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ORIGINAL RESEARCH

Sampling methods for multi-stage robust optimization problems

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Abstract

In this paper, we consider multi-stage robust optimization problems of the minimax type. We assume that the total uncertainty set is the cartesian product of stagewise compact uncertainty sets and approximate the given problem by a sampled subproblem. Instead of looking for the worst case among the infinite and typically uncountable set of uncertain parameters, we consider only the worst case among a randomly selected subset of parameters. By adopting such a strategy, two main questions arise: (1) Can we quantify the error committed by the random approximation, especially as a function of the sample size? (2) If the sample size tends to infinity, does the optimal value converge to the "true" optimal value? Both questions will be answered in this paper. An explicit bound on the probability of violation is given and chain of lower bounds on the original multi-stage robust optimization problem provided. Numerical results dealing with a multi-stage inventory management problem show that the proposed approach works well for problems with two or three time periods while for larger ones the number of required samples is prohibitively large for computational tractability. Despite this, we believe that our results can be useful for problems with such small number of time periods, and it sheds some light on the challenge for problems with more time periods.

Keywords Multi-stage robust optimization · Constraint sampling · Scenario approach in optimization · Randomized algorithms

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1 Introduction

We consider a multi-stage decision problem, i.e. a problem, where the decisions x_t have to be taken at discrete time instants $t = 1, \dots, H+1$, where $H+1$ is the horizon length. In our setup, some relevant parameters are not known at time t of decision and become revealed only at time $t+1$. We look for the optimal decision strategy under the objective to minimize the costs for the worst case among all possible parameter values. Thus we look for a multi-stage robust solution.

The usual robust optimization models deal with *static* problems, where all the decision variables have to be determined before the uncertain parameters are revealed. A vast literature focused on uncertainty structure to obtain computationally tractable problems is available, see for instance Dimitris Bertsimas (2004) and Soyster (1973) for polyhedral uncertainty sets and Ben-Tal & Nemirovski (1999) for ellipsoidal uncertainty sets, respectively. However, this approach cannot directly handle problems that are multiperiod in nature, where a decision at any period should take into account data realizations in previous periods, and the decision maker needs to adjust his/her strategy to the information revealed over time. This means that some of the variables (non-adjustable variables) must be determined before the realization of the uncertain parameters, while the other variables (adjustable variables) have to be chosen after the uncertainty realization. For a recent overview of multiperiod robust optimization, we refer to Bertsimas et al. (2011), Delage and Iancu (2015), Gabrel et al. (2012). In order to describe such a situation, and extend robust optimization to a dynamic framework, the concept of *Adjustable Robust Counterpart* (ARC) has been first introduced and analyzed in Ben-Tal et al. (2003). This approach opened up the research in several new application areas, such as portfolio optimization (Pınar & Tütüncü 2005, Tütüncü & Koenig 2004), inventory management (Ben-Tal et al., 2005; Bertsimas & Thiele, 2006), scheduling (Yamashita et al., 2007), facility location (Baron et al., 2011), revenue management (Perakis & Roels, 2010) and energy generation (Zhao et al., 2013). ARC is clearly less conservative than the static robust approach, but in most cases it turns out to be computationally intractable. One of the most recent methods to cope with this difficulty is obtained by approximating the adjustable decisions by *decision rules*, i.e. combinations of given basis functions of the uncertainty. A particular case is the *Affinely Adjustable Robust Counterpart* (AARC) (Ben-Tal et al., 2003), where the adjustable variables are affine functions of the uncertainty. The decision rule approximation often allows to obtain a formulation which is equivalent to a tractable optimization problem (such as linear, quadratic and second-order conic (Ben-Tal et al., 2002), or semidefinite (Ghaoui et al., 1998)), transforming the original dynamic problem into a static robust optimization problem whose decision variables are the coefficients of the linear combination. In Postek and den Hertog (2016) a methodology for constructing decision rules for integer and continuous decision variables has been provided. The authors show by iteratively splitting the uncertainty set into subsets, how one can determine the later-period decisions based on the revealed uncertain parameters.

However, in many practical cases, also the static robust optimization problem ensuing from the decision rule approximation is still numerically intractable. In these situations, one can recur to approximate solutions based on constraint sampling, which consists in taking into account only a finite set of constraints, chosen at random among the possible continuum of constraint instances of the uncertainty. The attractive feature of this method is to provide explicit random bounds on the measure of the original constraints of the static problem that are possibly violated by the randomized solution. The properties of the solutions provided by this approach, called scenario approach have been studied in Calafiore and Campi (2004), Campi and Garatti (2008), de Farias and Roy (2004), where it has been shown that most of the constraints of the original static problem are satisfied provided the number of scenarios sufficiently large. The constraint sampling method has been also extensively studied within the chance constraint approach through different directions by Erdogan and Iyengar (2006), Luedtke and Ahmed (2008), Pagnoncelli et al. (2009).

In Bertsimas and Dunning (2016), Calafiore and Nilim (2004), Vayanos et al. (2012) multi-stage convex robust optimization problems are solved by combining general nonlinear decision rules and constraint sampling techniques. This means that the dynamic robust optimization problem is transformed into a static one through decision rules approximation and then solved via a scenario counterpart. In practice, the novelty of Vayanos et al. (2012) is to introduce, besides polynomial decision rules, also trigonometric monomials and basis functions based on sigmoidal and Gaussian radial functions, thus allowing more flexibility. A rigorous convergence proof for the optimal value, based on the decision rule approximation and of the constraint randomization approach is also given. Convergence is proved when both the complexity parameter (the number of basis functions in the decision rule approximation) and the number of scenarios tend to infinity.

The work of Bertsimas & Dunning (2016) proposes a technique based on structured adaptability that results in sample complexity, i.e. the minimum number of samples required to achieve the desired probabilistic guarantees, that is polynomial in the number of stages. This allows to provide a hierarchy of adaptability schemes, not only for continuous problems, but also for discrete problems.

In the context of randomized methods for uncertain optimization control problems, the *scenario with certificates* approach has been proposed in Formentin et al. (2016), based on an original idea of Oishi (2006). This approach has been then extended and exploited for anti-windup augmentation problems (Formentin et al., 2016). The main idea of this approach is to distinguish between design variables (corresponding to non-adjustable variables) and certificates (corresponding to adjustable variables).

Linear decision rules have a long history also in stochastic programming (see, e.g., Garstka and Wets (1974)), and have been adapted to Multi-stage Linear Stochastic Programming (MSLP) in Shapiro and Nemirovski (2005), and in Kuhn et al. (2011) who analyzed their application in the dual of the MSLP. Under certain assumption such as stagewise independence, compact and polyhedral support, if uncertainty is limited to the right-hand side of the constraints, Kuhn et al. (2011) and Shapiro and Nemirovski (2005) have shown that the static approximations obtained restricting the primal and dual policies to be linear decision rules are both tractable linear programs. Better policies have been obtained in Bampou and Kuhn (2011), Chen et al. (2008) by considering polynomial decision rules and piecewise linear decision rules respectively, while binary decision rules have been considered in Bertsimas and Georghiou (2018). Recently Bodur and Luedtke (2018) present a new use of linear decision rules for MLSP named *two-stage linear decision rule approach* based on the idea of partitioning the decision variables into *state* and *recourse* decisions and applying linear decision rules only to the state variables. This approach allows to reduce the problem to a two-stage stochastic linear program with a potentially improved policy and bounds. The approach is also applied to the dual of an MSLP, imposing the restriction only on the dual variables associated with the state equations and they show to obtain better bounds and policies than the the ones provided by the standard static approach.

Other approaches not involving decision rule approximations have been proposed in the literature to solve multi-stage robust optimization problem under special problem structure: this is the case of the work Georghiou et al. (2019) which presents a Robust Dual Dynamic Programming (RDDP) scheme to solve multistage robust linear optimization problems with special structure such as uncertain technology matrices and/or constraint right-hand sides. The proposed algorithm decouples the problem into $H + 1$ two-stage subproblems that approximate the costs arising in future stages through bounds on the cost-to-go functions and iteratively solves the $H + 1$ subproblems in sequences of forward and backward passes.

In this paper, we consider randomized methods for robust multi-stage optimization problems. We approximate the given robust problem by a sampled subproblem via a scenario-tree approximation, where instead of looking for the worst case among the infinite and typically uncountable set of uncertain parameters, we consider only the worst case among a randomly selected subset of parameters. In this way, we establish a link between multi-stage robust optimization and multi-stage stochastic optimization. By adopting such a strategy, two main questions arise: (1) Can we quantify the error committed by the random approximation, especially as a function of the sample size and provide a bound on the violation probability of the ignored constraints? (2) If the sample size tends to infinity, does the optimal value converge to the "true" optimal value? Both questions will be answered in this paper.

To simplify the understanding, we first consider the two-stage case, and we show that the theoretical sample complexity depends only on the number of first-stage variables. For the multi-stage case, the contributions can be summarized as follows: (i) define the probability of violation at each decision stage; (ii) provide a bound on the probability of violation by a function of the number of nodes of the tree up to that stage, the number of decision variables at that stage and the pre-specified violation tolerance; (iii) come up with an iterative scheme to define a sufficiently large number of nodes of the tree at each stage; and lastly, (iv) define the total violation probability as the probability of violation at any stage. Moreover, lower bounds on the true optimal value by extending two commonly used relaxations from the stochastic programming literature such as the wait-and-see problem, and the two-stage relaxation are provided. The proposed ideas are illustrated on a simple inventory model. The way how the proposed algorithm works is shown by analyzing the optimality gaps and empirical violation probabilities of the scenario-problem solutions, for many levels of the violation threshold, for the two- and three-stage cases. While the main application considered deals with a piece-wise affine objective and linear constraints, our bounds are valid also in the more general setup of a convex objective and convex constraint sets. An example with convex objective function and linear constraints is also presented.

The main difference between the approach proposed in this paper and the one in Vayanos et al. (2012), is that we do not change the decision model to a simpler one restricting the decision functions spaces via decision-rule approximation. The asymptotic result they provide holds only, if the chosen function space is such large that any continuous function can be uniformly approximated with a sup-distance less than some chosen ϵ. By more sampling alone, the optimization gap cannot be brought to zero. In our setup, we keep the model as it is and approximate it by sampling. Moreover, the authors in Vayanos et al. (2012) consider only sampled paths from the uncertain parameters, while we consider complete sampled scenario trees, leading to a much stricter notion of the so called *violation probability*, as it will be explained in detail in Sect. 2.6. Furthermore, if the uncertainty set is finite and we have sampled all points, then our solution is exact, while the decision-rule approximation approach is typically not.

We may summarize the differences between our approch and that of Georghiou et al. (2019) as follows: We allow convex objective functions and convex constraints, which is not the case of Georghiou et al. (2019). Besides, our uncertainty sets can be arbitrary and need not to be polyhedral. What we only need is a method to sample with a density which is bounded from below. Advisable is to use a sampling method with constant density. In contrast, in the setup in Georghiou et al. (2019) the geometry of the uncertainty sets as polyhedra is crucial for the algorithm to work. There is an extension in Georghiou et al. (2019), where the authors talk about convex non-polyhedral sets which can be asymptotically approximated by polyhedral sets. In this case the problem cannot be reduced to a finite number of constraints but the approximation method decribed in Georghiou et al. (2019) is shown to converge to the true optimal value asymptotically. This is achievable only with gigantic sample sizes. No quality assertion can be given for a finite number of samples. In contrast, our notion of violation probability allows to make a statement about the quality of the result for finite sample size and also allows to calculate the required sample size for a predescibed required quality. Furthermore, in our approach, the assumption that the uncertainty set is a cartesian product can easily be relaxed. One may sample from the cartesian product and reject points which are outside the prescribed uncertainty set. In terms of the scenario trees which we will construct, we just cut off some subtrees which represent scenarios which are not in the uncertainty set.

Our work is also related to a recent stream of articles (see (Chen & Luedtke, 2022; Lam & Li, 2022; Liu, 2020)) on the use of sample average approximation for stochastic programming problems without the assumption of relatively complete recourse. However, the special nature of the robust optimization problem considered in this paper allows us to obtain stronger results than the ones in the articles mentioned above. In particular, for the case of two-stage stochastic linear programs, the strongest bounds in the literature on the required sample size are obtained in Chen and Luedtke (2022) and the bounds proposed in this paper for the robust version of the same case are stronger, not depending on the number of second-stage variables. In addition, of the three papers mentioned above, only Liu (2020) considers multi-stage problems.

The rest of the paper is organized as follows. Section 2 discusses the formulations of two-stage, multi-stage robust linear and convex programs, and provides a result on the probability of violation. Bounds on the number of scenarios needed to obtain a user-prescribed guarantee of violation is given. Section 3 provides a chain of inequalities among lower bounds on the optimal value of the multi-stage robust optimization problem. Section 4 presents numerical results dealing with a multi-stage inventory management problem. The conclusions follow.

2 Main results

2.1 Basic facts

We consider a multi-stage discrete-time decision problem, where the decisions at times $t = 1, \ldots, H + 1$, denoted by $x_t \in \mathbb{R}^{n_t}$ have to be made under the presence of parameters ξ_t, $t = 1, \ldots, H$. At time t, the values $\xi_1, \ldots, \xi_{t-1}$ are known, but for ξ_t it is known only that it lies in some uncertainty set Ξ_t. The problem is to find optimal decisions under a worst-case objective, making the problem of nested minimax type.

Typically, the uncertainty sets are uncountable and one of the possible ways of treating this problem is by approximating the large uncertainty set by well chosen finite one. Notice that, in this paper, we do not restrict the class of possible decisions.

The uncountable sets Ξ_t are replaced by finite subsets $\tilde{\Xi}_t$. These subsets may be chosen by minimizing the Pompeiu-Hausdorff distance between the large sets Ξ_t and the finite sets $\tilde{\Xi}_t$, i.e. by an optimal selection of points. However, in this paper we use the simplest way of extracting points from larger sets: we do random sampling.

Notice that for a sampling method we need to define a probability measure[1] $\mathbb{P}$ on $\Xi = \times_{t=1}^{H} \Xi_t$. While the proposed methods work for any probability measure on Ξ_t which has a Lebesgue density which is bounded away from zero, we recommend to use, if possible, a uniform distribution on Ξ_t for a "fair" treatment of all points in Ξ_t and the product measure on Ξ. Notice that we have to use the same probability measure for sampling and for calculation of the violation probability. Theoretically, one could also try to construct a probability measure which makes the "extremal" points more likely, but by treating some points less likely, we may have difficulties in interpretating the violation probability.

If one chooses a probability which is not stagewise independent, then the notion of violation probablity becomes difficult to interpret. In fact, the notion of the violation probability can be interpreted as the volume of the set of points, which would lead to a "surprise", when sampled because they lead to a change in the best optimal value found so far. The volume of "surprise" points is found be using independent sampling with probabilites with constant density. If we choose stagewise dependent sampling, it could happen that the calculated violation probability is quite low, but the volume of samples leading to "surprise" in the above sense is large.

By considering N_t independent random samples from $\mathbb{P}_t$, for each t, finite subsets $\hat{\Xi}_t^{N_t}$ of sizes N_t are extracted from Ξ_t, and the multi-stage worst-case problem is solved with

$$\Xi_1 \times \cdots \times \Xi_H ,$$

replaced by the finite sets

$$\hat{\Xi}_1^{N_1} \times \cdots \times \hat{\Xi}_H^{N_H} .$$

The technique to replace a possible infinite set of convex constraints by a random finite selection of these constraints was originally introduced by Calafiore and Campi (2004) and later improved independently by Calafiore (2010) as well as by Campi and Garatti (2008). The sampled sets $\hat{\Xi}_t^{N_t}$ are referred to as *sampled scenarios*. We remark that in the stochastic optimization literature the use of a finite number of scenarios to represent the infinite possible realization of the uncertain quantities ξ_t is rather popular, see e.g. Dupačová et al. (2003), Pflug & Pichler, (2014), Shapiro et al. (2009).

Consider the robust optimization problem:

$$\begin{aligned} \text{RO} \quad : \quad & \min_{x\in\mathbb{X}} c^\top x \\ & \text{s.t. } f(x,\xi) \le 0, \quad \forall \xi \in \Xi , \end{aligned} \tag{1}$$

or equivalently

$$\text{RO} \quad : \quad \min_{x\in\mathbb{X}} \left\{ c^\top x : \sup_{\xi\in\Xi} f(x,\xi) \le 0 \right\} , \tag{2}$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the optimization variable, $\mathbb{X}$ is convex and closed and $f(x,\xi) : \mathbb{X} \times \Xi \to \mathbb{R}$ is a convex function in x for all $\xi \in \Xi$. The optimal objective value $v(\cdot)$ of problem (1) is denoted by $v(\text{RO})$.

