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Can short-term memory processes be accurately detected? A reexamination of existing definitions

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One major inadequacy in using the sample autocorrelation function (ACF) is the results from sample properties. Hassani's $-\frac{1}{2}$ theorem demonstrates that the sum of the sample ACF is always $-\frac{1}{2}$ for any time series with any length. This result has led to doubts about methodologies that sum sample ACFs for diagnostics and analyses. Thus, the current tools and approaches fall short in detecting short-memory processes with due accuracy. Perhaps the larger question that looms here is about whether, with such definitions and methods, short-memory processes can really be picked up? Resolving this issue stands as a basic precursor to strong predictions and to precluding model mis-specification.

Keywords: Short-Term memory process; sum of sample autocorrelation function; Hassani's $-\frac{1}{2}$ theorem; spectral density; time series; autocorrelation.

1. Introduction

Short-memory time series are characterized by having an autocorrelation function (ACF) that decays to zero at an exponential rate as the lag increases. This means

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the correlation between observations at different time steps lags will vanish which indeed does not present any apparent regular pattern in time. Hence, this property makes it appealing for application in domains where modeling short-term dependencies and immediate past behavior provide enough modeling capability for accurate predictions. Short-memory series have been considered basic concepts for time-dependence analysis to short ranges (e.g., see Refs. 8, 2 and 17). Adequate long-range dependence has also been developed where short-memory processes are considered assertion to equivocation, as already laid down by founding works such as Refs. 16 and 15, and later investigation.[1,5]

Let us consider a weak stationary process $\{X_t\}$. In other words,

$$\mathbb{E}(X_t) = \mu, \quad \mathrm{Var}(X_t) = \sigma^2, \quad \mathrm{Cov}(X_t, X_{t+h}) = \gamma(h),$$

where $\mathbb{E}$ is the expected value operator; μ is the constant mean, σ^2 is the finite variance, and $\gamma(h)$ is the autocovariance function that depends only on the lag h.

Mathematically, a weak stationary process $\{X_t\}$ is said to be short-memory if the sum of the absolute values of the autocovariance function is finite[4]:

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty.$$

This ensures that the influence of past observations on the current value of the series diminishes quickly as the lag increases. The ACF, $\rho(h)$, which is normalized as

$$|\rho(h)| = \frac{\gamma(h)}{\gamma(0)}, \quad \text{where } \gamma(0) = \mathrm{Var}(X_t),$$

decays exponentially as $h \to \infty$:

$$|\rho(h)| \sim c\exp(-\lambda h), \quad c > 0,\ \lambda > 0. \tag{1}$$

The most common examples of short-memory processes include the Auto-Regressive Moving Average (ARMA) family of models. For instance, an AR(1) process is defined as

$$X_t = \phi X_{t-1} + \varepsilon_t, \quad |\phi| < 1,\ \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2), \tag{2}$$

where ϕ is the autoregressive parameter and ε_t is a white noise error term. The ACF of this process decays exponentially, given by Ref. 4

$$\rho(h) = \phi^h, \quad \text{for lag } h. \tag{3}$$

The importance of short-memory processes lies in being trackable and practically applicable. Their rapidly decaying ACF allows for simpler model estimation and forecasting since the contribution of distant observations from the past becomes negligible. For instance, most of the time in financial time series, short-memory models such as ARMA are used to capture short-term correlations in asset returns or volatility, while GARCH models are used to model conditional heteroscedasticity with short-term volatility clustering.

However, the correct diagnosis of short-memory processes remains one of the time series analysis's most critical challenges. Failure to recognize that a short-memory process is being approximated by long memory (or vice versa) may lead to incorrect model specification, unreliable forecasts, and flawed statistical inferences. The sum of the sample ACF (S_{ACF}) is often used as a diagnostic tool for short memory since it measures the decay rate at which autocorrelations go to zero. In practice, this quantity is usually calculated as follows:

$$S_{\text{ACF}} = \sum_{h=1}^{T-1} \hat{\rho}(h),$$

where $\hat{\rho}(h)$ is the sample autocorrelation at lag h. For short-memory processes, S_{ACF} should converge to a small value as $T \to \infty$, reflecting the finite summability of the autocovariances. However, in real-world applications, deviations between the sample and theoretical ACF can occur, raising questions about the robustness of S_{ACF} as a diagnostic measure.[9–12,14]

This paper discusses the difficulties in identifying short-memory processes in time series data and is therefore concerned with some of the theoretical properties of short-memory processes, a study into the behavior of the sample ACF, and a critical examination of diagnostic tools that are commonplace. The results of this study and the examples provided in the paper highlight the necessity of robust method development in short-memory process identification.

