven that the effect of pollution on individual health is strongly age-dependent, meaning that older people are suffering more. This, in turn, implies higher health expenditures to overcome these additional health issues (abstracting from other health treatment unrelated to environmental pollution) which can be considered as part of *c*(*t*, *a*) that therefore has to increase across age.

Different to the decentralized solution, the individual consumption cannot be formulated in closed form. This is because of the anticipation of the externality and spillover effects on the dynamics. Moreover, (38) is derived for a given total production, which is considered exogenous for an individual in the decentralized problem. In the social optimum, however, the effect is anticipated and included in the set of optimality conditions and adjoint equations.

The firms' behavior is not modeled explicitly but emerges as a byproduct of the socially optimal solution. Solving (49) with the age-structured Maximum Principle (see Feichtinger et al. (2003) or Veliov (2008)) and a suitable numerical solver (see Veliov (2003)) are established methods to obtain a reliable solution. As mentioned above, the social optimum will be used to consider taxes on individual consumption and emissions generated by the firm. On that basis we are able to explore under which conditions the social optimum can be attained by the different tax instruments.

5 Taxation

We have introduced two different types of taxes in the general model (13), i.e., individual taxes on consumption, $\tau_I(t,a)$, and firms' taxation levied on production, $\tau_F(t)$. In general, that gives the possibility of three different tax regimes shown in Table 2.

The question whether consumer taxes should really depend on time and age is a non-trivial issue and depends on the point of view. This will be discussed together with all cases of the table in the following subsections.

5.1 Case 1: Firm Taxation

Assuming that the government taxes only the production of firms it can be shown by the consumption Euler equations that the social optimum cannot be attained in general, that is, during the transitional period towards and within a steady state.

Proposition 6 Consider the full decentralized model (13) and the consumption utility function (24). Then the social optimum cannot be reached if only firms are taxed, i.e., with $\tau_F(t) \in [0,1)$ for $\forall t$ and $\tau_I(t,a) = 0$ for $\forall (t,a)$.

Proof From section 3.2 we know that firm taxes are transferred to individuals by the capital and wage income, i.e., both are reduced by the tax rate imposed by the government. While (a reduction of) the wage rate shifts the consumption path downwards, the (reduction of) tax rate decreases the discounted weighted rest life-time expectancy and shifts consumption to younger ages.

Both effects cannot compensate the differences in the consumption Euler equations (34) and (66), as they are weighted by $\frac{u_c(c)}{u_{cc}(c)}$ (which for (24) reduces to $\frac{c}{\sigma-1}$). To prove that consider a focal cohort born at t_0. For this cohort it is possible to chose $\tau_F(t)$ at any t such that the decentralized consumption $c(t_0+a,a)$ at all ages a ($a \in [0,\omega]$) the centralized consumption value. However, implementing the same tax rate for a different cohort (note that it only depends on t) in general does not lead to a socially optimal consumption path, as the interest and wage rate as well as total emissions change over time. □

This result lies at the heart of a critical issue inherent in the decentralized economy. Firms generate CO_2 emissions without bearing the consequences of the resulting emission stock, while individuals derive utility from consumption but incur disutility from the same emission stock. Importantly, individuals fail to internalize the impact of their consumption on the dynamics of aggregate CO_2 emissions, the resulting effects on other agents in the economy, and the broader implications for market equilibrium. This disconnect underscores a division of cause and effect between consumers and firms, with both exhibiting myopia in this regard. The proposition demonstrates that taxation on firms alone cannot resolve this issue, given the inherent asynchrony in the composition of the age-structured population. Moreover, this limitation persists even when an additional fiscal instrument, such as a tax on

Table 2 Summary of different taxes schemes

Tax regime	$\tau_I(t,a)$	$\tau_F(t)$
Case 1: Firm taxation	$=0$	x
Case 2: Individual taxation	x	$=0$
Case 3: Full taxation	x	x

interest income as proposed by Ono (1996), is introduced. The analysis thus reveals that the fundamental misalignment in time horizons across agents renders the problem resistant to resolution through conventional (age-independent) fiscal interventions, regardless of the combination of instruments employed.

Next important question is whether the above *impossibility* result is a structural problem appearing by the OLG structure, or whether there are situations in which a firm tax is sufficient for social optimality. As mentioned above, the analysis of the firm problem shows that firm taxes reduce the individual income by (42) and (44). The following proposition now formulates conditions under which the social welfare optimum can be reached by firm taxes.