Definition 1 (scenario approximation) Suppose that Ξ is a compact set and $\mathbb{P}$ is a probability measure on it with nonvanishing density. Let $\xi^{(1)}, \ldots, \xi^{(N)}$ be independent scenarios from Ξ, sampled according to $\mathbb{P}^N = \mathbb{P} \times \cdots \times \mathbb{P}$, N times. The "scenario" approximation of

[1] If historical data are available, we can use them, but we would formulate the problem differently.

problem (2) is defined as follows:

$$\widehat{\mathrm{RO}}^N \quad : \quad \min_{x \in \mathbb{X}} \left\{ c^\top x : \max_{i=1,\ldots,N} f(x, \xi^{(i)}) \leq 0 \right\}, \tag{3}$$

with the understanding that, in (3), we let the optimal solution $v(\widehat{\mathrm{RO}}^N) = \infty$ whenever the random extraction $\xi^{(1)}, \ldots, \xi^{(N)}$ leads to an infeasible problem.[2]

Definition 2 (violation probability) The "violation probability" $V(\cdot)$ of the sample $\hat{\Xi}^N := \{\xi^{(1)}, \ldots, \xi^{(N)}\}$ is defined as:

$$V(\hat{\Xi}^N) := \mathbb{P}\left\{ \xi^{(N+1)} : \min_{x \in \mathbb{X}} \left\{ c^\top x : \max_{i=1,\ldots,N+1} f(x, \xi^{(i)}) \leq 0 \right\} > v(\widehat{\mathrm{RO}}^N) \right\}, \tag{4}$$

where also $\xi^{(N+1)}$ is sampled from $\mathbb{P}$.

Notice that V is a random variable taking its values in $[0, 1]$.

We report here the Calafiore, Campi and Garatti main result, which is crucial for this paper.

Proposition 1 (CCG Theorem, Calafiore (2010) and Campi and Garatti (2008)) *Given an accuracy level $\epsilon \in (0, 1)$ and a sample $\hat{\Xi}^N$, the tail probability of the violation probability $V(\hat{\Xi}^N)$ under $\mathbb{P}$ can be bounded by*

$$\mathbb{P}\{V(\hat{\Xi}^N) > \epsilon\} \leq B(N, \epsilon, n) = \sum_{j=0}^{n} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j}. \tag{5}$$

For any probability level $\epsilon \in (0, 1)$ and confidence level $\beta \in (0, 1)$, let:

$$N(\epsilon, \beta, n) := \min \left\{ N \in \mathbb{N} : \sum_{j=0}^{n} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j} \leq \beta \right\}. \tag{6}$$

Then $N(\epsilon, \beta, n)$ is a sample size which guarantees that the ϵ-violation probability given in (5) lies below β.

Remark 1 (On the CCG Theorem) Notice that, in the CCG Theorem, we defined the violation probability in terms of the *cost function*, following the approach in Calafiore (2010). This allows to define the violation also for possible situations in which the ensuing scenario problem (3) turns out to be infeasible. It should also be remarked that the previous result holds under some assumptions on the scenario problem (3). Namely, the CCG Theorem requires that it is guaranteed that when problem (3) admits an optimal solution, this solution is unique (uniqueness), and that it is nondegenerate with probability one (nondegeneracy). These assumptions are of technical nature, and, as observed in Calafiore (2010), can be usually relaxed. For instance, uniqueness of the solution can essentially be always obtained by imposing some suitable tie-breaking rule. Regarding the definition degeneracy, we refer the reader to Calafiore (2010, §3.4) for a detailed discussion.

We remark that, in the literature, the minimum number of samples for which $B(N, \epsilon, n) \leq \beta$ holds for given $\epsilon \in (0, 1)$ and $\beta \in (0, 1)$ is referred to as *sample complexity*, see for instance

[2] Notice that $v(\widehat{\mathrm{RO}}^N) = \infty$ in (3) implies that also $v(\mathrm{RO}) = \infty$ in (2) and detecting this, one may stop sampling.

Tempo et al. (2013). There exist several results in the literature about bounding the sample complexity. In particular, in Lemma 1 and 2 in Alamo et al. (2015), it is proved that given $\epsilon \in (0, 1)$ and $\beta \in (0, 1)$:

$$N(\epsilon, \beta, n) \leq N^*(\epsilon, \beta, n) := \frac{1}{\epsilon}\frac{e}{e-1}\left(\ln\frac{1}{\beta} + n\right), \tag{7}$$

where e is the Euler constant and n is the dimension $dim(\cdot)$ of vector x, i.e. $n = dim(x)$. This bound gave a (numerically) significant improvement upon other bounds available in the literature (Calafiore, 2010; Calafiore et al., 2011). Notice that while bound (7) is certainty useful for estimating N, the problem in (6) can be solved by using bisection or ready-made tools such as Matlab betainv, in order to get the exact (tight) value of N.

The CCG Theorem can also be applied to more general situations, since some of them can be reformulated to fit into the form (2). A convex objective function $c(x)$ instead of the linear objective function can be handled as follows:

$$\min_{x \in \mathbb{X}} \left\{ c(x) : \sup_{\xi \in \Xi} f(x, \xi) \leq 0 \right\},$$

which can be rewritten as:

$$\min_{x \in \mathbb{X}, \gamma \in \mathbb{R}} \left\{ \gamma : \sup_{\xi \in \Xi} f(x, \xi) \leq 0;\ c(x) - \gamma \leq 0 \right\}.$$

This problem is now as in (2). Notice that the decisions are here the pair (x, γ) and therefore the dimension of the decision variable (vector) is $dim(x) + dim(\gamma) = n + 1$.

Also a supremum in the objective as in the following problem can be reformulated:

$$\min_{x} \sup_{\xi \in \Xi} \left\{ g(x, \xi) : x \in \mathbb{X}(\xi) \right\}, \tag{8}$$

where $g : x \mapsto g(x, \xi)$ is convex in x and $\mathbb{X}(\xi)$ are convex sets for all $\xi \in \Xi$. To see this, set:

$$f(x, \xi) := g(x, \xi) + \psi_{\mathbb{X}(\xi)}(x), \tag{9}$$

where

$$\psi_{\mathbb{B}}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{B} \\ \infty & \text{otherwise.} \end{cases} \tag{10}$$

Then f is convex in x and (8) can be written as:

$$\min_{x} \sup_{\xi \in \Xi} f(x, \xi).$$

Finally, observe that this problem is equivalent to:

$$\min_{x, \gamma} \left\{ \gamma : \sup_{\xi \in \Xi} f(x, \xi) - \gamma \leq 0 \right\}.$$

Again, this problem is of the standard form (2) with an augmented decision variable (vector) of dimension $dim(x) + 1$.

2.2 Two-stage robust linear case

To simplify our exposition, we first analyze a *two-stage robust linear program*, formally defined as follows:[3]

$$\begin{aligned} \mathrm{RO}_2 \quad : \quad & \min_{x_1} c_1^\top x_1 + \sup_{\xi_1 \in \Xi_1} \left[\min_{x_2(\xi_1)} c_2^\top(\xi_1) x_2(\xi_1) \right] \\ & \text{s.t. } Ax_1 = h_1, \quad x_1 \geq 0 \\ & \quad T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\xi_1) = h_2(\xi_1), \qquad x_2(\xi_1) \geq 0, \quad \forall \xi_1 \in \Xi_1 , \end{aligned} \tag{11}$$

where $c_1 \in \mathbb{R}^{n_1}$ and $h_1 \in \mathbb{R}^{m_1}$ are known vectors and $A \in \mathbb{R}^{m_1 \times n_1}$ is a given (known) matrix. The uncertain parameter vectors and matrices as functions of the uncertain factor ξ_1 are given by $h_2(\xi_1) \in \mathbb{R}^{m_2}$, $c_2(\xi_1) \in \mathbb{R}^{n_2}$, $T_1(\xi_1) \in \mathbb{R}^{m_2 \times n_1}$, and $W_2(\xi_1) \in \mathbb{R}^{m_2 \times n_2}$. Ξ_1 is a compact set in $\mathbb{R}^{k_1}$. The goal is to find a first-stage decision x_1 and a second-stage decision function $\xi_1 \mapsto x_2(\xi_1)$, such that the cost function in the worst-case realization of $\xi_1 \in \Xi_1$ is minimized. To this end, we first remark that problem (11) can equivalently be rewritten as follows:

$$\mathrm{RO}_2 \quad : \quad \min_{x_1 \in \mathbb{X}_1} \left\{ c_1^\top x_1 + \mathscr{R}(x_1) \right\}, \tag{12}$$

where

$$\mathbb{X}_1 := \{x_1 \geq 0 : Ax_1 = h_1\} , \tag{13}$$

and $\mathscr{R}(x_1)$ is the *worst-case recourse function*

$$\mathscr{R}(x_1) := \sup_{\xi_1 \in \Xi_1} \mathcal{Q}(x_1, \xi_1) \ ,$$

with $\mathcal{Q}(x_1, \xi_1)$ being the *recourse function*

$$\begin{aligned} \mathcal{Q}(x_1, \xi_1) := \min_{x_2(\xi_1)} \ & c_2^\top(\xi_1) x_2(\xi_1) \\ \text{s.t. } & T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\xi_1) = h_2(\xi_1) \\ & x_2(\xi_1) \geq 0 \, . \end{aligned} \tag{14}$$

Since RO_2 in (11) is of minimax type, we have to make the used notion of feasibility more precise. This is discussed in the next remark.

Remark 2 (On the feasibility of RO_2) Define the feasible set at stage 2 for given x_1 and ξ_1 as follows:

$$\mathbb{X}_2(x_1, \xi_1) := \{x_2 \geq 0 : T_1(\xi_1)x_1 + W_2(\xi_1)x_2 = h_2(\xi_1)\} \, . \tag{15}$$

Notice that the worst-case recourse function can be expressed in terms of (15):

$$\mathscr{R}(x_1) = \sup_{\xi_1 \in \Xi_1} \min_{x_2 \in \mathbb{X}_2(x_1, \xi_1)} c_2^\top(\xi_1) x_2(\xi_1) \, .$$

Define $\mathbb{X}_2(x_1) := \bigcap_{\xi_1 \in \Xi_1} \mathbb{X}_2(x_1, \xi_1)$. For all x_1 such that $\mathbb{X}_2(x_1) = \emptyset$, we set $\mathscr{R}(x_1) = \infty$. Then, we set $\mathrm{Feas} = \{x_1 \geq 0 : Ax_1 = h_1, \mathbb{X}_2(x_1) \neq \emptyset\}$. Notice that the problem has

[3] We adopt the convention of putting as lower indices the number of stages of the problem, e.g. RO_2 denotes a two-stage robust linear problem ($H = 1$).

relatively complete recourse iff Feas $= \mathbb{X}_1$. If Feas $= \emptyset$, we set the optimal objective value $v(\mathrm{RO}_2)$ to ∞.

It may happen that all second-stage problems are unbounded, i.e. $\mathscr{R}(x_1) = -\infty$ for some $x_1 \in$ Feas $\neq \emptyset$. In this case we assign the value $v(\mathrm{RO}_2) = -\infty$. However an infeasible first-stage (i.e. Feas $= \emptyset$ with $v(\mathrm{RO}_2) = \infty$) is not compensated by an unbounded second stage and gives the value $v(\mathrm{RO}_2) = \infty$. This resolves the problem about $\infty - \infty$. An infeasible second stage for some ξ_1 makes the problem infeasible, even if the first-stage would be unbounded.

If RO_2 is feasible and bounded, its optimal value is neither ∞ nor $-\infty$. In this case, the optimum $v(\mathrm{RO}_2)$ may be attained or not in general (but by our assumptions the optimum is always attained). Suppose that the optimal value $v(\mathrm{RO}_2)$ with $-\infty < v(\mathrm{RO}_2) < \infty$ is attained. Then a solution set consists of all pairs $(x_1, \xi_1 \mapsto x_2(\xi_1))$ such that $x_1 \in$ Feas and $x_2(\xi_1) \in \mathbb{X}_2(x_1, \xi_1)$ for all $\xi_1 \in \Xi_1$ such that

$$c_1^\top x_1 + \sup_{\xi_1 \in \Xi_1} c_2^\top(\xi_1) x_2(\xi_1) = v(\mathrm{RO}_2)\ .$$

Notice that we do not require that $x_2(\xi_1)$ are in the argmins of $\min_{x_2(\xi_1)}\{c_2^\top(\xi_1)x_2(\xi_1) : x_2(\xi_1) \in \mathbb{X}_2(x_1, \xi_1)\}$ for all $\xi_1 \in \Xi_1$.

It is immediate to observe that problem RO_2 rewrites as follows:

$$\mathrm{RO}_2 \quad : \quad \min_{x_1 \in \mathbb{X}_1} \left\{ c_1^\top x_1 + \gamma : \sup_{\xi_1 \in \Xi_1} \mathcal{Q}(x_1, \xi_1) - \gamma \leq 0 \right\}. \tag{16}$$

A key observation of this section is the fact that the above problem is exactly in the form of the CCG Theorem. Indeed, we remark that the function $\mathcal{Q}(x_1, \xi_1)$ as defined in (14) is a convex function in x_1. This follows from the structure of (14) and the fact that if $(x, y) \mapsto f(x, y)$ is jointly convex, then $x \mapsto \min_y f(x, y)$ is also convex.

The above observation justifies the adoption of a sampling approach, based on the random extraction of N_1 independent identically distributed (iid) scenarios:

$$\hat{\Xi}_1^{N_1} := \left\{ \xi_1^{(1)}, \ldots, \xi_1^{(N_1)} \right\},$$

of the random variable ξ_1, similarly to what is proposed in Vayanos et al. (2012). Recall that $\mathbb{P}$ has a nonvanishing density on the compact set Ξ. Let $T_1(\xi_1^{(i)})$, $h_2(\xi_1^{(i)})$, $c_2(\xi_1^{(i)})$ be the realization of $T_1(\xi_1)$, $h_2(\xi_1)$ and $c_2(\xi_1)$ under scenario $\xi_1^{(i)}$, $i = 1, \ldots, N_1$, and let $x_2^{(i)}$ be the second-stage (adjustable) design variables created for the scenarios $\xi_1^{(i)}$, $i = 1, \ldots, N_1$.

These scenarios are used to construct the following *sample-based approximation* based on N_1 instances of the uncertain constraints:

$$\widehat{\mathrm{RO}}_2^{N_1} \quad : \quad \min_{x_1 \in \mathbb{X}_1, \gamma} \left\{ c_1^\top x_1 + \gamma : \max_{i=1,\ldots,N_1} \mathcal{Q}\left(x_1, \xi_1^{(i)}\right) - \gamma \leq 0 \right\}.$$

We note that the above problem explicitly rewrites as follows:

$$\begin{aligned}
\widehat{\mathrm{RO}}_2^{N_1} : \min_{x_1 \in \mathbb{X}_1, \gamma, x_2^{(1)}, \ldots, x_2^{(N_1)}} \quad & c_1^\top x_1 + \gamma \\
\text{s.t. } & c_2^\top(\xi_1^{(i)}) x_2^{(i)} \leq \gamma, \quad i = 1, \ldots, N_1 \\
& T_1(\xi_1^{(i)}) x_1 + W_2(\xi_1^{(i)}) x_2^{(i)} = h_2(\xi_1^{(i)}), \quad i = 1, \ldots, N_1 \\
& x_2^{(i)} \geq 0, \ i = 1, \ldots, N_1\ .
\end{aligned} \tag{17}$$

We define now the violation probability V_1 for the two-stage case as:

$$V_1(\hat{\Xi}_1^{N_1}) := \mathbb{P}\left\{\xi^{(N_1+1)} : v(\widehat{\mathrm{RO}}_2^{N_1+1}) > v(\widehat{\mathrm{RO}}_2^{N_1})\right\} . \tag{18}$$

The interpretation of the violation probability is as follows: if we consider the sampled problem $\widehat{\mathrm{RO}}_2^{N_1}$, then $V_1(\hat{\Xi}_1^{N_1})$ is the probability that we encounter an (yet unseen) uncertainty realization $\xi_1^{(N_1+1)}$ leading to a cost $v(\widehat{\mathrm{RO}}_2^{N_1+1})$ larger than $v(\widehat{\mathrm{RO}}_2^{N_1})$. Notice that, in the light of Remark 2, a larger cost could also mean that the problem becomes infeasible at stage two (we have in this case that the cost is infinite). Hence, the smaller is V_1, the higher is the probability that the solution at stage one will lead to a feasible stage two problem, and that no cost increase is observed.