The following sections structure the rest of the paper. Section 2 provides an exhaustive theoretical background on short-memory processes with definitions and properties. The behavior of the sample ACF, and its limitations as a diagnostic tool are discussed in Sec. 3. Section 3 also compares existing approaches to short-memory detection. The effectiveness of these approaches is analyzed, both from a theoretical and practical point of view. Findings are critically discussed in Sec. 3, with the paper being concluded in Sec. 5 by the same through summary contributions and future research directions. This work advances the further development of time series analysis through more detailed insight into short-memory process identification, underlining robust diagnostic tools for accurate modeling and prediction.

2. The Sum of Sample Autocorrelation Function and Short-Term Memory Process

2.1. *The sum of the sample autocorrelation function*

For any process $\{X_t\}$, we can compute its sample autocorrelations at lag h:

$$\hat{\rho}(h) = \frac{\sum_{t=1}^{T-h}(X_t - \overline{X})(X_{t+h} - \overline{X})}{\sum_{t=1}^{T}(X_t - \overline{X})^2}, \tag{4}$$

with $\overline{X} = \frac{\sum_{t=1}^{T} X_t}{T}$ being the sample mean. The sum of sample ACF has a notable behavior, as shown by Ref. 9.

Theorem 1. *The sum of the sample ACF, S_{ACF}, with lag $h \geq 1$ is always $\frac{-1}{2}$ for any time series with arbitrary length $T \geq 2$:*

$$S_{\mathrm{ACF}}^{\mathrm{empirical}} = \sum_{h=1}^{T-1} \hat{\rho}(h) = -\frac{1}{2}. \tag{5}$$

Proof (Ref. 9). The sum of the sample ACF ($S_{\mathrm{ACF}}^{\mathrm{empirical}}$) for a time series has the following key properties:

(i) **Independence of Time Series Length:** $S_{\mathrm{ACF}}^{\mathrm{empirical}} = -\frac{1}{2}$ for $T \geq 2$, regardless of the time series length.
(ii) **Constant Value for any Processes:** For any process, including ARMA(p, q) or Gaussian white noise, $S_{\mathrm{ACF}}^{\mathrm{empirical}}$ is always equal to $-\frac{1}{2}$.
(iii) **Linear Dependence of ACF Values:** The sample autocorrelation values $\hat{\rho}(h)$ satisfy the linear relationship:

$$\hat{\rho}(i) = -\frac{1}{2} - \sum_{j \neq i=1}^{T-1} \hat{\rho}(j), \quad i = 1, \ldots, T-1.$$

This equation shows that the ACF values are not independent but systematically related.
(iv) **Presence of Negative ACF Values:** For any time series, there is always at least one negative $\hat{\rho}(h)$, even for autoregressive AR(p) models with predominantly positive ACF values. □

The constancy of $S_{\mathrm{ACF}}^{\mathrm{empirical}} = -\frac{1}{2}$ for any time series has significant implications for time series modeling and diagnostics (see, for example Refs. 13 and 14).

2.2. *Short-term memory process*

Let us now consider short-memory processes, which includes the stationarity, thus allows to compute the theoretical ACF and the spectral density function. We can define the concept of short-memory processes in various ways. One common definition is based on the summability of the ACF, as shown by[11]:

$$\sum_{h=-\infty}^{\infty} |\rho(h)| < \infty, \quad \sum_{h=-\infty}^{\infty} \rho(h) > 0, \tag{6}$$

where $\rho(h)$ is the autocorrelation coefficient at lag h. This condition ensures that the correlations between observations diminish rapidly, resulting in negligible dependence at large lags. Let us define

$$S_{\mathrm{ACF}}^{\mathrm{theory}} = \sum_{h=1}^{\infty} \rho(h). \tag{7}$$

Since $\sum_{h=1}^{\infty} \rho(h) = \sum_{h=-\infty}^{h=-1} \rho(h)$ then we then obtain that for a short-memory process, we have

$$S_{\mathrm{ACF}}^{\mathrm{theory}} > -\frac{1}{2}. \tag{8}$$

As an alternative approach, a short-memory process can be defined by the exponential decay of the autocovariance function; $\gamma(h)$ decreases at an exponential rate as the lag h increases:

$$|\rho(h)| \sim c\exp(-\lambda h), \quad \lambda > 0,\ c > 0, \quad \text{as } h \to \infty.$$

Here, λ determines the rate of decay, and c is a constant scaling factor. This exponential decay distinguishes short-memory processes from long-memory processes, where the decay is much slower (e.g., polynomial).