Proposition 7 Consider the full decentralized model (13) and the consumption utility function (24). The social optimum can be attained by firm taxes, i.e., with $\tau_F(t) \in [0, 1)$ for $\forall t$ and $\tau_I(t, a) = 0$ for $\forall(t, a)$, if the system has reached a steady state. If, additionally,

(i) $\kappa = 0$, or
(ii) $\bar{d}(a) = 0$,a smooth consumption path (i.e., constant along life and cross cohorts) is optimal.

Proof For the proof we compare the consumption Euler equations of the decentralized and social welfare solutions and draw on the discussion within the proof of Proposition 6 that the weight $\frac{u_c}{u_{cc}}$. This implies that the social optimum can only be attained in a steady state, i.e., when the additional terms in the Euler equation of the social welfare problem vanish.

Applying the steady state conditions, we obtain

$$\text{decentralized:} \qquad c_t + c_a = \frac{u_c}{u_{cc}} \left(- \frac{\kappa \alpha d}{1 - \kappa + \kappa d} \frac{\bar{d}_a}{\bar{d}} \right), \tag{69}$$

$$\text{social welfare:} \qquad c_t + c_a = \frac{u_c}{u_{cc}} \left(- \frac{\kappa \alpha d}{1 - \kappa + \kappa d} \frac{\bar{d}_a}{\bar{d}} \right). \tag{70}$$

The steady state value of the tax τ_F must be chosen such that the consumption value of the decentralized solution equals that of the socially optimal one. The equality of (69) and (70) proves that both solutions are equal, as identical consumption implies identical steady state values of $E(t)$, $K(t)$, and $L(t)$.
Inserting $\kappa = 0$ or $\bar{d}_a = 0$ to (69) and (70) proves the second assertion. □
According to this result, in a steady state firm taxes are sufficient to implement the socially optimal outcome in the decentralized market. As obvious from the comparison of the Euler equations, the social optimal solution anticipates the change in the marginal effect of total capital and emissions normalized by the marginal damage of total consumption. As, however, firm taxes does not only shift the consumption level, but also change the path, the previously mentioned cross-cohort externalities become age-dependent and firm taxes cannot adjust in general (but only in a steady state where externalities become nil). In addition, Proposition 7 shows that under certain conditions even consumption smoothing is optimal. Here, the age-structure is obviously key. Either an age-independent disutility of $E(t)$ or an additive disutility is necessary to overcome consumption differences across age-groups.

Finally in this section it can be proven that firm taxes are sufficient to reach socially optimal production at every t. Although the result will not be socially optimal, it pushes (aggregated) CO_2 emissions to the socially optimal path.

Proposition 8 Consider the full decentralized model (13) and the consumption utility function (24). The socially optimal production can be attained by firm taxes, i.e., with $\tau_F(t) \in [0, 1)$ for $\forall t$ and $\tau_I(t, a) = 0$ for $\forall (t, a)$ at any time.

Proof To prove the proposition, we have to show that the production in the decentralized model equals that of the social optimal one. Then, consequently, $E(t)$ (as well as $K(t)$ and $L(t)$) equals the same in both cases.

By (29) and the functional choice (24) the decentralized consumption of any individual can be explicitly derived as

$$c = \left((1 + \tau_I) \frac{1}{S\sigma} \frac{\lambda^A}{1 - \kappa + \kappa d(a, E)} \right)^{\frac{1}{\sigma - 1}}, \tag{71}$$

and aggregated across cohorts to obtain

$$\int_0^\omega N_0 S(a) c(t, a) \mathrm{d}a, \tag{72}$$

which is the total consumption in the decentralized market solution. From the firm's optimality conditions (42) it follows that a positive/negative tax decreases/increases the income and therefore decreases/increases the consumption. And, in fact, this relation holds for all cohorts. To obtain the socially optimal production, the tax rate has to be chosen, such that (72) as well as interest and wage income across cohorts are adapted such that the total capital stock of the decentralized market solution equals that of the social welfare problem. Socially optimal $E(t)$ is equivalent to socially optimal $L(t)$ (by constant N_0 the dynamics of the survival probability), which proves existence. □