We are hence in the position of providing a rigorous result connecting the violation probability to the number of scenarios N_1 adopted in the construction of the $\widehat{\mathrm{RO}}_2^{N_1}$ problem.

An easy consequence of the basic Proposition 1 is the following result (see Formentin et al. 2016).

Theorem 2 (two-stage robust linear case) *Given an accuracy level $\epsilon \in (0, 1)$, the violation probability of the sample-based problem $\widehat{RO}_2^{N_1}$, based on the random extraction of N_1 iid scenarios of ξ_1, is bounded as:*

$$\mathbb{P}\left\{V_1(\hat{\Xi}_1^{N_1}) > \epsilon\right\} \leq B(N_1, \epsilon, n_1 + 1) , \tag{19}$$

where $B(N_1, \epsilon, n_1 + 1)$ is as in (5) with $n_1 = dim(x_1)$ and $1 = dim(\gamma)$.

Note that Eq. (7) can be used to obtain a priori the number of scenarios N_1 (i.e. the sample complexity) necessary to guarantee the desired level of confidence β that the violation probability $V_1(\hat{\Xi}_1^{N_1})$ is less than a pre-determined desired level ϵ. It is important to highlight that the number of scenarios N_1 in formula (7) depends only on the dimension of first-stage variables (non-adjustable variables); thus it reduces the number of scenarios needed to satisfy a prescribed level of violation with respect to that proposed in Vayanos et al. (2012).

2.3 Connections with scenario with certificates approach

It is interesting to observe that problem RO_2 can be restated as the following *robust with certificates* RwC_2 problem:

$$\begin{aligned}
\mathrm{RwC}_2 \quad : \quad & \min_{x_1 \in \mathbb{X}_1, \gamma} c_1^\top x_1 + \gamma \\
& \text{s.t. } \forall \xi_1 \in \Xi_1, \ \exists \, x_2(\xi_1) \text{ satisfying} \\
& \quad c_2^\top(\xi_1)\, x_2(\xi_1) \leq \gamma \\
& \quad x_2(\xi_1) \geq 0, \ T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\xi_1) = h_2(\xi_1) ,
\end{aligned}$$

where we distinguish between *design variables* (x_1, γ) and certificates $x_2(\xi_1)$. This problem does not contain a nested optimization as RO_2. It is just a standard optimization problem with possibly infinitely many variables and infinitely many constraints. RwC_2 is feasible, if its constraint set is non-empty (otherwise we set its optimal value to ∞). It is bounded, if it is feasible and its optimal value is not $-\infty$. We set the optimal value $v(\mathrm{RwC}_2)$ to ∞ if RwC_2 is not feasible and to $-\infty$ if it is unbounded.

It should be noted that the two formulations are equivalent, as formally proved in the next Theorem. We remark that a similar result can be found in Takeda et al. (2008). We provide the proof in Appendix A for completeness.

Theorem 3 *RO_2 and RwC_2 are equivalent formulations, i.e. RO_2 is feasible and bounded if and only if RwC_2 is feasible and bounded. In the case of feasibility and boundedness, the optimal value is either attained by both or by none. If the optimal value is attained, the optimal solution values coincide.*

We note that in problem $\widehat{\mathrm{RO}}_2^{N_1}$, a *certificate* $x_2^{(i)}$ is constructed for every scenario $\xi_1^{(i)}$. The rationale behind this approach is the following: We are not interested in the explicit knowledge of the function $x_2(\xi_1)$, we are content with the fact that for every possible value of the uncertainty *there exists* a possible choice of x_2 compatible with the ensuing realization of the constraints. Note that this represents a key difference with respect to other sampling based approaches. In particular, in Vayanos et al. (2012) different explicit parametrizations of the decision function $x_2(\xi_1)$ forming an M-dimensional subspace are introduced. It is easy to infer how this latter approach is bound to being more conservative, since the an extra constraint on the solution space is introduced.

It is clear that the approximate solution returned by problem $\widehat{\mathrm{RO}}_2^{N_1}$ is optimistic, since it considers only a subset of possible scenarios. That is, the following bound holds for all N_1:

$$v(\widehat{\mathrm{RO}}_2^{N_1}) \leq v(\mathrm{RO}_2) \,. \tag{20}$$

Hence, we have derived a lower bound, which by construction is better than bounds derived using wait-and-see approaches, as discussed in Sect. 3. Moreover, it is easy to show that the formulation is consistent, that is:

$$\lim_{N_1 \to \infty} v(\widehat{\mathrm{RO}}_2^{N_1}) = v(\mathrm{RO}_2) \qquad a.s.$$

This will be discussed in a more general setting in Sect. 2.5.

2.4 Multi-stage robust case

We are now ready to introduce the multi-stage generalization of RO_2, see (11) with convex objective functions and linear constraints. We denote by $\underline{\xi}_t := (\xi_1, \ldots, \xi_t)$ the history of the uncertainty up to time t. We consider the following robust convex program over $H+1$ stages:

$$\begin{aligned}
\mathrm{RO}_{H+1} \quad : \quad & \min_{x_1} c_1(x_1) + \\
& + \sup_{\xi_1 \in \Xi_1} \left[\min_{x_2(\xi_1)} c_2\left(x_2, \xi_1\right) + \sup_{\xi_2 \in \Xi_2} \left[\cdots + \sup_{\xi_H \in \Xi_H} \left[\min_{x_{H+1}\left(\underline{\xi}_H\right)} c_{H+1}^{\top}\left(x_{H+1}, \xi_H\right) \right] \right] \right] \\
& \text{s.t. } Ax_1 = h_1, \; x_1 \geq 0 \\
& \quad T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\xi_1) = h_2(\xi_1), \; \forall \xi_1 \in \Xi_1 \\
& \quad \quad \vdots \\
& \quad T_H(\xi_H)x_H(\underline{\xi}_{H-1}) + W_{H+1}(\xi_H)x_{H+1}(\underline{\xi}_H) = h_{H+1}(\xi_H), \; \forall \xi_H \in \Xi_H \\
& \quad x_t(\underline{\xi}_{t-1}) \geq 0 \quad \forall \xi_{t-1} \in \Xi_{t-1}; \quad t = 2, \ldots, H+1 \,,
\end{aligned} \tag{21}$$

where $h_1 \in \mathbb{R}^{m_1}$ is a known vector and $A \in \mathbb{R}^{m_1 \times n_1}$ is a known matrix. The uncertain parameter vectors and matrices depending on the parameters $\xi_t \in \Xi_t$ are then given by $h_t \in \mathbb{R}^{m_t}$, $T_{t-1} \in \mathbb{R}^{m_t \times n_{t-1}}$, and $W_t \in \mathbb{R}^{m_t \times n_t}$, $t = 2, \ldots, H+1$. Ξ_t are compact sets in $\mathbb{R}^{k_t}$ and $c_t : \mathbb{R}^{n_t} \times \mathbb{R}^{k_{t-1}} \to \mathbb{R}$, $i = 1, \ldots, H+1$, are convex in x_t and continuous in (x_t, ξ_{t-1}).

There is an important difference between the two-stage case and the three- (or more-) stage case, due to the dynamic character of the multi-stage model: Since the optimization problems in (21) over stages are nested, they cannot be written as one big optimization problem unless new additional constraints are formulated, that is decisions at stage t are not allowed to depend on ξ-values from later stages. This property of *non-anticipativity* requires to reconsider the notion of random sampling and constraint violation. Indeed, the correct data structure for a multi-stage (non-anticipative) robust optimization problem is a tree of ξ-values and not just a collection of ξ-vectors. This tree has height H. A relevant random draw from the uncertainty set $\Xi = \times_{t=1}^{H} \Xi_t$ is the collection of independently sampled values:

$$\begin{aligned}
\hat{\Xi}_1^{N_1} &= \{\xi_1^{(1)}, \ldots, \xi_1^{(N_1)}\} , \\
\hat{\Xi}_2^{N_2} &= \{\xi_2^{(1)}, \ldots, \xi_2^{(N_2)}\} , \\
&\vdots \\
\hat{\Xi}_H^{N_H} &= \{\xi_H^{(1)}, \ldots, \xi_H^{(N_H)}\} ,
\end{aligned}$$

which can be organized as a tree $\hat{\mathcal{T}}^{N_1,\ldots,N_H}$, where $\{\xi_1^{(1)}, \ldots, \xi_1^{(N_1)}\}$ are the successors of the root, and recursively all nodes at stage $t+1$ get all values from $\Xi_t^{N_t}$ as successors. Notice that this is a random tree (as it depends on the sample) and that the number of nodes at stage $t+1$ of the tree is $\bar{N}_t := \prod_{s=1}^{t} N_s$. The total number of nodes of the tree is hence:

$$N_{\text{tot}} := 1 + \sum_{i=1}^{H} \bar{N}_i = 1 + N_1 + N_1 N_2 + \cdots + N_1 N_2 \cdots N_H \ .$$

For each node of the finite tree $\hat{\mathcal{T}}^{N_1,\ldots,N_H}$ one has to consider a decision variable x. Let $\xi_1^{(i_1)}, \xi_2^{(i_2)}, \ldots, \xi_H^{(i_H)}$ with $i_1 = 1, \ldots, N_1,\ i_2 = 1, \ldots, N_2, \ldots, i_H = 1, \ldots, N_H$ be a path of the tree and let $x_1^{(i_1)}, x_2^{(i_1 i_2)}, \ldots, x_H^{(i_1 \ldots i_H)}$ the corresponding decision variables[4]. The finite problem on the sampled tree can be written as:

$$\begin{aligned}
\widehat{\text{RO}}_{H+1}^{N_1,\ldots,N_H} \ :& \min_{x_1} c_1(x_1) + \\
& + \max_{i_1} \left[\min_{x_2^{(i_1)}} c_2\left(x_2^{(i_1)}, \xi_1^{(i_1)}\right) + \max_{i_2} \left[\cdots + \max_{i_H} \left[\min_{x_{H+1}^{(i_1 \ldots i_H)}} c_{H+1}\left(x_{H+1}^{(i_1 \ldots i_H)}, \xi_H^{(i_H)}\right) \right] \right] \right] \\
& \text{s.t. } A x_1 = h_1,\ x_1 \geq 0 \\
& \quad T_1(\xi_1^{(i_1)}) x_1 + W_2(\xi_1^{(i_1)}) x_2^{(i_1)} = h_2^{(i_1)},\ \forall i_1 = 1, \ldots, N_1 \\
& \quad \vdots \\
& \quad T_H(\xi_H^{(i_H)}) x_H^{(i_1 \ldots i_{H-1})} + W_{H+1}(\xi_H^{(i_H)}) x_{H+1}^{(i_1 \ldots i_H)} = h_{H+1}(\xi_H^{(i_H)}),\ \forall i_H = 1, \ldots, N_H \\
& \quad x_t^{(i_1 \ldots i_t)} \geq 0\ ,\ t = 1, \ldots, H+1; \quad \forall i_1, \ldots, i_H \ .
\end{aligned} \tag{22}$$

[4] Because of the general setup of the model, the decisions may be path dependent.

The purpose of this paper is to compute a bound for the violation probability of the optimal value obtained by the sampled version $\widehat{\mathrm{RO}}_{H+1}^{N_1 \cdots N_H}$ given by (22), and to show that the optimal values of the sampled version converge to those of the basic problem RO_{H+1} (see (21)), when the sampling rates tend to infinity. Notice that if the dimension of the sampled model $\widehat{\mathrm{RO}}_{H+1}^{N_1 \cdots N_H}$ is too large, in special cases, decomposition techniques can be applied.

2.5 Convergence analysis

In this section we show that by letting the sample sizes $N_1, N_2, \ldots, N_H$ tend to infinity, the optimal value of the sampled problem (22) converges almost surely to the optimal value of the basic problem (21). To this end, introduce the following assumptions:

(i) The sets Ξ_t are compact with nonempty interior.
(ii) The probability $\mathbb{P}$ defined on $\Xi = \Xi_1 \times \cdots \times \Xi_H$ has a nonvanishing density, such that the probability of all relative open sets in Ξ is positive.
(iii) The functions $\xi_t \mapsto c_{t+1}(\xi_t)$, $\xi_t \mapsto T_t(\xi_t)$, $\xi_t \mapsto W_{t+1}(\xi_t)$, $\xi_t \mapsto h_{t+1}(\xi_t)$ defined on Ξ are continuous and therefore uniformly continuous for $t = 1, \ldots, H$.
(iv) There is a constant K such that the optimal values of (21) are uniformly bounded, i.e. $\|x_t\| \leq K, t = 1, \ldots, H + 1$.
(v) The rank of the matrices $W_{t+1}(\xi_t)$ is m_{t+1} for all $\xi_t \in \Xi_t$.

Recall that we sample N_t independent replications from the distribution $\mathbb{P}_t$ in Ξ_t. It is important to emphasize that the choice of the sampling probability is at our discretion. Any distribution on $\Xi = \Xi_1, \times \cdots \times \Xi_H$ with nonvanishing Lebesgue density can be used. However, there is no reason for using dependent samples, this would only complicate the convergence proof. Notice also that the violation probabilites must be formulated using the same probability adopted for sampling. The best choice for $\mathbb{P}$ would be a sampling measure which concentrates on the values of ξ_t, which are maximizers of the involved functions. However these maximizers are unknown when the samples have to be selected.

Lemma 4 *Under Assumptions (i)–(v),*

$$\lim_{N_t \to \infty} \max_{\xi_t \in \Xi_t} \min_{\xi_t^{(i)} \in \widehat{\Xi}_t^{N_t}} \|\xi_t^{(i)} - \xi_t\| = 0,$$

almost surely for all t.

Corollary 5 *If g is a continuous functions on Ξ_t, then $\min_{\xi_t^{(i)} \in \widehat{\Xi}_t^{N_t}} g(\xi_t^{(i)})$ converges to $\min_{\xi_t \in \Xi_t} g(\xi_t)$ a.s. for $N_t \to \infty$. The same is true for the maximum.*

Proposition 6 *Under Assumptions (i)-(v), if $\min(N_1, \ldots, N_H) \to \infty$, then the optimal value of the sampled problem (22) converges to the optimal value of the basic problem (21).*

We provide the proofs of Lemma 4 and Proposition 6 in Appendix C for completeness.

2.6 The violation probability at stage *t*

First note that there is a one-to-one correspondence between a multisample $\hat{\Xi}_1^{N_1} \times \cdots \times \hat{\Xi}_H^{N_H}$ and the sample scenario tree $\hat{\mathcal{T}}^{N_1,\ldots,N_H}$. For a fixed tree $\hat{\mathcal{T}}^{N_1,\ldots,N_H}$, the robust optimization problem (22) may be solved, leading to an optimal value of:

$$v(\hat{\mathcal{T}}^{N_1,\ldots,N_H}) := v(\widehat{\mathrm{RO}}_{H+1}^{N_1 \cdots N_H}) .$$

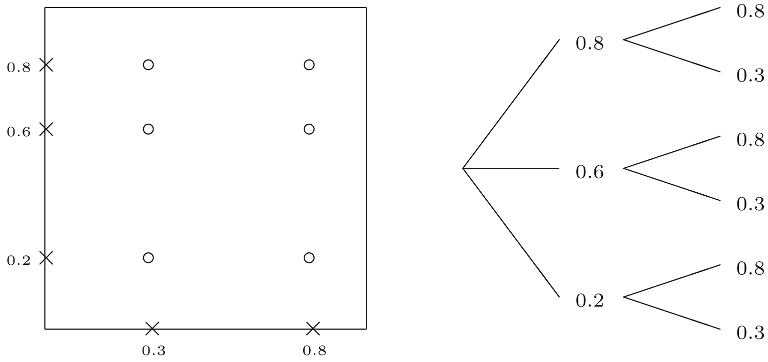

Fig. 1 The original sampled tree $\hat{\mathcal{T}}^{3,2}$

Notice that this value is by construction a lower bound to the optimal value $v(RO_{H+1})$ of the original infinite problem (21).

As a first step, we define the probability of violation at stage t. To this end, we add a new scenario $\xi_t^{(N_t+1)}$ from Ξ_t to the original data set, and form the associated tree $\hat{\mathcal{T}}^{N_1,\dots,N_t+1,\dots,N_H}$. Then, a violation at stage t occurs, if solving the finite problem on this new extended tree leads to a higher value than for the smaller tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$. Given the previously sampled tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$, the probability of stage t violation is therefore given by:

$$V_t(\hat{\mathcal{T}}^{N_1,\dots,N_H}) = V_t(\hat{\Xi}_1^{N_1},\dots,\hat{\Xi}_H^{N_H}) := \mathbb{P}\left\{\xi_t^{(N_t+1)} : v(\hat{\mathcal{T}}^{N_1,\dots,N_t+1,\dots,N_H}) > v(\hat{\mathcal{T}}^{N_1,\dots,N_H})\right\}. \tag{23}$$

Before discussing how to derive bounds on the probability distribution of V_t, we illustrate with a simple example the meaning of the concepts introduced so far.