Another way to define short-memory processes is through their spectral density function $f(\lambda)$. In the frequency domain, short-memory processes exhibit a spectral density that remains bounded and nonsingular as the frequency λ approaches zero:

$$f(\lambda) \sim C, \quad \text{as } \lambda \to 0,\ C > 0.$$

This highlights the lack of strong dependence at low frequencies, which is a key property of short-memory processes.

Furthermore, a short-memory process can be explained using the so-called Wold decomposition; the process can be expressed as a linear combination of past innovations with rapidly decreasing weights:

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} = \psi(B)\varepsilon_t,$$

where $\psi_0 = 1$, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, and ε_t is a white noise process with variance σ^2. For short-memory processes, the weights ψ_j decay exponentially:

$$|\psi_j| \sim c\exp(-\lambda j), \quad \lambda > 0,\ c > 0, \quad \text{as } j \to \infty.$$

This rapid decay ensures that the influence of past shocks on the current values of the process is short-lived.

The absence of long-range dependence reflects the core property of short-memory processes defined above. It should be noted that unlike the long-memory processes, short-memory processes have correlations that decay quickly, both in the time and frequency domains. For further discussions and theoretical comparisons, refer to Refs. 7 and 8.

3. Empirical Versus Theoretical Results

Let us consider three widely used processes: white noise, the AR(1) process, and a broader class of short-term memory processes. This examination provides a general overview and helps establish a clearer understanding of the key concepts and issues involved.

3.1. *Sum of the theoretical ACF for white noise*

The white noise process is the simplest form of a stationary time series. It is defined as

$$X_t = \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2),$$

where ε_t is a sequence of independent and identically distributed random variables with mean zero and variance σ^2. For a white noise process, the ACF is given by

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{if } h \neq 0. \end{cases} \tag{9}$$

This reflects the fact that there is no temporal dependence in a white noise process: observations at different time steps are uncorrelated.

If we sum the theoretical ACF for lags $h = 1$ to $T - 1$, the sum is

$$S_{\text{ACF}}^{\text{theory, finite}} = \sum_{h=1}^{T-1} \rho(h).$$

Given that $\rho(h) = 0$ for all $h \geq 1$, the sum simplifies to:

$$S_{\text{ACF}}^{\text{theory, finite}} = 0.$$

From the property of the ACF for a white noise, given in Eq. (9), we have $S_{\text{ACF}}^{\text{theory}} = 0$, which satisfies the short-memory property given in Eq. (8).

For white noise, the expected value of the sample ACF is approximately zero for all $h > 0$. However, due to finite sample size and random variations, the empirical sum of the ACF ($S_{\text{ACF}}^{\text{empirical}}$) does not exactly equal zero and may exhibit small random deviations. Let us now compare the theoretical and empirical ACF Sums.

- **Theoretical Sum:** The theoretical sum of the ACF for white noise is exactly zero:

$$S_{\text{ACF}}^{\text{theory, finite}} = 0.$$

- **Empirical Sum:** For finite sample size T, the empirical sum of the ACF, using Hassani $-1/2$ Theorem is however $-\frac{1}{2}$.

$$S_{\text{ACF}}^{\text{empirical}} = -\frac{1}{2}.$$

For a white noise process, the theoretical ACF is zero for all lags $h > 0$, but the empirical sum of the ACF is always $-\frac{1}{2}$.