The proposition offers the possibility of the implementation of a solution (into the decentralized market) that is socially optimal in terms of environmental pollution but discriminates across cohorts. I.e., some cohorts will be better off, others worse. In this respect the solution is only second-best but opens up a pleasant opportunity if it is combined with Proposition 7 as follows: A policy maker, opting for the highest possible social welfare, can set firm taxes according to the socially optimal total emissions. If the solution approaches the steady state, the result of Proposition 7 then automatically implies that the cohorts get closer and closer to the socially optimal consumption path. This, in a sense, means that the policy maker takes into account cross-cohort discrimination initially, which reduces over time (i.e., at least after some initial adaptation period, and not necessarily at equal pace for all cohort) until they disappear in the long run.

5.2 Case 2: Individual Taxation

The opposite case contemplates only the tax on atomistic consumers. In the definition of the decentralized problem (13), the individual tax rate $\tau_I(t,a)$ has already been defined to depend in general on time and age. The following proposition collects its implications.

Proposition 9 Consider the full decentralized model (13) without firm taxation at any time, i.e., $\tau_F(t)=0$ for $\forall t$. Then the social optimum can be attained if individuals are taxed according to

$$\tau_I(t,a) = -\frac{\zeta^D(t)S(t,a)+\lambda^A(t,a)}{\lambda^A(t,a)}, \tag{73}$$

where $\lambda^A(t,a)$ and $\zeta^D(t)$ correspond to the shadow prices of the corresponding decentralized or social optimum. $S(t, a)$ depicts the socially optimal survival probability. If a decision maker restricts the individual tax rate to depend on time or age only (i.e., $\tau_I=\tau_I(a)$ or $\tau_I=\tau_I(t)$), the social optimum cannot be obtained in general.

Proof Equating the first order conditions of the decentralized and the social optimal solutions ((29) and (57), respectively) and isolating individual taxes proves (73).

If the individual tax rate depends on time only the social optimum cannot be obtained analogously to the firm tax rate (see also discussion in the previous subsection). An age-dependent tax rate (depending on time) cannot adjust to the transitional path and, consequently, cannot be optimal. □

Continuing the discussion of the previous subsection, it is obvious why a purely time- or age-dependent individual tax rate on consumption does not lead to the socially optimal consumption path. The above proposition proves the existence of (and explicitly states) an individual tax rate according to which the social optimal solution (i.e., socially efficient consumption path for all cohorts and socially efficient total emissions) will be attained. This notwithstanding, the existence property does not completely solve the decision makers problem, but implies the question whether it is implementable, as it means that the consumption of the same product has a different market price for individuals at different ages. E.g., one liter milk may cost 1 Euro for a 20 year old individual, while it may cost 1.50 Euro for a 40 year old one. This can hardly be argued but shows the core issue of the asynchronous time horizon of different cohorts at t. Firstly, 20 year old individuals have (approximately) 20 years more residual life-expectancy than 40 year old individuals. This time-effect can only be adjusted by time and age dependent taxes. Secondly, individuals have to compensate the loss of the individual net utility by $E(t)$ by higher consumption, which is not only time- but primarily an age-specific effect. I.e., individuals are smoothing their net utilities as far as possible at the cost of a cross-cohort discrimination with a potentially negative effect along time.

5.3 Case 3: Full Taxation

Instead of using taxes alternatively charged upon either individuals or firms, it is also possible to tax both market sides, as it is usually the case in reality. Apart from political reasons, such a parallel taxation can be used to overcome several problematic implementation issues such as passing on firm taxes to individuals or keeping back tax reliefs on the other hand.

The following proposition derives the relation between individual and firm taxation that must hold to implement the social optimum in general.

Proposition 10 Consider the full decentralized model (13) and the consumption utility function (24). Then the social optimum can be attained if individuals are taxed according to

$$\tau_I(t,a) = -\frac{\zeta^D(t)S(t,a) + \bar{\lambda}^A(t,a)}{\bar{\lambda}^A(t,a)}, \tag{74}$$

where $S(t, a)$ and $\zeta^D(t)$ correspond to the social optimum. $\bar{\lambda}^A(t,a)$ depicts the adjoint variable of individual assets shifted by firm taxes $\tau_F(t) \in [0,1)$. Moreover, the two tax rates are negatively related, i.e., $\frac{\partial \tau_I(t,a)}{\partial \tau_F(t)} < 0$ (pointwise for every t) for admissible values of $\tau_F(t)$.