Illustration. For a simple illustration, assume that $\Xi_1 = \Xi_2 = [0, 1]$. We sampled from Ξ_1 the three values $\xi_1^{(1)} = 0.2$, $\xi_1^{(2)} = 0.6$, $\xi_1^{(3)} = 0.8$ and from Ξ_2 the two values $\xi_2^{(1)} = 0.3$, $\xi_2^{(2)} = 0.8$. The corresponding sampled tree $\hat{\mathcal{T}}^{3,2}$ is shown in Fig. 1. As before, we denote the optimal value based on the (random) data of tree $\hat{\mathcal{T}}^{3,2}$ by $v(\hat{\mathcal{T}}^{3,2})$. In order to define the stage 1 violation probability, we sample a new point $\xi_1^{(4)} = 0.4 \in \Xi_1$ and form the new, extended tree $\hat{\mathcal{T}}^{4,2}$ (see Fig. 2). A violation V_1 occurs, if $v(\hat{\mathcal{T}}^{4,2}) > v(\hat{\mathcal{T}}^{3,2})$ and the stage 1 violation probability is the probability of the random draw ξ_1^4 which results in a violation. Similarly, we may define the stage 2 violation probability. Sample a new point $\xi_2^{(3)} = 0.5$ and form the tree given in Fig. 3. A stage 2 violation occurs, if the optimal value on tree $\hat{\mathcal{T}}^{3,3}$ is larger than the optimal value on tree $\hat{\mathcal{T}}^{3,2}$, i.e. whenever $v(\hat{\mathcal{T}}^{3,3}) > v(\hat{\mathcal{T}}^{3,2})$.

There is an important difference between the work of Vayanos et al. (2012) and our approach. We assume that the total uncertainty set is the cartesian product set $\Xi = \times_{t=1}^{H} \Xi_t$ and consider therefore trees, while Vayanos et al. (2012) consider always paths. To illustrate it, suppose that we sample N paths from the product set $\Xi = \Xi_1, \times \cdots \times \Xi_H$ denoted by $(\xi_1^{(i)}, \dots, \xi_H^{(i)})$ for $i = 1, \dots, N$. Since Ξ has product structure, any combination of points from Ξ_t, $t = 1, \dots, H$ is a valid point in Ξ. Thus by sampling N path, we get in fact N^H points

$$(\xi_1^{(i_1)}, \dots, \xi_t^{(i_t)}, \dots, \xi_H^{(i_H)}), \qquad i_t \in \{1, \dots, N\},$$

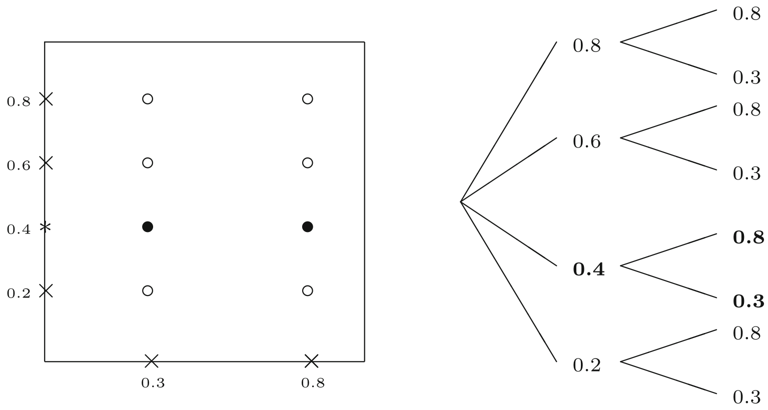

Fig. 2 The randomly extended tree $\hat{\mathcal{T}}^{4,2}$. The new nodes are in bold

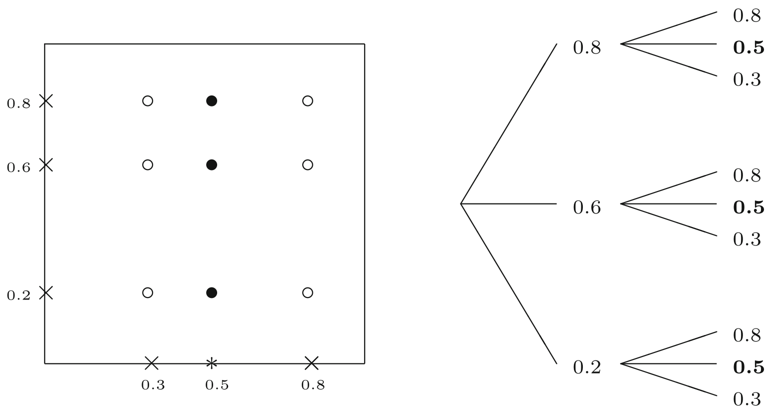

Fig. 3 The randomly extended tree $\hat{\mathcal{T}}^{3,3}$. The new nodes are in bold

i.e. all possible selections of N points from $\hat{\Xi}_t, t = 1, \ldots, H$. These selections can be organised in a tree with N^H leaves. Since our tree contains N^H paths and not just N, a violation is much more likely to occur and be detected earlier than in the path-oriented approach.

The calculation ignoring the combined samples leads to an underestimation of the true value (the true costs) of the underlying problem and also the violation probability. Compared to Vayanos et al. (2012), our notion of violation is different and stronger. The tree structure as we consider it in this paper guarantees the best (lower) bound of the true value, which is obtainable from all the samples obtained so far.

Moreover, we allow different sample sizes N_t for different stages, which may be important, since often the sizes of the uncertainty sets vary and increase by stages (see Remark 3).

Notice that the assumption that the uncertainty set Ξ is a cartesian product can easily be relaxed. It is sufficient that the uncertainty set is an arbitrary subset of $\times_{t=1}^{H} \Xi_t$. One may sample from $\Xi \subseteq \times_{t=1}^{H} \Xi_t$ by sampling from $\times_{t=1}^{H} \Xi_t$ and reject points which are not from Ξ. In terms of the scenario trees which we will construct, we just cut off some subtrees which represent scenarios which are not in Ξ.

In Appendix B, we further illustrate the difference between the path-oriented approach as in Vayanos et al. (2012) and the tree structured model here proposed.

In order to prove the main result of this section, we use a variable-split notation for the tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$. The tree has $\bar{N}_H = \prod_{s=1}^{H} N_s$ leaves, indexed by $\ell = 1, \dots, \bar{N}_H$. Notice that the leaves represent the random scenarios, i.e. there is a one-to-one correspondence between the scenarios and the tree leaves. Hence, to select a sample path $\underline{\xi}_H$, we may equivalently select a leaf index ℓ.

For every leaf ℓ the index of the predecessor at stage t is denoted by $p_t(\ell)$. Moreover, we introduce the relation $\ell_1 \sim_t \ell_2$, to denote the fact that the leaves ℓ_1, ℓ_2 share the same predecessors at stage t, i.e. $p_t(\ell_1) = p_t(\ell_2)$. We denote by $x_{t,\ell}$ the decision variable at stage t for the scenario ℓ. Note that, in principle, a different decision $x_{t,\ell}$ has to be made at each stage $t \in 1, \dots, H+1$ and for each scenario $\ell \in 1, \dots, \bar{N}_H$. However, the non-anticipativity condition requires that $x_{t,\ell_1} = x_{t,\ell_2}$, if $\ell_1 \sim_t \ell_2$, that is if the leaves ℓ_1, ℓ_2 share the same predecessor at stage t.

Formally, at each stage t we have an $\bar{N}_H$-vector $x_{t,\cdot}$, containing the different decisions at stage t corresponding to the different scenarios/sample-paths. However, as observed, the relation $\sim_t$ – and the related non-anticipativity constraints – dissects the set $\{1, 2, \dots, \bar{N}_H\}$ into equivalence classes. The constraint that decisions in the same equivalence class must share the same value can be expressed by the condition $(x_{t,\cdot}) \in I_t$ where I_t is a linear subspace. For instance, all vectors $(x_{1,\cdot}) \in I_1$ have all identical components, and for $t > 1$ some subgroups of components must share the same value.

These considerations allow us to reformulate our basic multi-stage robust problem. To this end, let, $\mathbb{X}_1 = \{x_1 \geq 0 : Ax_1 = h_1\}$ as in (13), and define:

$$\mathbb{X}_t(x_{t-1}, \xi_{t-1}) := \{x_t \geq 0 : T_{t-1}(\xi_{t-1})x_{t-1} + W_t(\xi_{t-1})x_t = h_t(\xi_{t-1})\}\,.$$

The basic problem $\widehat{\mathrm{RO}}_{H+1}^{N_1\cdots N_H}$ formulated on the sampled tree can be written in a compact form as:

$$\widehat{\mathrm{RO}}_{H+1}^{N_1\cdots N_H} : \min_{(x_{1,\cdot})\in I_1} \max_{\ell} \min_{(x_{2,\cdot})\in I_2} \max_{\ell} \min_{(x_{3,\cdot})\in I_3} \dots \max_{\ell} f_\ell(x_{1,\ell}, \dots, x_{H,\ell}, x_{H+1,\ell})\,, \tag{24}$$

where the functions f_ℓ are defined as follows, for $\ell = 1, \dots, \bar{N}_H$

$$f_\ell(x_{1,\ell}, \dots, x_{H+1,\ell}) := c_1(x_{1,\ell}) + \psi_{\mathbb{X}_1}(x_{1,\ell}) + \sum_{t=2}^{H+1} (c_t(x_{t,\ell}) + \psi_{\mathbb{X}_t(x_{t-1,\ell},\xi_{p_t(\ell)})}(x_{t,\ell}))\,,$$

where $\psi_{(\cdot)}$ is defined in (10). Notice that, in (24), the minima can be taken over all paths ℓ, since if $\ell_1 \sim_t \ell_2$, then the function values for ℓ_1 and ℓ_2 are identical.

Let us now introduce the first-stage objective function:

$$\bar{f}(x_1, \ell) = \min_{(x_{2,\cdot})\in I_2} \max_{\ell} \min_{(x_{3,\cdot})\in I_3} \dots \max_{\ell} f_\ell(x_{1,\ell}, \dots, x_{H+1,\ell})\,.$$

Two crucial observations can be made about the function $\bar{f}(x_1, \ell)$:

1. The function $\bar{f}(x_1, \ell)$ is constant on the equivalence classes given by I_2. In particular, one may write it as $\bar{f}(x_1, \xi_1)$.
2. The function is convex in the variable x_1 for given ℓ (and hence, for given ξ_1).
   In particular, the convexity of $\bar{f}$ in x_1 can be seen from the following two facts:

   (i) If $(x, y) \mapsto f(x, y)$ is jointly convex, then $x \mapsto \min_y f(x, y)$ is also convex.
   (ii) The maximum of convex functions is convex.

These properties hold for our case at hand (and it is the basic underlying reason in the scenario with certificates results in Formentin et al. (2016)).

Now, we keep the samples $\hat{\Xi}_2, \ldots, \hat{\Xi}_H$ fixed and look only at the dependency on ξ_1. In particular, to analyze the first-stage violation, we introduce a previously unobserved random value $\xi_1^{(N_1+1)}$ at stage one, keeping all the other stages fixed. It is immediate to observe that we are again in the standard setup of Proposition 1. Indeed, we find that the violation probability at stage one is the violation probability of the problem:

$$\min_{x_1} \max_{\xi_1} \bar{f}(x_1, \xi_1) \ .$$

Therefore, we get the estimate:

$$\mathbb{P}\left\{ V_1(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}) > \epsilon \right\} \leq B(N_1, \epsilon, n_1 + 1) \ , \tag{25}$$

where $n_1 = dim(x_1)$ and $1 = dim(\gamma_1)$ with

$$\gamma_1 := \max_{\ell} \min_{(x_{2,\cdot}) \in I_2} \max_{\ell} \min_{(x_{3,\cdot}) \in I_3} \ldots \max_{\ell} f_\ell(x_{1,\ell}, \ldots, x_{H,\ell}, x_{H+1,\ell}) \ .$$

Similarly, at stage t, there are $\bar{N}_{t-1} = \prod_{s=1}^{t-1} N_s$ nodes of the tree. Denoting with $\hat{\mathcal{T}}_j^{N_t, \ldots, N_H}$ the sub-tree born from node j, the violation probability at stage t and a fixed node j defined as

$$V_{t,j}(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}) := \mathbb{P}\left\{ \xi_t^{(N_t+1)} : v(\hat{\mathcal{T}}_j^{N_t+1, \ldots, N_H}) > v(\hat{\mathcal{T}}_j^{N_t, \ldots, N_H}) \right\} \ , \tag{26}$$

and it follows that

$$\mathbb{P}\left\{ V_{t,j}(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}) > \epsilon \right\} \leq B(N_t, \epsilon, n_t + 1),$$

where as before $n_t = dim(x_t)$ and $1 = dim(\gamma_{t-1})$ with

$$\gamma_{t-1} := \min_{(x_{t,\cdot}) \in I_t} \ldots \max_{\ell} f_\ell(x_{1,\ell}, \ldots, x_{H,\ell}, x_{H+1,\ell}) \ .$$

Notice that this bound does not depend on j. Now:

$$\begin{aligned} V_t(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}) &= \mathbb{P}\left\{ \text{ Violation at any node at stage } t | \hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H} \right\} \\ &\leq \sum_{j=1}^{\bar{N}_{t-1}} \mathbb{P}\left\{ \text{ Violation at node } j \text{ at stage } t | \hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H} \right\} \\ &= \sum_{j=1}^{\bar{N}_{t-1}} V_{t,j}(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}) \ , \end{aligned}$$

where the inequality follows by the fact that $V_{t,j}$, $j = 1, \ldots, \bar{N}_{t-1}$ are possibly dependent random variables. Now, we use the following result (whose proof is reported in Appendix C):

Lemma 7 *Let $Z_1, \ldots, Z_K$ be a sequence of identically distributed, but possibly dependent random variables. Then*

$$\mathbb{P}\left\{ \sum_{i=1}^{K} Z_i \geq z \right\} \leq K \mathbb{P}\{Z_i \geq z/K\} \ . \tag{27}$$

This gives:

$$\mathbb{P}\left\{\sum_{j=1}^{\bar{N}_{t-1}} V_{t,j}(\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H})>\epsilon\right\} \leq \bar{N}_{t-1} B(N_t, \epsilon/\bar{N}_{t-1}, n_t+1)\ ,$$

assuming that the probability distribution $\mathbb{P}$ has nonvanishing Lebesgue density.
The above line of reasoning proves the following theorem, which constitutes the main result of this paper.

Theorem 8 (Violation probability at stage t of sampled scenario tree) *Given an accuracy level $\epsilon \in (0, 1)$, let $\bar{N}_{t-1} = \prod_{s=1}^{t-1} N_s$ and $\epsilon_t := \epsilon/\bar{N}_{t-1}$. Then, the probability of violation at stage t, $V_t(\hat{\Xi}^{N_1},\ldots,\hat{\Xi}^{N_H})$ defined in (23), is bounded as:*

$$\mathbb{P}\left\{V_t(\hat{\Xi}^{N_1},\ldots,\hat{\Xi}^{N_H})>\epsilon\right\} \leq \bar{N}_{t-1} B(N_t, \epsilon_t, n_t+1)\ , \tag{28}$$

where $n_t = dim(x_t)$.

Remark 3 In the light of the above result, we can derive the required sample size to guarantee an ϵ-exceedance of the stagewise violations V_t, $t = 1,\ldots,H$ being smaller than β:

- N_1 has to be chosen larger than $N_1^* = \frac{1}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta} + n_1 + 1)$.
- Given N_1^*, the number N_2 has to be at least $N_2^* = \frac{N_1^{*2}}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta} + n_2 + 1)$.
- Given the values $N_1^*,\ldots,N_{t-1}^*$, form $\bar{N}_{t-1}^* = \prod_{s=1}^{t-1} N_s^*$ and choose N_t at least $N_t^* = \frac{\bar{N}_{t-1}^{*2}}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta} + n_t + 1)$.

Example Notice that these sample sizes are calculated under a worst-case setup. In practical cases one needs much fewer samples. Here is a table for the required sample size resulting from our above calculations, assuming that $n_t = 2$.