3.2. *Theoretical versus empirical sum of the autocorrelation function for AR(1)*

To illustrate the differences between the theoretical and empirical sum of the ACF (S_{ACF}), let us now consider the first-order autoregressive process (AR(1)), which is defined as

$$X_t = \phi X_{t-1} + \varepsilon_t, \quad |\phi| < 1, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma^2),$$

where X_t is the time series, ϕ is the autoregressive parameter, and ε_t is a white noise process with mean zero and variance σ^2. For a stationary AR(1) process, we must have $|\phi| < 1$. Then the theoretical ACF $\rho(h)$, given in Eq. (3), decays exponentially with lag h:

$$\rho(h) = \phi^h, \quad h = 0, 1, 2, \ldots$$

Thus $|\rho(h)|$ has the exponential form given in Eq. (1), with $\lambda = -\ln(|\phi|) > 0$ and $c = 1$. When we calculate the sum of the theoretical ACF from lag $h = 1$ to $h = T - 1$, the finite sum of the ACF is

$$S_{\text{ACF}}^{\text{theory, finite}} = \sum_{h=1}^{T-1} \rho(h) = \sum_{h=1}^{T-1} \phi^h.$$

The above expression is a finite geometric series and thus:

$$S_{\text{ACF}}^{\text{theory, finite}} = \phi \frac{1 - \phi^{T-1}}{1 - \phi}, \quad |\phi| < 1. \tag{10}$$

Therefore, for small values of T, the sum $S_{\text{ACF}}^{\text{theory, finite}}$ depends significantly on both ϕ and the truncation limit T. It worth noting that as $T \to \infty$, the sum converges to the infinite geometric series result:

$$S_{\text{ACF}}^{\text{theory}} = \frac{\phi}{1 - \phi}. \tag{11}$$

This result demonstrates that for an AR(1) process, the sum of the theoretical ACF depends directly on the autoregressive coefficient ϕ, and it satisfies the short-memory property given in Eq. (8), $S_{\text{ACF}}^{\text{theory}} > -\frac{1}{2}$, as soon as $|\phi| < 1$.

In practice, however, the empirical ACF $\hat{\rho}(h)$ is estimated from a finite sample of size T. Let us now compare the theoretical and empirical ACF Sums for AR(1) processes.

- **Theoretical Sum:** The theoretical sum of the ACF for white noise is exactly zero:

$$S_{\text{ACF}}^{\text{theory, finite}} = \phi \frac{1 - \phi^{T-1}}{1 - \phi}.$$

- **Empirical Sum:** For finite sample size T, the empirical sum of the ACF, using Hassani $-1/2$ Theorem is however $-\frac{1}{2}$.

$$S_{\text{ACF}}^{\text{empirical}} = -\frac{1}{2}.$$

For a AR(1) process then, the theoretical ACF is zero for all lags $h > 0$, but the empirical sum of the ACF is always $-\frac{1}{2}$.

Theorem 2. (i) *Let $(X_t)_t$ be a stationary AR(1) process ($|\phi| \neq 1$). We have*

$$S_{\mathrm{ACF}}^{\text{theory, finite}} = S_{\mathrm{ACF}}^{\text{empirical}},$$

if and only if $1 + \phi - 2\phi^T = 0$.

(ii) *As a consequence, if $(X_t)_t$ is a stationary causal AR(1) process ($|\phi| < 1$). If ϕ satisfies either:*

- *$\phi = 0$, i.e., $(X_t)_t$ is a white noise,*
- *or $0 < \phi < 1$,*
- *or $-1 < \phi < 0$ and T is odd,*

then $S_{\mathrm{ACF}}^{\text{theory, finite}} \neq S_{\mathrm{ACF}}^{\text{empirical}}$.

But if T is an even number, there exists one value of $\phi \in\,]-1, 0[$ such that the equality holds.

Proof. See Appendix A. □

3.3. *Deviation between theoretical and empirical sums of the autocorrelation function*

The theoretical and empirical sums of the ACF, however, differ significantly due to the following factors:

(i) Finite sample size: For stationary processes, empirical ACF converge toward the theoretical ones.[4] While the theoretical ACF $\rho(h)$ implicitly assumes infinite sample ($T \to \infty$), the empirical ACF $\hat{\rho}(h)$ is computed from a finite sample. As a result, the empirical ACF does not perfectly match the theoretical ACF, especially for small sample sizes T, or at lags h close to T, where sampling variability increases. Then $S_{\mathrm{ACF}}^{\text{empirical}}$ deviates from $S_{\mathrm{ACF}}^{\text{theory, finite}}$ and also from $S_{\mathrm{ACF}}^{\text{theory}}$.

