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X and Y Operators for General Linear Transport Processes

(radiative transfer/atmospheric physics/scattering processes/Riccati operators/dimensionality reduction)

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ABSTRACT **This note presents the derivation of generalized Ambartsumian–Chandrasekhar X and Y functions for stationary transfer in a plane-parallel slab. An algebraic formula relating these functions to the usual reflection function is also presented, together with the appropriate generalization of the Chandrasekhar H-equations for the semi-infinite medium. The planetary problem will also be briefly discussed.**

1. Problem statement

We consider the plane parallel slab $\Pi(a,r)$, $r > a$, having boundaries $z = a$ and $z = r$. The distribution of radiation in the direction of increasing and decreasing z is represented by $I^{\pm}(z)$, respectively. These quantities take into account frequency, degree of polarization, direction, and so forth. Thus, $I^{\pm}(z)$ take on values in a reproducing cone K of nonnegative functions in a suitable separable Banach space B.

To each subslab $\Pi(z,z')$, $(z,z') \subset (a,r)$, there is associated reflection operators $R^{\pm}(z,z')$ and transmission operators $Q^{\pm}(z,z')$, which assume values from the Banach algebra $\mathcal{B}$ of bounded linear operators acting in B. The signs of $\pm$ refer to illumination of the subslab from the left and right, respectively (Fig. 1).

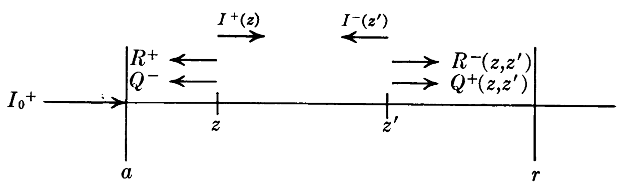

FIG. 1. Plane-parallel slab.

In the medium, we assume $\|Q^{\pm} + R^{\pm}\| \leq 1$ (no fission) and $Q^{\pm}(z,z') \to I$, $R^{\pm}(z,z') \to 0$ for $z' \to z + 0$. We also assume the existence of the limits

$$T^{\pm}(z) \equiv \lim_{z' \to z+0} \frac{I - Q^{\pm}(z,z')}{z - z'}, \qquad [1]$$

$$Z^{\pm}(z) \equiv \lim_{z' \to z+0} \frac{R^{\pm}(z,z')}{z' - z}.$$

In general, $T^{\pm}$, $Z^{\pm}$ are nonnegative operators. For an homogeneous medium, $T^{\pm}$, $Z^{\pm}$ are independent of z, while for a locally isotropic medium, $T^+ = T^-$ and $Z^+ = Z^-$.

On the medium $\Pi(a,r)$, let the flow I_0^+ be incident from the left. Then consideration of the regimes on the boundaries of the sub-slab $\Pi(z,z')$ shows that $I^{\pm}(z)$ satisfy the equations (ref. 1):

$$\pm \frac{dI^{\pm}}{dz} = -A^{\pm}I^{\pm} + Z^{\pm}(I^+(z) + I^-(z)), \qquad [2]$$

$$I^+(z) = I_0^+, \qquad I^-(r) = 0,$$

where $A^{\pm}(z) = T^{\pm}(z) + Z^{\pm}(z)$.

In concrete transfer problems, the operators $A^{\pm}(z)$, $Z^{\pm}(z)$ are known and we are interested in methods for determining $R^{\pm}$ and $Q^{\pm}$.

2. Reflection, transmission, and X-Y operators

Consideration of Fig. 1 shows that for $z' = r$, we have

$$I^-(z) = R(z)I^+(z), \qquad R(z) \equiv R^+(z,r). \qquad [3]$$

Substitution of [3] into [2] leads to the Cauchy problem for the operator R:

$$\frac{-dR}{dz} = Z^-(z) - T^-(z)R - RT^+(z) + RZ^+(z)R, \qquad [4]$$

$$R(r) = 0.$$

Knowledge of $R(z)$ allows us to simultaneously solve a family of different problems with different values of a. We determine $I^+(z)$ from the Cauchy problem

$$\frac{dI^+}{dz} = (Z^+R - T^+)I^+, \qquad I^+(a) = I_0^+, \qquad [5]$$

while $I^-(z)$ is determined from [3].

Since the pioneering work of Chandrasekhar (2) and Ambartsumian (3), it is well known that, in some cases, the solutions to the operator Riccati Eq. 4 may be expressed by an algebraic combination of lower-dimensional operators, the so-called X and Y operators. Our main result shows when this may be expected.