ϵ	β	N_1^*	N_2^*
0.2	0.1	42	74088
0.1	0.1	84	592704
0.2	0.05	48	110592
0.1	0.05	95	857375

Remark 4 Notice that the sample complexity result given in Theorem 8 can be easily extended with similar considerations to multi-stage convex robust programs of the following type:

$$\begin{aligned} \mathrm{CRO}_{H+1} \quad : \quad & \min_{x_1, x_2(\xi_1),\ldots,x_{H+1}(\underline{\xi}_H)} \ \sup_{\underline{\xi}_H\in\Xi} g(x_1, x_2(\xi_1),\ldots,x_{H+1}(\underline{\xi}_H), \underline{\xi}_H) \\ & \text{s.t. } h(x_1, x_2(\xi_1),\ldots,x_{H+1}(\underline{\xi}_H), \underline{\xi}_H) \leq 0,\ \forall \underline{\xi}_H \in \Xi \\ & x_1 \geq 0\ , \quad x_t(\underline{\xi}_{t-1}) \geq 0\ ,\ t = 2,\ldots,H+1\ , \end{aligned}$$

where $g : \mathbb{R}^{\sum_{t=1}^{H+1} n_t} \times \Xi \to \mathbb{R}$ and $h : \mathbb{R}^{\sum_{t=1}^{H+1} n_t} \times \Xi \to \mathbb{R}$ are convex in $x_t \in \mathbb{R}_+^{n_t}$, $t = 1,\ldots,H+1$ and continuous in $(x_t, \underline{\xi}_H)$.

2.7 The "total" violation probability

Theorem 8 provides a way to bound the probability of violation at stage t. This result can be used to bound the probability of "total" violation, which we define as the probability of violating *at any* stage t. Formally we write:

$$V_{\text{TOT}}(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}) := \mathbb{P}\left\{\xi_1^{(N_1+1)}, \ldots, \xi_H^{(N_H+1)} : \exists t \text{ s.t. } v(\hat{\mathcal{T}}^{N_1,\ldots,N_t+1,\ldots,N_H}) > v(\hat{\mathcal{T}}^{N_1,\ldots,N_H})\right\}.$$

We note that that the above quantity may be immediately bounded as follows:

$$V_{\text{TOT}}(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}) \le V_1(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}) + \cdots + V_H(\hat{\Xi}_1^{N_1}, \ldots, \hat{\Xi}_H^{N_H}).$$

3 Lower bounds for multi-stage linear robust optimization problems

Due to the large number of required samples, problem $\widehat{\text{RO}}_{H+1}^{N_1,\ldots,N_H}$ is typically difficult to solve. Consequently, it is advisable to solve simpler problems allowing to obtain at least guaranteed bounds for it. Notice that $\widehat{\text{RO}}_{H+1}^{N_1,\ldots,N_H}$ gives a lower bound for the original problem RO_{H+1} for any size of the random extractions. As to upper bounds, any feasible decision of the original problem gives an upper bound. Thus, by extending the solution of the sampled subproblem to a solution of the original problem, guaranteed upper bounds are obtained. An extension would assign the decision x_t to a history $(\xi_1, \ldots, \xi_t)$, by taking the same value as assigned to the nearest history $(\xi_1^{(i_1)}, \xi_2^{(i_1 i_2)}, \ldots, \xi_t^{(i_1,\ldots,i_t)})$ in the sample.

Several construction principles for lower bounds are known in the context of stochastic programming, see for instance Maggioni et al. (2014, 2016); Maggioni & Pflug (2019, 2016). Here we adapt them for the sampled scenario approach and compare them in terms of optimal objective function values for the case of robust multi-stage linear programs. We remark that a general principle for obtaining lower bounds is to relax some of the constraints. Relaxing non-anticipativity constraints leads typically to a computationally much simpler problem, especially for the sampled approximations (see later).

First, we introduce the *robust multi-stage wait-and-see* problem RWS_{H+1}, where the realizations of all the history of the random parameters $\underline{\xi}_H = (\xi_1, \ldots, \xi_H)$ are assumed to be known at the first-stage. This problem takes the following form:

$$\begin{aligned}
\text{RWS}_{H+1} \quad : \quad \sup_{\underline{\xi}_H} \; & \min_{x_1(\underline{\xi}_H),\ldots,x_{H+1}(\underline{\xi}_H)} c_1^\top x_1(\underline{\xi}_H) + \cdots + c_{H+1}^\top(\xi_H) x_{H+1}(\underline{\xi}_H) \\
& \text{s.t. } A x_1(\underline{\xi}_H) = h_1, \; x_1(\underline{\xi}_H) \ge 0 \\
& \quad T_1(\xi_1) x_1(\underline{\xi}_H) + W_2(\xi_1) x_2(\underline{\xi}_H) = h_2(\xi_1) \\
& \quad \vdots \\
& \quad T_H(\xi_H) x_H(\underline{\xi}_H) + W_{H+1}(\xi_H) x_{H+1}(\underline{\xi}_H) = h_{H+1}(\xi_H) \\
& \quad x_t(\underline{\xi}_H) \ge 0, \; t = 2, \ldots, H+1.
\end{aligned} \tag{29}$$

Notice that, in the above setup, the minimum and supremum have been exchanged. Hence, the decision process has become *anticipative*, since the decisions $x_1, x_2, \ldots, x_{H+1}$ depend on a given realization of $\underline{\xi}_H$.

We introduce 3 definition, which is an immediate extension of the concept of *Expected Value of Perfect Information* for stochastic programs:

Definition 3 The difference

$$\mathrm{RVPI}_{H+1} := v(\mathrm{RO}_{H+1}) - v(\mathrm{RWS}_{H+1}) \,, \tag{30}$$

denotes the *Robust Value of Perfect Information* and compares robust multi-stage wait-and-see RWS_{H+1} with robust multi-stage RO_{H+1}.

Note that the RVPI_{H+1} can be interpreted as a measure of the advantage of reaching perfect information in advance: A small RVPI_{H+1} indicates a small advantage for reaching the perfect information since all possible realizations of uncertainty have similar costs. In particular, the following inequality can be proven.

Proposition 9 (lower bound for RO_{H+1}) *Given the robust multi-stage linear optimization problem RO_{H+1} defined in (21), and the robust multi-stage wait-and-see problem RWS_{H+1} defined in (29), the following inequality holds true:*

$$v(RWS_{H+1}) \leq v(RO_{H+1}) \,. \tag{31}$$

The proof is given in Appendix D.

A second lower bound for problem RO_{H+1} can be obtained by relaxing the non-anticipativity constraints only at stages $2, \ldots, H$ and replacing the future from stage 2 with a single sample path (see Maggioni et al. (2014)). The ensuing program is the so-called *robust two-stage relaxation* RT_{H+1}. Formally, consider the discrete random process as follows:

$$\underline{\tilde{\xi}}_t := (\xi_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_t), \; t = 2, \ldots, H \,,$$

where $\tilde{\xi}_t$, is a deterministic realization of the random process ξ_t. We denote the robust two-stage relaxation problem RT_{H+1}, as follows:

$$\begin{aligned}
\mathrm{RT}_{H+1} \;:\; & \min_{x_1} c_1^\top x_1 + \\
& \sup_{\xi_1} \left[\min_{x_2,\ldots,x_{H+1}} c_2^\top(\xi_1) x_2(\underline{\tilde{\xi}}_H) + c_3^\top(\tilde{\xi}_2) x_3(\underline{\tilde{\xi}}_H) + \ldots + c_{H+1}^\top(\tilde{\xi}_H) x_{H+1}(\underline{\tilde{\xi}}_H) \right] \\
& \text{s.t. } A x_1 = h_1, \; x_1 \geq 0 \\
& \quad T_1(\xi_1) x_1 + W_2(\xi_1) x_2(\underline{\tilde{\xi}}_H) = h_2(\xi_1), \; \forall \xi_1 \in \Xi_1 \\
& \quad \vdots \\
& \quad T_H(\tilde{\xi}_H) x_H(\underline{\tilde{\xi}}_H) + W_{H+1}(\tilde{\xi}_H) x_{H+1}(\underline{\tilde{\xi}}_H) = h_{H+1}(\tilde{\xi}_H), \forall \xi_1 \in \Xi_1 \\
& \quad x_t(\underline{\tilde{\xi}}_H) \geq 0, \; t = 2, \ldots, H+1, \; \forall \xi_1 \in \Xi_1 \,.
\end{aligned} \tag{32}$$

There are no non-anticipativity conditions here (except for the first-stage decisions).

Finally we remark that one may introduce intermediate relaxation steps by just relaxing some of the later non-anticipativities (or moving the max-operators to left only for stages later than a given stage P). Relaxing the non-anticipativity constraints in stages $P, \ldots, H$ with $P = 3, \ldots, H-1$ and replacing the future from stage P with a single sample path, hence considering a discrete random process:

$$\underline{\tilde{\xi}}_{P,H} := (\xi_1, \ldots, \xi_{P-1}, \tilde{\xi}_P, \ldots, \tilde{\xi}_H) \,,$$

we can get a sequence of lower bounds by stepwise relaxation from the end to the beginning. Denoting by $v(\mathrm{RO}_{P,H+1})$ the value of this robust P-stage relaxation, and following reasons similar to those in the proof of Proposition 9, the following bounds can be proven. In particular, it is clear that $\mathrm{RO}_{1,H+1} = \mathrm{RWS}_{H+1}$ and $\mathrm{RO}_{2,H+1} = \mathrm{RT}_{H+1}$.

Proposition 10 (Chain of lower bounds for RO_{H+1}) *Given the robust multi-stage linear optimization problem RO_{H+1} (21), the robust multi-stage wait-and-see problem RWS_{H+1} (29), the robust two-stage relaxation problem RT_{H+1} (32) and the robust P-stage relaxation problem $RO_{P,H+1}$, $P = 3, \dots, H-1$, the following inequalities hold true:*

$$v(RWS_{H+1}) = v(RO_{1,H+1}) \le v(RT_{H+1}) = v(RO_{2,H+1}) \le \dots \le v(RO_{P,H+1}) \le \dots \le v(RO_{H+1}).$$

The above results have a clear theoretical meaning. However, it should be remarked that, in the general case, problems $\mathrm{RO}_{P,H+1}$ may be hard to solve in practice. In such case, it becomes of great interest to introduce and study the sampled versions of them. In particular, given $\widehat{\mathrm{RO}}_{H+1}^{N_1,\dots,N_H}$ and a collection of independently sampled values $\hat{\Xi}_1^{N_1}, \dots, \hat{\Xi}_H^{N_H}$, we can introduce the *sampled robust wait-and-see* problem $\widehat{\mathrm{RWS}}_{H+1}^{\bar{N}_H}$, based on the extraction of $\bar{N}_H = N_1 \cdot N_2 \cdot \ldots \cdot N_H$ iid samples $\underline{\xi}_H^{(1)}, \dots, \underline{\xi}_H^{(\bar{N}_H)}$ from $\hat{\Xi}_1^{N_1}, \hat{\Xi}_2^{N_2}, \dots, \hat{\Xi}_H^{N_H}$:

$$\begin{aligned}
\widehat{\mathrm{RWS}}_{H+1}^{\bar{N}_H} : \max_{i=1,\dots,\bar{N}_H} \quad \min_{x_1(\underline{\xi}_H^{(i)}),\dots,x_H(\underline{\xi}_H^{(i)})} \quad & c_1^\top x_1(\underline{\xi}_H^{(i)}) + \dots + c_{H+1}^\top(\xi_H^{(i)}) x_{H+1}(\underline{\xi}_H^{(i)}) \\
\text{s.t. } & A x_1(\underline{\xi}_H^{(i)}) = h_1, \; x_1(\underline{\xi}_H^{(i)}) \ge 0 \\
& T_1(\xi_1^{(i)}) x_1(\underline{\xi}_H^{(i)}) + W_2(\xi_1^{(i)}) x_2(\underline{\xi}_H^{(i)}) = h_2(\xi_1^{(i)}) \\
& \vdots \\
& T_H(\xi_H^{(i)}) x_H(\underline{\xi}_H^{(i)}) + W_{H+1}(\xi_H^{(i)}) x_{H+1}(\underline{\xi}_H^{(i)}) = h_{H+1}(\xi_H^{(i)}) \\
& x_t(\underline{\xi}_H^{(i)}) \ge 0\,, \; t = 2, \dots, H+1, \; i = 1, \dots, \bar{N}_H\,.
\end{aligned} \tag{33}$$

Similarly, one can extract N_1 iid scenarios $\xi_1^{(i)}, i = 1, \dots, N_1$ and keep the rest $\tilde{\xi}_2, \dots, \tilde{\xi}_H$ deterministic such that $\underline{\tilde{\xi}}_H^{(i)} := (\xi_1^{(i)}, \tilde{\xi}_2, \dots, \tilde{\xi}_H)$ and construct the *sampled robust two-stage relaxation* problem $\widehat{\mathrm{RT}}_{H+1}^{N_1}$ given by:

$$\begin{aligned}
\widehat{\mathrm{RT}}_{H+1}^{N_1} \quad : \quad & \min_{x_1,\gamma_1} c_1^\top x_1 + \gamma_1 \\
& \text{s.t. } A x_1 = h_1, \; x_1 \ge 0 \\
& \quad \mathcal{Q}_1(x_1, \underline{\tilde{\xi}}_H^{(i)}) \le \gamma_1, \quad i = 1, \dots, N_1\,,
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\mathcal{Q}_1(x_1, \underline{\tilde{\xi}}_H^{(i)}) := \min_{x_2,\dots,x_{H+1}} \; & c_2^\top(\xi_1^{(i)}) x_2(\xi_1^{(i)}) + c_3^\top(\tilde{\xi}_2) x_3(\underline{\tilde{\xi}}_2^{(i)}) + \dots + c_{H+1}^\top(\tilde{\xi}_H) x_{H+1}(\underline{\tilde{\xi}}_H^{(i)}) \\
\text{s.t. } & T_1(\xi_1^{(i)}) x_1 + W_2(\xi_1^{(i)}) x_2(\xi_1^{(i)}) = h_2(\xi_1^{(i)}) \\
& \vdots \\
& T_H(\tilde{\xi}_H) x_H(\underline{\tilde{\xi}}_H^{(i)}) + W_{H+1}(\tilde{\xi}_H) x_{H+1}(\underline{\tilde{\xi}}_H^{(i)}) = h_{H+1}(\tilde{\xi}_H) \\
& x_t(\underline{\tilde{\xi}}_{t-1}^{(i)}) \ge 0, \qquad t = 2, \dots, H+1\,.
\end{aligned} \tag{35}$$

The violation probability at stage one, $V_1(\hat{\Xi}_1^{N_1}, \tilde{\xi}_2, \dots, \tilde{\xi}_H)$, of the objective function value returned by $\widehat{\mathrm{RT}}_{H+1}^{N_1}$ depends only on $dim(x_1) + dim(\gamma_1) = n_1 + 1$, i.e.:

$$\mathbb{P}\left\{ V_1(\hat{\Xi}_1^{N_1}, \tilde{\xi}_2, \dots, \tilde{\xi}_H) > \epsilon \right\} \le B(N_1, \epsilon, n_1 + 1)\,.$$

The dimension of the variables at stages greater than 1 is irrelevant, which is a quite remarkable fact.

Similarly a *sampled robust P-stage relaxation* $\widehat{\mathrm{RO}}_{P,H+1}^{N_1\ldots N_{P-1}}$ of problem $\mathrm{RO}_{P,H+1}$ can be defined. Again, probabilistic guarantees of the solution of problem $\widehat{\mathrm{RO}}_{P,H+1}^{N_1\ldots N_{P-1}}$ can be obtained on the same lines of Theorem 8.

We conclude this section by providing the following proposition, which shows the relationship between the various lower bounds based on sampling presented in this paper.