(ii) Hassani's $-1/2$ Theorem: For any stationary time series, the empirical sum of the ACF is constant and equal to $-\frac{1}{2}$, regardless of the underlying process:

$$S_{\mathrm{ACF}}^{\text{empirical}} = -\frac{1}{2}.$$

This result highlights a fundamental limitation of the empirical ACF in practice as an estimate of theoretical ACF. Even for an AR(1) process with a positive theoretical $S_{\mathrm{ACF}}^{\text{theory}}$, the empirical sum remains constant at $-\frac{1}{2}$, leading to discrepancies between theoretical expectations and observed behavior. The difference between the theoretical and empirical sums of the ACF is significant for model building and diagnostics. For instance, these findings emphasize the need for caution when relying on the empirical ACF for time series diagnostics and highlight the importance of understanding its limitations.

4. Numerical Results

4.1. *Simulations*

Figure 1 as an example, compares the empirical and theoretical ACFs for an AR(1) process with $\phi = 0.7$ or $\phi = -0.7$ and series lengths of $T = 20$ or $T = 100$. From Eq. (3), we recall that $\rho(h) = \phi^h$, so that $|\rho(h)|$ has an exponential decreasing evolution toward its null limit, when h increases. The empirical ACF $\hat{\rho}(h)$ is computed as in Eq. (4) It is derived from 5000 simulated AR(1) processes. The median value is represented by the blue dashed line with triangular markers and the interquartile area is filled in light blue. First, we remark that both empirical and theoretical ACFs converge rather quickly to the theoretical null limit. Moreover, we note that the empirical ACFs are significantly far from the theoretical ones, for short series ($T = 20$) and when ϕ is positive. On the contrary, we observe that the theoretical and empirical ACFs are very close when ϕ is negative, even for small sample sizes T.

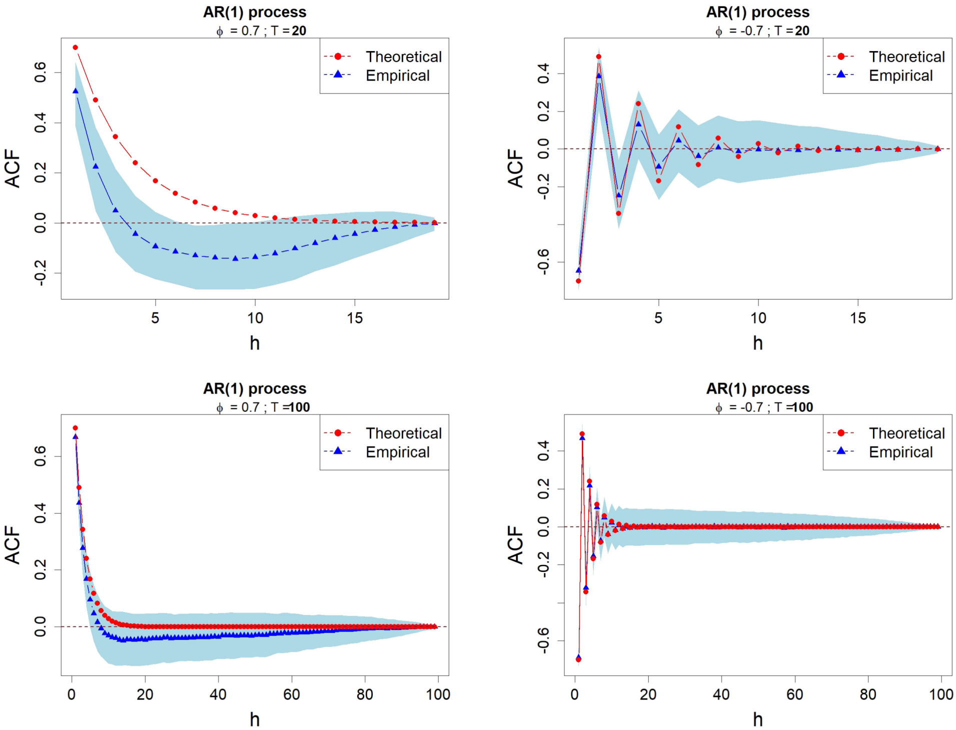


Fig. 1. ACF for AR(1) process, at lags h varying from 1 to $T-1$. The red dashed line with circular markers represents the theoretical ACF $\rho(h)$, as expressed in Eq. (3). The blue dashed line with triangular markers represents the empirical ACF $\hat{\rho}(h)$, as defined in Eq. (4). The dark red-dotted horizontal line represents the limit of $\rho(h)$ as h tends to $+\infty$, i.e., 0. In the left figures, we have $\phi = 0.7$, whereas in the right figures, we take $\phi = -0.7$. In the top figures, we have sample size $T = 20$, whereas in the bottom figures, we take $T = 100$.