Proof The explicit form of individual taxes can be shown as in the proof of Proposition 9, where only the adjoint variable for individual assets is adjusted for firm taxes.

For the second part of the proposition, we can show by Lemma 3 that consumption at all ages depends negatively on firm taxes as follows: Firm taxes decrease both $r(t)$ and $w(t)$ in the decentralized model. $w(t)$ determines $\Psi_I(t,a)$ (given by (39)) with $\frac{\partial \Psi_I(t,a)}{\partial \tau_F(t)} < 0$. Similarly, $r(t)$ enters $\Psi_{LE}(t,a)$ (given by (40)) with $\frac{\partial \Psi_{LE}(t,a)}{\partial \tau_F(t)} > 0$. Consequently, as this derivation holds for any t and a, consumption in the decentralized solution depends negatively on firm taxes, i.e., $\frac{\partial c(t,a)}{\partial \tau_F(t)} < 0$.

As a result (by the individual first order condition (29)), $\lambda^A(t)$ depends positively on $\tau_F(t)$, i.e., $\frac{\partial \lambda^A(t,a)}{\partial \tau_F(t)}$. In fact, $\bar{\lambda}^A(t,a) > \lambda^A(t,a)$ if $\bar{\lambda}^A(t,a)$ and $\lambda^A(t,a)$ denote the adjoint with and without firm taxes. Application of the explicit form of the individual taxes proves $\frac{\partial \tau_I(t,a)}{\partial \tau_F(t)} < 0$. $\square$

This proposition closes the gap between subsections 5.1 and 5.2. The purely individual tax rate given by (73) can be obtained by (74) in absence of firms' taxation. The tax levied on firms is generally weaker in terms of implementing a solution meeting less optimality criteria and, therefore, cannot be seen as a special case of (74). The main message of Proposition 10 is that $\tau_I(t,a)$ and $\tau_F(t)$ are not uniquely defined, but that there exists a continuum of tax rates that attain social optimality. The choice of the individual taxes according to Proposition 10 depends on the value of firm taxes at t, which means that there is one degree of freedom from the viewpoint of the government. However, up to now the balance sheet of the taxes has not been considered. All taxes are supposed to be positive implying a profit that has to be redistributed by the government. The following proposition integrates a tax

Table 3 Summary of results with different tax rates

		General case	Steady state
Case 1: Firm taxation	Social optimality	No	Yes
	Budget balance	No	No
Case 2: Individual taxation	Social optimality	Yes	
	Budget balance	No	
Case 3: Full taxation	Social optimality	Yes	
	Budget balance	Yes	

redistribution condition based on fixed (exogenous) governmental expenditures that fixes firm and individual taxes to single values.

Proposition 11 Consider the full decentralized model (13) and the consumption utility function (24). If exogenous governmental expenditures at t are denoted by $G(t) \geq 0$, the individual and firm taxes (if admissible) can be chosen to attain social optimality (by relation (74)) as well as a tax redistribution at t if they fulfill the following implicit condition

$$G(t) = \tau_F(t)F(K(t), L(t)) + \int_0^{\omega} N_0 S(t,a)\tau_I(t,a)c(t,a)\mathrm{d}a. \tag{75}$$

Proof Equation (75) fixes implicitly the non-unique individual taxes, for which in Proposition 10 only a relation (74) has been defined. Existence of an admissible firm tax rate follows from the continuity of the optimal solution when tax rates are shifted according to (74) and the mean value theorem from real analysis. Whether the resulting firm tax rate lies in the admissible region cannot be shown theoretically but has to be checked for the specific problem. □

This final proposition concludes the analysis by proving the existence of a balanced (i.e., redistributive) tax scheme based on an equilibrium relationship between two tax rates. Both together implement the socially optimal solution accounting for individual time- and age-specific consumption and aggregate emissions. From (75), it becomes obvious why a single tax rate cannot achieve the same outcome in presence of an exogenous governmental expenditure $G(t)$. As for firm taxes, it suffices to look at Proposition 6. The proposition, moreover, clarifies that in this tax scheme for sufficiently small values of $G(t)$ at least a continuum of cohorts exist for which consumption is subsidized (i.e., the individual tax rate is negative). This highlights the different mechanisms behind individual and firm taxation. While firm taxes can be used *first* to implement the socially optimal amount of total emissions, individual taxes *additionally* redistribute the available supply such that the cross-cohort consumption decomposition is socially optimal at t.