Proposition 11 (Chain of sampling-based lower bounds for RO_{H+1}) *Given the robust multi-stage linear optimization problem $\widehat{RO}_{H+1}$ (21), the sampled robust optimization problem $\widehat{RO}_{H+1}^{N_1,\ldots,N_H}$ (22), the sampled robust multi-stage wait-and-see problem $\widehat{RWS}_{H+1}^{\bar{N}_H}$ (33), the sampled robust two-stage relaxation $\widehat{RT}_{H+1}^{N_1}$ (34) and the sampled robust P-stage relaxation $\widehat{RO}_{P,H+1}^{N_1\ldots N_{P-1}}$ for a fixed collection of independently sampled values $\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_N^{N_H}$, the following chain of inequalities holds true:*

$$v(\widehat{RWS}_{H+1}^{\bar{N}_H}) \le v(\widehat{RT}_{H+1}^{N_1}) \le \ldots \le v(\widehat{RO}_{P,H+1}^{N_1\ldots N_{P-1}}) \le \ldots \le v(\widehat{RO}_{H+1}^{N_1\ldots N_H}) \le v(RO_{H+1}) \,. \tag{36}$$

4 Numerical results: an inventory management problem

In this section, to show the effectiveness of the proposed approach, we consider a problem from inventory management which was originally considered in Ben-Tal et al. (2005), describing the negotiation of flexible contracts between a retailer and a supplier in the presence of uncertain orders from customers. In particular, we analyze the performance of the approach proposed in this paper on simplified version discussed in Bertsimas et al. (2011) and in Vayanos et al. (2012) (see Sect. 4.1) and on a modified version having non-linear objective function and linear constraints (see Sect. 4.2). The problems derived from the case studies have been formulated and solved under *AMPL* environment along with *CPLEX* 22.1.1.0 solver. Computations have been performed on a 64-bit machine with 32 GB of RAM and an Intel Core i7-1065G7 CPU 1.30 GHz processor and on AMD EPYC 7302 16-Core processor-based cluster with a base frequency of 3.00 GHz and 512 GB of RAM.

4.1 The piece-wise affine objective function case

The problem setting can be summarized as follows: A retailer received orders ξ_t at the beginning of each time period $t \in \{1,\ldots,H\}$, $\underline{\xi}_t$ represents the demand history up to time t. The demand needs to be satisfied from an inventory with filling level s_t^{inv} by means of orders x_t^o at a cost d_t per unit of product. Unsatisfied demand may be backlogged at cost p_t and inventory may be held in the warehouse with a per-unit holding cost h_t. Lower and upper bounds on the orders x_t^o ($\underline{x}_t^o$ and $\bar{x}_t^o$) at each period as well as on the cumulative orders s_t^{co} ($\underline{s}_t^{co}$ and $\bar{s}_t^{co}$) up to period t are imposed. We assume that there is no demand at time $t = 1$ and that the demand at time t lies within an interval centered around a nominal value $\bar{\xi}_t$ and uncertainty level $\rho \in [0,1]$ resulting in a box uncertainty set as follows: $\Xi = \times_{t=1}^{H}\left\{\xi_t \in \mathbb{R} : \left|\xi_t - \bar{\xi}_t\right| \le \rho\bar{\xi}_t\right\}$. Denoting with x_t^c the retailer's cost at stage t, the problem with *Cumulative Order Constraints* ($\mathcal{COC}$) can be modeled as a convex problem of the following form, having piece-wise affine convex objective function and linear constraints:

Table 1 Input data for the inventory management problem

Parameters of the Problem $\mathcal{COC}$	$t=1$	$t=2$
(p_t, d_t, h_t)	(11,1,10)	(11,1,10)
s_1^{inv}	0	
$(\underline{x}_t^o, \bar{x}_t^o)$	$(0, \infty)$	$(0, \infty)$
$\underline{s}_t^{co}$	47	134
$\bar{s}_t^{co}$	94	248
$\bar{\xi}_t = 100\left(1+\frac{1}{2}\sin\left(\frac{\pi(t-2)}{6}\right)\right)$	75	100

$$
\begin{aligned}
\mathrm{RO}_{H+1}(\mathcal{COC}):\ & \min_{x_t^o, x_t^c, s_t^{co}, s_t^{inv}} \left[x_1^c + \max_{\underline{\xi} \in \Xi} \sum_{t \in \mathbb{T}} x_{t+1}^c(\underline{\xi}_t) \right] && \text{(37a)}\\
& \text{s.t. } x_1^c \geq d_1 x_1^o + \max\left\{ h_1 s_1^{inv}, -p_1 s_1^{inv} \right\} && \text{(37b)}\\
& x_{t+1}^c(\underline{\xi}_t) \geq d_{t+1} x_{t+1}^o(\underline{\xi}_t) + \\
& + \max\left\{ h_{t+1} s_{t+1}^{inv}(\underline{\xi}_t), -p_{t+1} s_{t+1}^{inv}(\underline{\xi}_t) \right\},\ t = 1, \ldots, H-1 && \text{(37c)}\\
& x_{H+1}^c(\underline{\xi}_H) \geq \max\left\{ h_{H+1} s_{H+1}^{inv}(\underline{\xi}_H), -p_{H+1} s_{H+1}^{inv}(\underline{\xi}_H) \right\} && \text{(37d)}\\
& s_2^{inv}(\underline{\xi}_1) = s_1^{inv} + x_1^o - \xi_1 && \text{(37e)}\\
& s_{t+1}^{inv}(\underline{\xi}_t) = s_t^{inv}(\underline{\xi}_{t-1}) + x_t^o(\underline{\xi}_{t-1}) - \xi_t\ , \quad t = 2, \ldots, H && \text{(37f)}\\
& s_2^{co}(\underline{\xi}_1) = s_1^{co} + x_1^o && \text{(37g)}\\
& s_{t+1}^{co}(\underline{\xi}_t) = s_t^{co}(\underline{\xi}_{t-1}) + x_t^o(\underline{\xi}_{t-1})\ , \quad t = 2, \ldots, H && \text{(37h)}\\
& \underline{x}_1^o \leq x_1^o \leq \bar{x}_1^o, \quad \underline{s}_1^{co} \leq s_1^{co} \leq \bar{s}_1^{co} && \text{(37i)}\\
& \underline{x}_t^o \leq x_t^o(\underline{\xi}_{t-1}) \leq \bar{x}_t^o,\ \underline{s}_t^{co} \leq s_t^{co}(\underline{\xi}_{t-1}) \leq \bar{s}_t^{co},\ t = 2, \ldots, H+1\ . && \text{(37j)}
\end{aligned}
$$

The objective function (37a) corresponds to minimizing the worst-case cumulative cost. Constraints (37b)–(37d) define the stagewise costs $x_{t+1}^c(\underline{\xi}_t)$, $t = 1, \ldots, H$ while constraints (37e), (37f) and (37g), (37h) respectively define the dynamics of the inventory level and cumulative orders. Finally, constraints (37i), (37j) denote the lower and upper bounds on the instantaneous and cumulative orders. Notice that the decision process is non-anticipative.

We remark that the considered numerical problem is such that the optimal solution of the original multi-stage robust optimization problem can be assessed: This allows to evaluate the performance of the scenario tree based approach.

We consider specific instances of problem $\mathrm{RO}_{H+1}(\mathcal{COC})$ as summarized in Table 1 under the assumption of two-stage ($H = 1$) and three-stage ($H = 2$) and uncertainty level $\rho = 30\%$ meaning that for a given value v, the uncertainty set is $[v(1-\rho), v(1+\rho)]$. The data presents some slight modifications of the data presented in Vayanos et al. (2012).

We define optimality gaps of the scenario problem $\widehat{\mathrm{RO}}_{H+1}^{N_1 \ldots N_H}(\mathcal{COC})$ as:

$$
\mathit{optimality\ gap} := \frac{v(\widehat{\mathrm{RO}}_{H+1}^{N_1 \ldots N_H}(\mathcal{COC})) - v(\mathrm{RO}_{H+1}(\mathcal{COC}))}{v(\mathrm{RO}_{H+1}(\mathcal{COC}))}\ . \tag{38}
$$

Table 2 Vertices of Ξ for the inventory management problem in the two-stage ($H = 1$) and three-stage ($H = 2$) cases

Vertex	Ξ_1
1	52.5
2	97.5

Vertex	Ξ_1	Ξ_2
1	52.5	70
2	52.5	130
3	97.5	70
4	97.5	130

We note that the optimality gap in (38) can be computed, since problem $\mathrm{RO}_{H+1}(\mathcal{COC})$ can be solved exactly by using a scenario tree that consists of the vertices of the polytopic uncertainty set Ξ reported in Table 2 (see Bertsimas et al. 2011).

To assess the performance of our approach, we compute the *empirical violation probability* $\hat{V}_t(\hat{\mathcal{T}}^{N_1,\dots,N_H})$ at stage $t = 1, \dots, H$ of the solution of a given scenario tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$ associated with the scenario problem $\widehat{\mathrm{RO}}_{H+1}^{N_1\dots N_H}(\mathcal{COC})$, defined as:

$$\hat{V}_t(\hat{\mathcal{T}}^{N_1,\dots,N_H}) := \sum_{i=1}^{1000} \frac{\mathbb{1}\left(v(\hat{\mathcal{T}}_i^{N_1,\dots,N_t+1,\dots,N_H}) - v(\hat{\mathcal{T}}^{N_1,\dots,N_H})\right)}{1000}, \quad t = 1, \dots, H\ , \tag{39}$$

where $\hat{\mathcal{T}}_i^{N_1,\dots,N_t+1,\dots,N_H}$, $i = 1, \dots, 1000$ is a new scenario tree with one new independent scenario $\xi_t^{(N_t+1)}$ from Ξ_t with respect to the tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$ and

$$\mathbb{1}\,(\alpha) := \begin{cases} 1 \text{ if } \alpha > 0 \\ 0 \text{ otherwise}\ ; \end{cases}$$

notice that the extended tree $\hat{\mathcal{T}}_i^{N_1,\dots,N_t+1,\dots,N_H}$ contains $\prod_{j=t+1}^{H} N_j$ new independent scenarios belonging to the sub-tree generated by the new scenario $\xi_t^{N_t+1}$ at stage t. In the two-stage case the extended tree $\hat{\mathcal{T}}_i^{N_1+1}$ contains only one new independent scenario extracted from Ξ_1.

The numerical results are obtained as follows:

- we fix a confidence level of $\beta = 0.01$ for the two-stage case and $\beta = 0.1$ for the three-stage case;
- we select the target violation probability $\epsilon = 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1, 0.2, 0.3$;
- we compute the corresponding sample size $N_1^* = \frac{1}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta} + n_1 + 1)$ and $N_2^* = \frac{N_1^{*2}}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta} + n_2 + 1)$.
- we solve 100 instances of problem $\widehat{\mathrm{RO}}_{H+1}^{N_1^*\dots N_H^*}$ each based on a different scenario tree $\hat{\mathcal{T}}^{N_1^*,\dots,N_H^*}$;
- for each instance, we compute the optimality gap given in formula (38) and empirical violation probability given in formula (39);
- we compute statistics over 100 instances.

First, we evaluate the performance of the sample-based approximation $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ in the two-stage case ($H = 1$). Figure 4 displays the optimality gaps of problem $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ with

Table 3 Number of scenarios N_1^*, constraints and variables for decreasing values of ϵ (%) in the two-stage case ($H = 1$) for the inventory management problem

ϵ (%)	N_1^*	# of const.	# of var.
30	35	420	246
20	53	636	372
10	105	1260	736
5	209	2508	1464
1	1045	12540	7316
0.5	2090	25080	14631
0.1	10450	125400	73151
0.05	20899	250788	146294

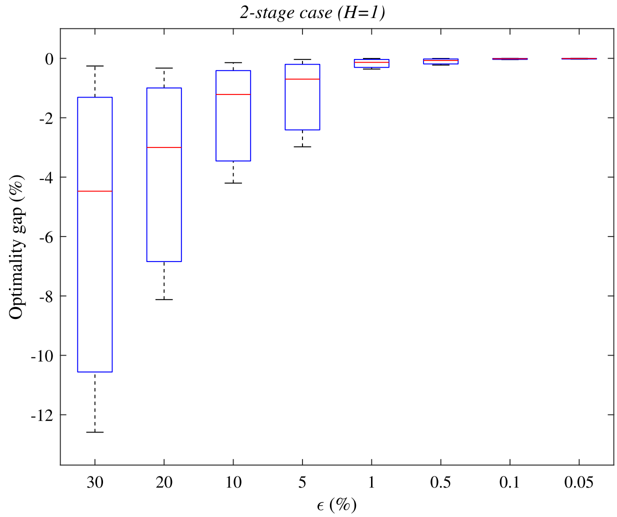

Fig. 4 Optimality gaps for $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ (boxes and whiskers) for decreasing values of ϵ for the two-stage ($H = 1$) case

respect to $\mathrm{RO}_2(\mathcal{COC})$ for different values of violation probability ϵ (%) ranging from 30% down to 0.05%. The number of scenarios N_1^*, constraints and variables of the corresponding optimization models are reported in Table 3.

From the results shown in Fig. 4 we can observe that the variance of $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ decreases substantially as ϵ decreases as well as the optimality gaps passing from -4.4% (in average) to $-10^{-5}\%$. The distribution of the empirical violation probability as function of ϵ is plotted in Fig. 5, for the two-stage case. As expected, as ϵ decreases, the violation converges to 0. We also note that the empirical violation probability is smaller than ϵ in all the considered cases.

Finally, Fig. 6 shows the average solver time (dashed lines) and the number of scenarios (solid lines) for problem $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ as a function of $Log(1/\epsilon)$. In particular, they are considerably lower than those used in Vayanos et al. (2012), where the number of scenarios depends on the size of the basis and on the number of decision variables at each stage. On the other hand, we should remark that the number of variables used in our approach is larger, due to the introduction of sample-dependent certificates (or second-stage decision variables).

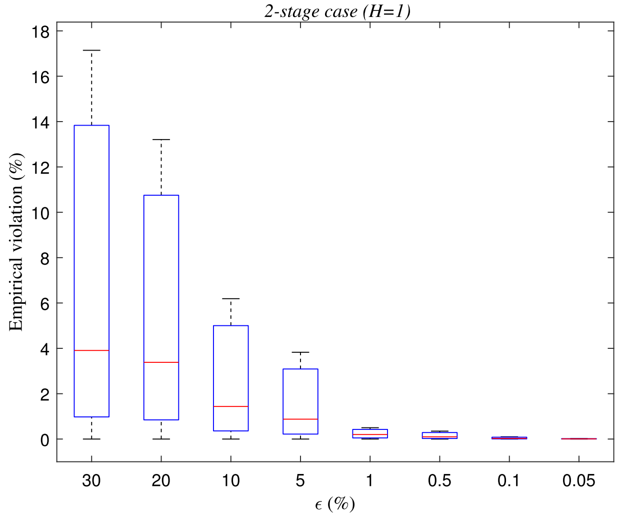

Fig. 5 Empirical violation probability for $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ (boxes and whiskers) for decreasing values of ϵ for the two-stage ($H = 1$) case

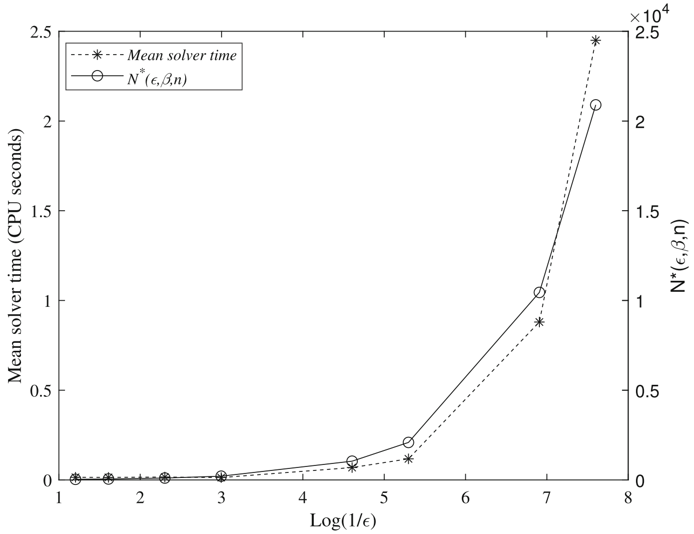

Fig. 6 Mean solver times (dashed line) and number of scenarios (solid line) as a function of $\mathrm{Log}(1/\epsilon)$ for problem $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ in the two-stage ($H = 1$) case

Table 4 Number of scenarios at first period N_1^*, at second period N_2^* and in total $\bar{N}_2$, constraints, variables and average CPU time (in seconds) for $\epsilon = 30, 20$ (%) in the three-stage case ($H = 2$) for the inventory management problem

ϵ (%)	N_1^*	N_2^*	$\bar{N}_2$	# of const.	# of var.	average CPU time
30	23	12003	276069	1380488	828304	1586
20	35	41691	1459185	7296140	4377700	42149.7

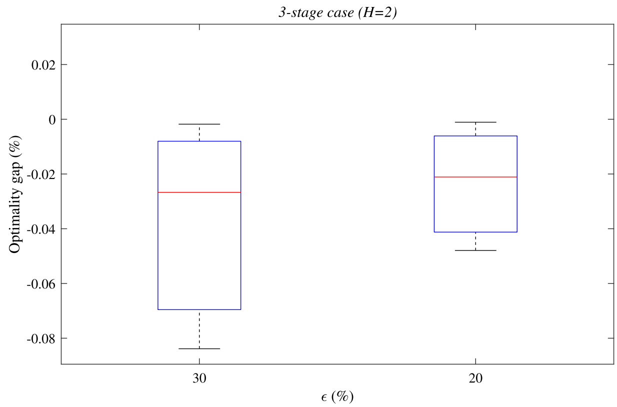

Fig. 7 Optimality gaps for $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ (boxes and whiskers) for decreasing values of ϵ for the three-stage ($H = 2$) case

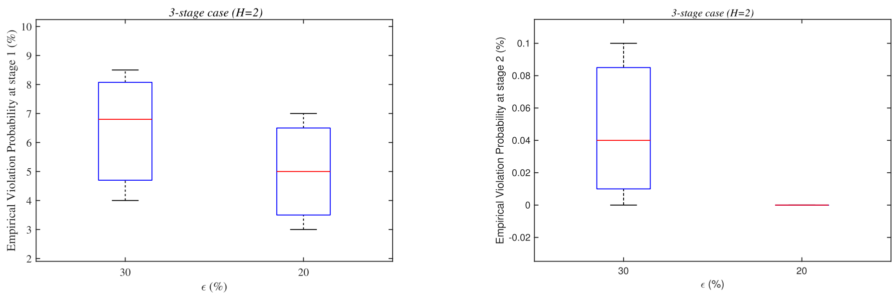

Fig. 8 Empirical violation probabilities $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ (on the left) and $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$ (on the right) for $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ for decreasing values of ϵ for the three-stage ($H = 2$) case

Secondly, we evaluate the performance of the sample-based approximation $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ in the three-stage case ($H = 2$). The number of scenarios N_1^*, N_2^* and $\bar{N}_2$, constraints and variables of the corresponding optimization models with average CPU time over 100 instances are reported in Table 4 for $\epsilon = 20\%$ and 30%. Notice that if the dimension of the sampled model is too large, decomposition techniques can be applied to speed up the computation. However this is out of the scope of this paper. Results shows that the average solver time to solve problem $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ pass from 1586 CPU seconds (with $\epsilon = 30\%$) with a scenario tree with $N_1^* = 23$ and $N_2^* = 12003$ to 42149.7 CPU seconds (with $\epsilon = 20\%$), for a tree with $N_1^* = 35$, $N_2^* = 41691$ and $\overline{N}_2 = N_1^* N_2^* = 1459185$ scenarios.