THEOREM 1. *Assume the medium is homogeneous, i.e.,* $T^{\pm}$, $Z^{\pm}$ *are independent of* z. *Further, assume*

(*i*) dim range $Z^- = p < \infty$
(*ii*) dim range $Z^+ = q < \infty$

and that $Z^{\pm}$ *are factored as* $Z^- = MN$, $Z^+ = UV$, *where dim range* $N = p =$ *dim domain* M, *dim range* $V = q =$

dim domain V. Then R admits the algebraic representation

$$T^-R(z) + RT^+(z) = Z^- + X_1(z)X_2(z) - Y_1(z)Y_2(z),$$

where Y_1, Y_2, X_1, X_2 satisfy the equations

$$\frac{dY_1(z)}{dz} = (T^- - X_1(z)V)Y_1, \qquad Y_1(r) = -M,$$

$$\frac{dY_2(z)}{dz} = Y_2(T^+ - UX_2(z)), \qquad Y_2(r) = N,$$

$$\frac{dX_1(z)}{dz} = -Y_1Y_2U, \qquad X_1(r) = 0,$$

$$\frac{dX_2(z)}{dz} = -VY_1Y_2, \qquad X_2(r) = 0.$$

Proof: We follow the proof of ref. 4, which was given for a special case of Eq. **4**. Differentiate Eq. **4** with respect to z. This yields the following homogeneous equation for the operator dR/dz:

$$\frac{d}{dz}\left(\frac{dR}{dz}\right) = (T^- - RZ^+)\frac{dR}{dz} + \frac{dR}{dz}(T^+ - Z^+R),$$

$$\left.\frac{dR}{dz}\right|_{z=r} = -Z^- = -MN.$$

We make the definitions $X_1(z) = RU$, $X_2(z) = VR$, and use the representation

$$\frac{dR}{dz} = -\alpha MN\beta,$$

where

$$\frac{d\alpha}{dz} = (T^- - X_1V)\alpha, \qquad \alpha(r) = I,$$

$$\frac{d\beta}{dz} = \beta(T^+ - UX_2), \qquad \beta(r) = I.$$

The theorem follows with $Y_1 = \alpha M$, $Y_2 = N\beta$.

Remarks: (*i*) For an isotropically scattering medium, $Z^+ = Z^-$ and $T^+ = T^-$, with $T^\pm$ being self-adjoint. Thus, $Y_1 = Y_2^*$, $X_1 = X_2^*$, and the usual situation of a single X and a single Y operator is recovered.

(*ii*) For slabs with a reflecting surface at $z = r$, the Riccati Eq. **4** has a nonzero initial condition at $z = r$, say $R(r) = F$. If F is independent of z, the foregoing arguments carry through, replacing assumption (*i*) of the *Theorem* by (*i'*) dim range $[-(Z^- - T^-F - FT^+ + FZ^+F)] < p < \infty$. For a specific application of this case to an atmosphere bounded by a Lambert law reflector, see ref. 5.

(*iii*) The finiteness of p and q is not essential. All that is required is that Z^+ and Z^- project into lower dimensional subspaces of B. However, for computational considerations, the finite case is the most appropriate.

3. Semi-infinite media

We now treat the case of a semi-infinite medium. In order to derive an equation for the operators $X_1(-\infty)$, $X_2(-\infty)$, we utilize the following lemma:

LEMMA 1. *Let P,A,Q be bounded linear operators of B to B. Then*

$$\sigma(PAQ) = (Q^* \otimes P)\sigma(A), \qquad [6]$$

where σ: $L(B,B) \rightarrow C^{(\dim B)^2}$ is the operator of "stacking" the "columns" of an element of $L(B,B)$, and $\otimes$ is the usual tensor product of two operators.

Proof: Using the separability of B, the proof follows by a coordinate-wise comparison of the left and right sides of [**6**].

The result, which generalizes the Chandrasekhar H-equation for the semi-infinite medium, is

THEOREM 2. *Let $X_1(-\infty) = H_1$, $X_2(-\infty) = H_2$. Then H_1 and H_2 satisfy the equations*

$$\sigma(H_1) = (U^* \otimes I)(I \otimes T^- + (T^+)^* \otimes I)^{-1}\sigma(Z^- + H_1H_2),$$

$$\sigma(H_2) = (I \otimes V)(I \otimes T^- + (T^+)^* \otimes I)^{-1}\sigma(Z^- + H_1H_2).$$

Proof: From the Riccati Eq. **4**, we have

$$T^-R(-\infty) + R(-\infty)T^+ = Z^- + H_1H_2.$$

Applying σ to both sides of this equation and using the identities

$$\sigma(H_1) = \sigma(RU) = (U^* \otimes I)\sigma(R),$$

$$\sigma(H_2) = \sigma(VR) = (I \otimes V)\sigma(R),$$

the theorem easily follows.

Remarks: (*i*) *Theorem 2* assumes that $\lambda_i + \mu_j \neq 0$, where $\{\lambda_i\}$ are the characteristic roots of T^- and $\{\mu_j\}$ are the roots of $(T^+)^*$; (*ii*) in both *Theorems 1* and *2*, considerable simplification occurs if Z^- and Z^+ are self-adjoint, while $T^+ = T^{-*}$, since in this case $X_1 = X_2^*$, $Y_1 = Y_2^*$, and $H_1 = H_2^*$. This is the situation that prevails in the classical plane-parallel, isotropic scattering, homogeneous case.

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