Table 3 compares the results presented in this section. Although firm taxation carries over to individual prices, we see that the social optimal solution cannot be reached generally in the decentralized market equilibrium. If individual taxation is considered (alone or in addition to firm taxes) social optimality is possible. A balanced budget from the governmental point of view is only possible with full taxation as this implies the necessary flexibility from the governmental side.

6 Conclusions

We have presented an economic framework comprising a production and consumption sector, both influenced by a negative environmental externality. Consumption decisions are made optimally by an overlapping generations (OLG) population that bears the disutility of aggregate emissions, while the production sector, composed of firms, optimizes output instantaneously without internalizing the environmental externality. Both consumers and firms exhibit myopia: individuals act atomistically, ignoring their collective impact, and firms fail to account for the consequences of their production plans. We have shown under which conditions the decentralized and socially optimal solutions differ at the individual level along time and across age. Particularly interesting are (i) the comparison of the different tax schemes levied on consumption and production, and (ii) the identification of fiscal interventions necessary to achieve the social optimum under various dynamic scenarios. Although implementing age-structured tax rates may be impractical in real-world settings, modeling such mechanisms in an OLG framework, especially one that incorporates a detailed description of the supply side, remains crucial for understanding the dynamics at play. However, the model suggests a way out of the dilemma to the policy maker by imposing firm taxes to attain socially optimal emissions. Although this implies cross-cohort discrimination, it decreases over time towards to the steady state. We acknowledge that our model is, by design, a simplified "toy" model. Nonetheless, its simplicity serves a purpose: isolating and elucidating effects that are obscured in models without an age-structured population or buried within complex, large-scale numerical simulations, such as agent-based models. This clarity underscores the value of our approach in advancing the theoretical understanding of intergenerational environmental policy challenges.

This study is subject to several limitations. One significant constraint is the assumption that individuals act atomistically, excluding any bequest motive from their consumption decisions. This simplification may be questionable, as it neglects the altruistic concerns that many people have for the welfare of their children, relatives, and, ideally, other members of society. Wrzaczek et al. (2014) demonstrate in a simplified framework that incorporating offspring's utility into an individual's welfare function does not eliminate the inefficiency inherent in the system. Even under the most favorable scenario-where individuals fully internalize the wellbeing of all others in their optimization decisions-the model retains its environmental externality.

One obvious extension will address the unfortunate fact that taxes borne by consumers must be age-specific. We plan to consider alternative tax and redistribution schemes which share similar properties as the proposed tax scheme of Proposition 11. Although an additional lump-sum tax imposed onto individuals to punish excessive consumption should basically work, the firm sector as well as the tax redistribution scheme complicates the analysis considerably.

Another appealing extension is to acknowledge that climate is expected not only to change gradually but also disruptively at climate tipping points. As the exact time of the tipping event is unknown the disruptive change of the climate has to be included by a random switch point that depends on past CO_2 emissions. In the context of economic-ecological optimal control models with single decision-maker, this question has been addressed by a number of papers, see e.g., Polasky et al. (2011), Van der Ploeg and de Zeeuw (2018, 2019), or Tsur and Zemel (1998, 2006, 2016), from a single decision makers point of view (e.g., a firm or policy maker). In contrast to these papers Wrzaczek (2021) considers a continuum of individuals with OLG structure (without firm sector) facing the possibility of a tipping point. A combination of these two strands of the literature results in a (degenerate) multi-stage differential game with random switching time.[9]

Funding Open access funding provided by International Institute for Applied Systems Analysis (IIASA).

Compliance with Ethical Standards

Manuscript Originality The manuscript is the authors' own original work and is not considered for publication elsewhere.

Original Research The paper reflects the authors' own research.

Sources and Citation All sources used are properly disclosed (correct citation).

Ethical Approval for Research In this research no human subjects or animals have been involved.

Conflicts of Interest The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this manuscript.

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[9] An analytical characterization is only possible by considering the stochastic tipping event as an age-structured optimal control problem (see Wrzaczek et al. (2020) for details).

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