From the results shown in Fig. 7 we can observe that the optimality gaps of $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ decrease as ϵ decreases passing from -0.03% (in average) when $\epsilon = 30\%$ to -0.02% when $\epsilon = 20\%$.

The distribution of the empirical violation probabilities $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ and $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$ as function of ϵ are plotted in Fig. 8, for the three-stage case. We note that both the empirical violation probabilities are always smaller than ϵ. Results on $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ and $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$ show that as ϵ decreases from 30% to 20%, both the empirical violation probabilities decrease passing from an average value of 6.8% to 5% and of 0.04% to 0%, respectively. Results also show that $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ is always larger than $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$.

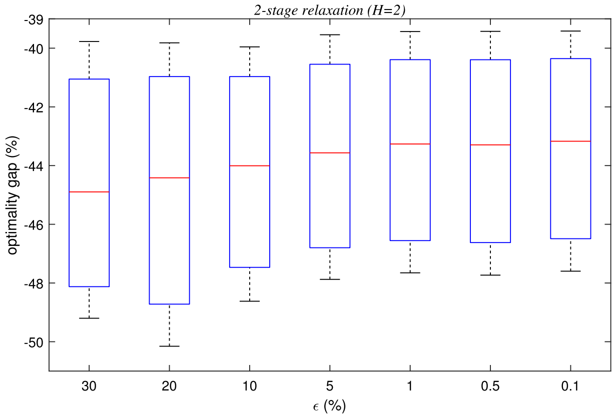

Fig. 9 Optimality gaps for $\widehat{\mathrm{RT}}_3^{N_1^*}(\mathcal{COC})$ (boxes and whiskers) for decreasing values of ϵ for the three-stage $(H = 2)$ case

4.1.1 Bounds for the inventory management with cumulative orders constraints

In this section, we evaluate possible relaxations to problem $\mathrm{RO}_{H+1}(\mathcal{COC})$ as described in Sect. 3. In particular we consider the multi-stage wait-and-see problem $\mathrm{RWS}_{H+1}(\mathcal{COC})$ for problem $\mathrm{RO}_{H+1}(\mathcal{COC})$, and the robust two-stage relaxation problem $\mathrm{RT}_H(\mathcal{COC})$ where the non-anticipativity constraints are relaxed in stages $2, \ldots, H$. Again, we remark that for the case at hand these two problems can be computed exactly by considering only the vertices of Ξ. Similarly to formula (38), we define optimality gaps of the problem $\mathrm{RWS}_{H+1}(\mathcal{COC})$ as:

$$(optimality\ gap)_{\mathrm{RWS}_{H+1}(\mathcal{COC})} := \frac{v(\mathrm{RWS}_{H+1}(\mathcal{COC})) - v(\mathrm{RO}_{H+1}(\mathcal{COC}))}{v(\mathrm{RO}_{H+1}(\mathcal{COC}))}, \quad (40)$$

and in the same way for $\mathrm{RT}_{H+1}(\mathcal{COC})$.

The optimality gap of $\mathrm{RWS}_3(\mathcal{COC})$ turned out to be equal to -68%, passing from an objective function value of 725.35 for $\mathrm{RO}_3(\mathcal{COC})$ to 227.5; consequently the Robust Value of Perfect Information $\mathrm{RVPI}_3((\mathcal{COC}))$ is 497.85.

The optimality gap of $\mathrm{TP}_3(\mathcal{COC})$ turned out to be equal to -39%, passing from an objective function value of 725.35 for $\mathrm{RO}_3(\mathcal{COC})$ to 439.64.

We now compute the optimality gaps by using the scenario approach. Figure 9 shows that the optimality gaps of $\widehat{\mathrm{RT}}_3^{N_1^*}(\mathcal{COC})$ with respect to $\mathrm{RO}_3(\mathcal{COC})$ slightly decrease as ϵ decreases passing from -44% (in average) to -43%. Notice that the best optimality gap which can be attained by the sampled two-stage relaxation $\widehat{\mathrm{RT}}_3^{N_1^*}(\mathcal{COC})$ is given by the two-stage relaxation itself $\mathrm{RT}_3(\mathcal{COC})$ i.e., -39%.

The distribution of the empirical violation probability $\hat{V}_1(\widehat{RT}_3^{N_1^*})$ as function of ϵ is plotted in Fig. 10, for the three-stage case. We note that the empirical violation probability $\hat{V}_1(\widehat{RT}_3^{N_1^*})$ is always smaller than ϵ and it decreases as ϵ decreases from 30% to 0.1%, passing from an average value of 6.6% to 0.06%. On the other hand, the empirical violation probability at stage 2, $\hat{V}_2(\widehat{RT}_3^{N_1^*})$, is equal to 1, independently on the value of ϵ showing the inappropriateness of the two-stage relaxation consisting in just one scenario per sub-tree at stage two.

Finally Fig. 11 presents the average solver time (solid line) and the number of scenarios (dashed line). We again note that the number of required scenarios is considerably smaller

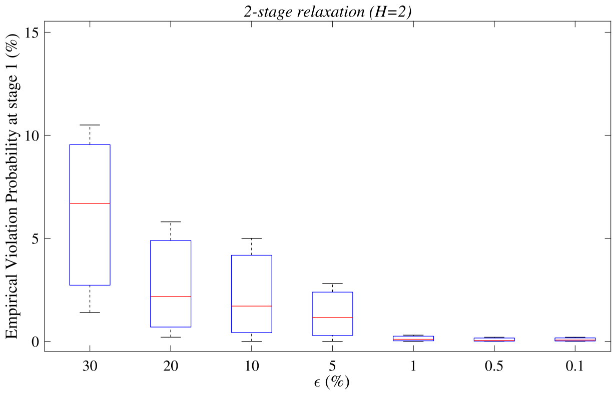

Fig. 10 Empirical violation probabilities $\hat{V}_1(\widehat{RT}_3^{N_1^*})$ for the two-stage relaxation $\widehat{RT}_3^{N_1^*}(\mathcal{COC})$ for decreasing values of ϵ for the three-stage ($H = 2$) case

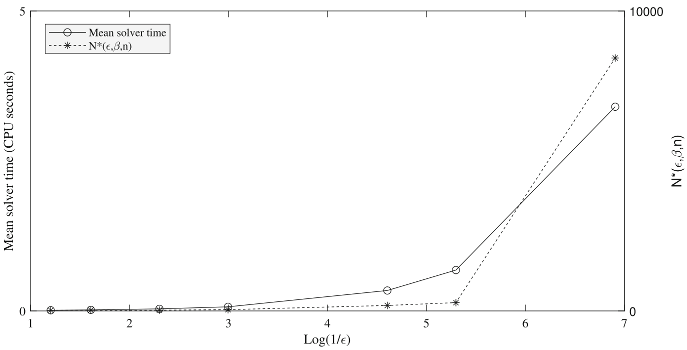

Fig. 11 Mean solver times (solid lines) and number of scenarios (dashed lines) as a function of $\mathrm{Log}(1/\epsilon)$ for problem $\widehat{\mathrm{RT}}_3^{N_1^*}(\mathcal{COC})$ for the three-stage ($H = 2$) case

than the one corresponding to the sampled robust problem $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}$ allowing us to solve the approximated problem $\widehat{\mathrm{RT}}_3^{N_1^*}$ in a reasonable amount of time (4 CPU seconds in the case of $\beta = 0.1$, $\epsilon = 0.1\%$ and $N_1^* = 6807$) at expenses of larger optimality gaps.

4.2 The non linear objective function case

In this section, we consider a modified version of the problem $RO_{H+1}(\mathcal{COC})$ presented in Section 4.1. Specifically Eq. (37b), (37c) and (37d) are replaced by the following ones, penalizing positive inventories in stages $t = 1, \ldots, H$ in a quadratic way:

$$x_1^c \geq d_1 x_1^o + \max\{h_1 (s_1^{inv})^2, -p_1 s_1^{inv}\} \tag{41a}$$

$$x_{t+1}^c(\underline{\xi}_t) \geq d_{t+1} x_{t+1}^o(\underline{\xi}_t) + + \max\left\{h_{t+1}(s_{t+1}^{inv}(\underline{\xi}_t))^2, -p_{t+1} s_{t+1}^{inv}(\underline{\xi}_t)\right\}, \; t = 1, \ldots, H-1 \tag{41b}$$

$$x_{H+1}^c(\underline{\xi}_H) \geq \max\left\{h_{H+1}(s_{H+1}^{inv}(\underline{\xi}_H))^2, -p_{H+1} s_{H+1}^{inv}(\underline{\xi}_H)\right\}. \tag{41c}$$

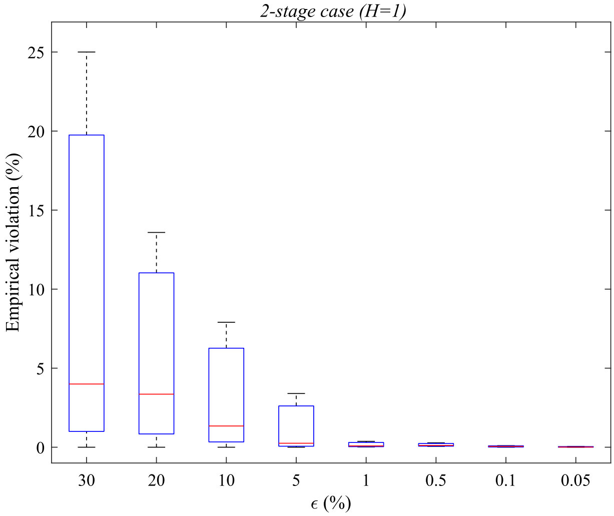

Fig. 12 Empirical violation probability for $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{NCOC})$ (boxes and whiskers) for decreasing values of ϵ for the two-stage ($H = 1$) case

This can be justified by the fact that more inventory needs new costly storage space. The resulting optimization problem $RO_{H+1}(\mathcal{NCOC})$ has a non linear convex objective function, linear constraints and non-anticipative decision process. Notice that the method presented in Georghiou et al. (2019) is not applicable in this case.

We consider the same instances reported in Table 1 under the assumption of two-stage ($H = 1$), three-stages ($H = 2$) and uncertainty level $\rho = 30\%$.

To assess the performance of our approach, we compute the empirical violation probability given in Eq. (39), following the same procedure described in Sect. 4.1. The number of scenarios N_1^* and N_2^* are reported in Tables 3 and 4.

The distribution of the empirical violation probabilities for the sample-based approximations $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{NCOC})$ and $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{NCOC})$ as function of ϵ are plotted in Figs. 12 and 13 for the two-stage and the three stage cases, respectively. As before, we note that the empirical violation probability is smaller than ϵ in all the considered cases even if slightly larger than in the linear case presented in Sect. 4.1, and as ϵ decreases, it converges to 0. Furthermore, in the three-stage case, the box-plots confirm that $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ is always larger than $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$. In conclusion, the results show that the approach works well also for a generic convex optimization problem.

5 Conclusions

In this paper probabilistic guarantees for constraint sampling in multi-stage convex robust optimization problems have been proposed. A sampled-based problem taking into account the non-anticipativity of the decision process has been considered. For this approach, which avoids the conservative use of parametrization through decision rules, a bound on the probability of violation of the randomized solution and a proof of convergence have been provided. Chains of lower bounds by relaxing the non-anticipativity constraints and sampling are also discussed. The considered approach is general, allowing convex objective functions, con-

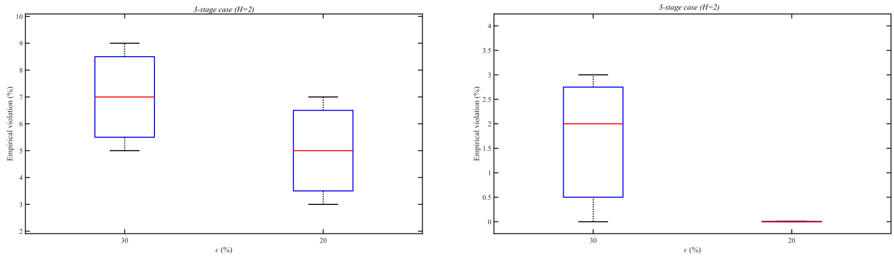

Fig. 13 Empirical violation probability $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ (on the left) and $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$ (on the right) for $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{NCOC})$ (boxes and whiskers) for decreasing values of ϵ for the three-stage ($H = 2$) case

vex constraints and the choice of arbitrary uncertainty sets. Because we use a worst-case approach, the numbers for needed sample sizes are prohibitively large for computational tractability for problems that have more than just two or three time periods. Despite this, we believe that our results can be useful for problems with such small number of time periods, and it sheds some light on the challenge for problems with more time periods. However, our numerical results show that the empirical violation probabilities are much smaller than their predetermined values used in the calculation of the sample sizes both in the case of a piece-wise affine objective function and convex non linear objective function. Moreover, for the first time in the literature we distinguish between violations at different stages. It was observed that violation in earlier stages are more probable than in later ones. It was also observed that for a three stage problem with 2-stage relaxation, the approximation provides good decisions at stage 1, but at stage 2 the violation probability is always 1, independently of the chosen ϵ. This shows that such a relaxation may be inappropriate for later stage decisions.

It is possible that the upper bounds derived in the paper could be improved significantly by exploiting special problems structures and decomposition techniques. This deserves a further investigation that will be addressed in future researches.

Appendix A Proof of Theorem 2.2

Proof Let $v(\mathrm{RO}_2) = -\infty$. Fix any $\gamma \in \mathbb{R}$. Then there is a $x_1 \in$ Feas such that $\mathscr{R}(x_1) = -\infty$, meaning that there are functions $\xi_1 \mapsto x_2(\xi_1)$ such that:

$$\sup_{\xi_1 \in \Xi_1} \{c_2^\top(\xi_1)x_2(\xi_1) : x_2(\xi_1) \in \mathbb{X}_2(x_1, \xi_1)\} \leq \gamma - \nu \,,$$

where $\nu = c_1^\top x_1 < \infty$. Notice that:

$$\text{Feas} = \{x_1 \geq 0 : Ax_1 = h_1; \forall \xi_1 \in \Xi_1 \text{ there exists a } x_2(\xi_1) \in \mathbb{X}_2(x_1, \xi_1)\} \,.$$

Thus γ together with $(x_1, \xi_1 \mapsto x_2(\xi_1))$ is feasible for RwC_2 and since γ is arbitrary, $v(\mathrm{RwC}_2) = -\infty$. The same argumentation shows that $v(\mathrm{RwC}_2) = -\infty$ implies that $v(\mathrm{RO}_2) = -\infty$.

Suppose now that $v(\mathrm{RO}_2) = \infty$. This means that either the first-stage problem or at least one second-stage problem is infeasible and this implies and is implied by the fact that RwC_2 is infeasible.

It remains to show what happens in the case $-\infty < v(\mathrm{RO}_2) < \infty$. In case the the optimal value is attained, let $(x_1, \xi_1 \mapsto x_2(\xi_1))$ be in the solution set of RO_2, then $(x_1, \gamma, \xi_1 \mapsto x_2(\xi_1))$

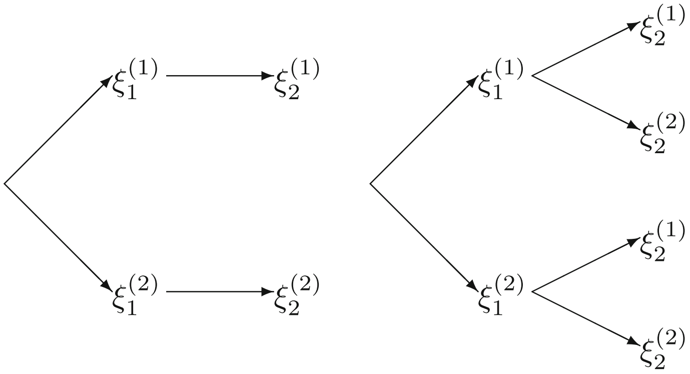

Fig. 14 Left: A fan with 2 paths. Right: The pertaining tree using the same data has $2^2 = 4$ paths

is feasible for RwC_2, iff $\gamma \geq v(\mathrm{RO}_2)$ and it is in the solution set of RwC_2, if $\gamma = v(\mathrm{RO}_2)$. Conversely, if a $(\gamma, x_1, \xi_1 \mapsto x_2(\xi_1))$ is feasible for RwC_2, then $(x_1, \xi_1 \mapsto x_2(\xi_1))$ is feasible for RO_2 and $v(\mathrm{RO}_2) \leq \gamma$. The optimal γ equals $v(\mathrm{RwC}_2)$. $\square$

Appendix B Difference between the path-oriented approach and the tree structured model

Example Consider a three-stage problem ($H = 2$) with two sampled points $\xi_1^{(1)}, \xi_1^{(2)} \in \Xi_1$ and two sampled points $\xi_2^{(1)}, \xi_2^{(2)} \in \Xi_2$. In Fig. 14, the path-oriented problem as in Vayanos et al. (2012) and our tree-structured problem are depicted for illustration.

W.l.o.g. we set $A = I$; $h_1 = 0$ and therefore $x_1 = 0$ and $T_1(\xi_1) = 0$. Set

$$\begin{aligned}
c_{ji} &:= c_j(\xi_{j-1}^{(i)}), \qquad i = 1, 2; \quad j = 2, 3\\
T_{ji} &:= T_j(\xi_j^{(i)}), \qquad i = 1, 2; \quad j = 2\\
W_{ji} &:= W_j(\xi_{j-1}^{(i)}), \qquad i = 1, 2; \quad j = 2, 3\\
h_{ji} &:= h_j(\xi_{j-1}^{(i)}), \qquad i = 1, 2; \quad j = 2, 3\\
x_{ji} &:= x_j(\xi_j^{(i)}), \qquad i = 1, 2; \quad j = 2, 3\,.
\end{aligned}$$

We consider a problem including equality and inequality constraints, which can be brought to the form (21) by introducing slack variables:

$$\begin{aligned}
\min \quad & (\max(c_{21}x_{21}, c_{22}x_{22}) + \max(c_{31}x_{31}, c_{32}x_{32}))\\
\text{s.t.} \quad & W_{21}x_{21} = h_{21}\\
& W_{22}x_{22} = h_{22}\\
& T_{21}x_{21} + W_{31}x_{31} \leq h_{31}\\
& T_{22}x_{22} + W_{32}x_{32} = h_{32}\\
& T_{21}x_{21} + W_{32}x_{32} = h_{32} \qquad\qquad \text{(B1)}\\
& T_{22}x_{22} + W_{31}x_{31} \leq h_{31}\\
& x_{ji} \geq 0, \qquad i = 1, 2;\ j = 2, 3\,. \qquad\qquad \text{(B2)}
\end{aligned}$$

If one argues pathwise then there are only two paths: $(\xi_1^{(1)}, \xi_2^{(1)})$ and $(\xi_1^{(2)}, \xi_2^{(2)})$ and the constraints (B1) and (B2) disappear.

For the concrete choice consider:

$$\begin{aligned}(W_{21}, W_{22}, W_{31}, W_{32}) &= (1, 1, 1, 1)\\ (T_{21}, T_{22}) &= (1, 1)\\ (h_{21}, h_{22}, h_{31}, h_{32}) &= (1, 2, 2, 2)\ .\end{aligned}$$

One can see that the path-oriented problem without constraints (B1) and (B2) is feasible, while the tree-structured problem including these two constraints is infeasible. Therefore the tree-structured problem may detect violation earlier than the path-oriented problem since we allow violation to occur at every stage by adding a new point from $\xi_t^{N_t+1} \in \Xi_t$ and not only sampling a complete new path $\left(\xi_1^{(N+1)}, \ldots, \xi_H^{(N+1)}\right)$ as is done in Vayanos et al. (2012).

Appendix C Proof of convergence

Proof of Lemma 2.1

Proof We prove that for every fixed $\xi_t \in \Xi_t$

$$\mathbb{P}\left\{ \lim_{N_t \to \infty} \min_{\xi_t^{(i)} \in \hat{\Xi}_t^{N_t}} \|\xi_t^{(i)} - \xi_t\| > 0 \right\} = 0.$$

Suppose that this assertion is wrong. Then there exist $\eta > 0$ and $\epsilon > 0$ such that

$$\mathbb{P}\left\{ \lim_{N_t \to \infty} \min_{\xi_t^{(i)} \in \hat{\Xi}_t^{N_t}} \|\xi_t^{(i)} - \xi_t\| \geq \eta \right\} > \epsilon.$$

Let B_η be the open ball with center ξ_t and radius η. Since $\mathbb{P}$ has a density bounded from below, $0 < \mathbb{P}(B_\eta) = \delta$ (say) with $\delta > 0$. Now, by independence sampling,

$$\mathbb{P}\{\xi_t^{(1)} \notin B_\eta, \ldots, \xi_t^{(N_t)} \notin B_\eta\} = (1 - \delta)^{N_t}.$$

Choosing N_t such large that $(1 - \delta)^{N_t} < \epsilon$ leads to a contradiction. The argument is true for every fixed rational $\xi_t \in \Xi_t$ and therefore for the union of the exeptional null sets. Since the rationals are dense in Ξ_t, this implies the assertion of the Lemma. □

Proof of Proposition 2.2

Proof For the sake of simplicity we give the proof for the three-stage problem, i.e. we assume that $H = 2$. The proof of the general case is analogous. The optimization problem (21) can be written as follows:

$$\begin{aligned}\mathrm{RO}_3 := \min_{x_1, \gamma_1}\ & c_1(x_1) + \gamma_1\\ \text{s.t. } & Ax_1 = h_1,\ x_1 \geq 0\\ & \mathcal{Q}_1(x_1, \xi_1) \leq \gamma_1, \quad \forall \xi_1 \in \Xi_1\ ,\end{aligned} \tag{C3}$$

where the function $\mathcal{Q}_1(x_1, \xi_1)$ can be written as

$$\begin{aligned}\mathcal{Q}_1(x_1, \xi_1) := \min_{x_2, \gamma_2}\ & c_2(x_2, \xi_1) + \gamma_2\\ \text{s.t. } & T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\xi_1) = h_2(\xi_1)\end{aligned}$$

$$\begin{aligned} & \mathcal{Q}_2(x_2, \xi_2) \leq \gamma_2, \quad \forall \xi_2 \in \Xi_2 \\ & x_2 \geq 0\,, \end{aligned} \tag{C4}$$

with

$$\begin{aligned} \mathcal{Q}_2(x_2, \xi_2) := & \min_{x_3} c_3(x_3, \xi_2) \\ & \text{s.t. } T_2(\xi_2)x_2 + W_3(\xi_2)x_3 = h_3(\xi_2) \\ & \quad x_3 \geq 0\,. \end{aligned} \tag{C5}$$

Now let $\hat{\Xi}_1^{N_1} = \{\xi_1^{(i_1)} : i_1 = 1, \ldots, N_1\}$ resp. $\hat{\Xi}_2^{N_2} = \{\xi_2^{(i_2)} : i_2 = 1, \ldots, N_2\}$, be independent random scenarios from of Ξ_1 resp. Ξ_2. We set:

$$\begin{aligned} \widehat{\text{RO}}_3 := & \min_{x_1, \gamma_1} c_1(x_1) + \gamma_1 \\ & \text{s.t. } Ax_1 = h_1,\ x_1 \geq 0 \\ & \quad \widehat{\mathcal{Q}}_1(x_1, \xi_1) \leq \gamma_1, \quad \forall \xi_1 \in \widehat{\Xi}_1^{N_1}\,, \end{aligned} \tag{C6}$$

where the function $\widehat{\mathcal{Q}}_1(x_1, \xi_1)$ can be written as

$$\begin{aligned} \widehat{\mathcal{Q}}_1(x_1, \xi_1) := & \min_{x_2, \gamma_2} c_2(x_2, \xi_1) + \gamma_2 \\ & \text{s.t. } T_1(\xi_1)x_1 + W_2(\xi_1)x_2 = h_2(\xi_1) \\ & \quad \mathcal{Q}_2(x_2, \xi_2) \leq \gamma_2, \quad \forall \xi_2 \in \widehat{\Xi}_2^{N_2} \\ & \quad x_2 \geq 0\,, \end{aligned} \tag{C7}$$

with

$$\begin{aligned} \mathcal{Q}_2(x_2, \xi_2) := & \min_{x_3} c_3(x_3, \xi_2) \\ & \text{s.t. } T_2(\xi_2)x_2 + W_3(\xi_2)x_3 = h_3(\xi_2) \\ & \quad x_3 \geq 0\,. \end{aligned} \tag{C8}$$

as before.

Notice that the functions $\mathcal{Q}_2$ are identical for the original problem and the sampled problem. We show that the functions $\widehat{\mathcal{Q}}_t(x_t, \xi_t)$, $t = 1, 2$ are continuous in x_t and ξ_t. We recall that the function $(c, \mathbb{X}) \mapsto \min_{x \in \mathbb{X}} c(x)$ is continuous in c and $\mathbb{X}$, where c comes from a family of continuous functions and $\mathbb{X}$ are compact sets, if we measure the distances for c by the supremum norm and for $\mathbb{X}$ by the Hausdorff-distance. Consider the compact polyhedrons $\mathbb{X}_3(x_2, \xi_2) = \{x_3 \geq 0 : W_3(\xi_2)x_3 = h_3(\xi_2) - T_2(\xi_2)x_2\}$. The fact that $W_3(\xi_2)$ has maximal rank for all ξ_2 implies that the extremals are continuous in $W_3(\xi_2)$ as well as in the r.h.s. $h_3(\xi_2) - T_2(\xi_2)x_2$. Therefore, the extremals are continuous in ξ and in x. This implies that $(x_2, \xi_2) \mapsto \mathbb{X}_3(x_2, \xi_2)$ is continuous in Hausdorff metric. Consequently $\mathcal{Q}_2(x_2, \xi_2)$ is continuous in both arguments and since both x_2 and ξ_2 lie in a compact set, it is uniformly continuous, that is:

$$\sup_{\|x_2\| \leq K} \Big| \sup_{\xi_2 \in \Xi_2} \mathcal{Q}_2(x_2, \xi_2) - \sup_{\xi_2 \in \widehat{\Xi}_2^{N_2}} \mathcal{Q}_2(x_2, \xi_2) \Big| \to 0 \tag{C9}$$

almost surely as $N_2 \to \infty$.

Now consider the function $\widehat{\mathcal{Q}}(x_1, \xi_1)$. As in the previous case, the extremals of $\mathbb{X}_2(x_1, \xi_1) = \{x_2 \geq 0 : W_2(\xi_1)x_2 = h_2(\xi_1) - T_1(\xi_1)x_1\}$ are continuous in x_1 and ξ_1.

Together with (C9) this implies that both $\mathcal{Q}_1$ and $\widehat{\mathcal{Q}}_1$ are continuous in x_1 and ξ_1 and therefore:

$$\sup_{\|x_1\|\le K} \,|\, \sup_{\xi_1\in\Xi_1} \mathcal{Q}_1(x_1,\xi_1) - \sup_{\xi_1\in\widehat{\Xi}_1^{N_1}} \widehat{\mathcal{Q}}_1^{N_2}(x_1,\xi_1)| \to 0 \tag{C10}$$

almost surely as $\min(N_1, N_2) \to \infty$.

The function $\mathcal{Q}_2(x_2,\xi_2)$ is uniformly continuous in ξ_2 and x_2. Therefore for $\epsilon > 0$ there is an $\eta > 0$ such that $\|\xi_2^1 - \xi_2^2\| \le \eta$ implies that $|\mathcal{Q}_2(x_2,\xi_2^1) - \mathcal{Q}_2(x_2,\xi_2^2)| \le \epsilon$. Thus by the previous Lemma:

$$\max_{\xi_2\in\Xi_2} \mathcal{Q}_2(x_2,\xi_2) - \max_{\xi_2\in\hat{\Xi}_2^{N_2}} \mathcal{Q}_2(x_2,\xi_2) \to 0\,.$$

The same argument applies also to the function $\mathcal{Q}_1(x_1,\xi_1)$. □

Appendix D Proof of Lemma 2.2

Proof

$$\mathbb{P}\left\{\sum_{i=1}^{K} Z_i \ge z\right\} \le \mathbb{P}\left(\bigcup_{i=1}^{K}\{Z_i \ge z/K\}\right) \le K\mathbb{P}\{Z_i \ge z/K\}.$$

This inequality is sharp: To see this, consider a discrete probability space having $K > 1$ atoms $\{\omega_1,\ldots,\omega_K\}$, each with same probability $P\{\omega_i\} = 1/K$. On ω_i define the random variables $Z_1,\ldots,Z_K$ as

$$Z_i = (z+K-1)/K; \qquad Z_j = (z-1)/K \text{ for } j \ne i.$$

Then the Z_i have all identical distributions and $\sum_i Z_i = z$. Consequently

$$P\left\{\sum_{i=1}^{K} Z_i \ge z\right\} = 1 = K\cdot(1/K) = K\cdot P\{Z_i \ge z/K\}.$$

□

Appendix E Proof of Proposition 3.1

Proof Since in RWS_{H+1} the non-anticipativity constraints are relaxed, we get the inequality (31). More formally, denoting by $f\left[\left(x_1(\underline{\xi}_H),\ldots,x_{H+1}(\underline{\xi}_H)\right),\underline{\xi}_H\right]$ in a compact way the objective function and constraints of problem (29), we can write:

$$\text{RWS}_{H+1} \quad : \quad \sup_{\underline{\xi}_H} \min_{(x_1(\underline{\xi}_H),\ldots,x_{H+1}(\underline{\xi}_H))} f\left[\left(x_1(\underline{\xi}_H),\ldots,x_{H+1}(\underline{\xi}_H)\right),\underline{\xi}_H\right].$$

For every realization, $\underline{\xi}_H$, we have the relation:

$$f\left[\left(\tilde{x}_1(\underline{\xi}_H),\ldots,\tilde{x}_{H+1}(\underline{\xi}_H)\right),\underline{\xi}_H\right] \le f\left[\left(x_1^*,\ldots,x_{H+1}^*(\underline{\xi}_H)\right),\underline{\xi}_H\right],$$

where $\left(x_1^*, \ldots, x_{H+1}^*(\underline{\xi}_H)\right)$ denotes an optimal solution to the RO_{H+1} problem (21), and $\left(\tilde{x}_1(\underline{\xi}_H), \ldots, \tilde{x}_{H+1}(\underline{\xi}_H)\right)$ denotes the optimal solution for each realization of $\underline{\xi}_H$. Taking the supremum of both sides yields the required inequality. $\square$

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Declarations

Conflict of interest Francesca Maggioni declares that she has no conflict of interest. Fabrizio Dabbene declares that he has no conflict of interest. Georg Ch. Pflug declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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