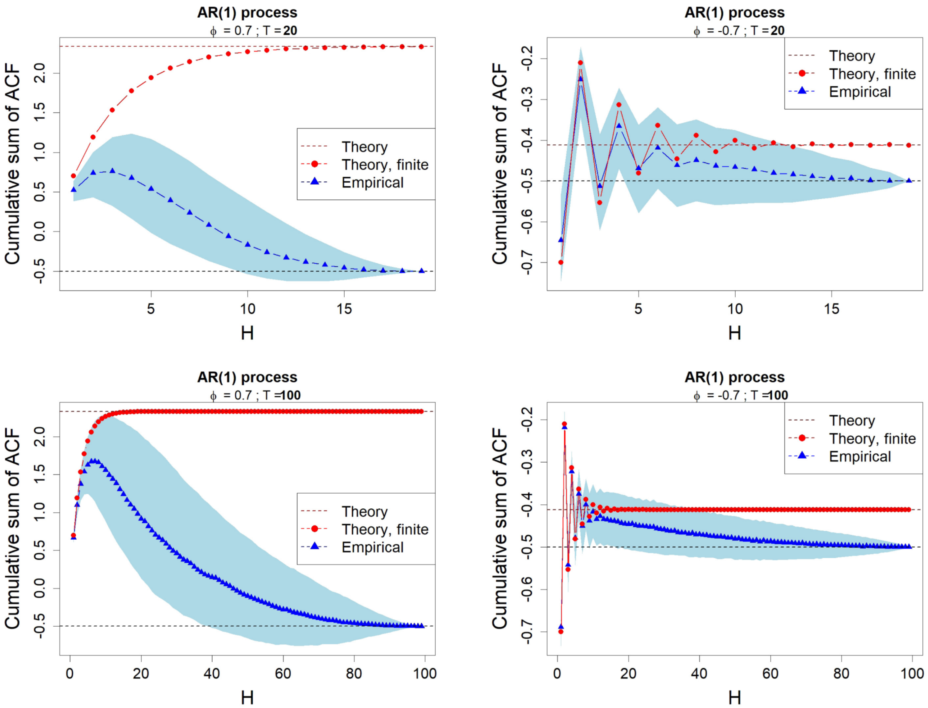


Fig. 2. Cumulative sums of ACF for AR(1) process, at lags H varying from 1 to $T-1$. The red dashed line with circular markers represents the theoretical finite cumulative sums $S_{\text{ACF}}^{\text{theory, finite}}(H)$, as expressed in Eq. (12). The blue dashed line with triangular markers represents the empirical finite $S_{\text{ACF}}^{\text{empirical}}(H)$, as expressed in Eq. (13). The dark red-dotted horizontal line represents $S_{\text{ACF}}^{\text{theory}}$, as expressed in Eq. (11), the theoretical target value. The black-dotted horizontal line represents $-1/2$, the empirical target value, as expressed in Eq. (5). In the left figures, we have $\phi = 0.7$, whereas in the right figures, we take $\phi = -0.7$. In the top figures, we have sample size $T = 20$, whereas in the bottom figures, we take $T = 100$.

Figure 2 compares the empirical and theoretical cumulative sums of the ACF for an AR(1) process with $\phi = 0.7$ or $\phi = -0.7$ and series lengths of $T = 20$ or $T = 100$. The theoretical cumulative sums

$$S_{\text{ACF}}^{\text{theory, finite}}(H) = \sum_{h=1}^{H} \rho(h) = \phi\frac{1-\phi^H}{1-\phi}, \quad H = 1, \ldots, T-1. \tag{12}$$

are shown as the red dashed line with circular markers, calculated using the geometric series formula for the ACF of an AR(1) process, expressed in Eq. (10). We also compute the empirical cumulative sums

$$S_{\text{ACF}}^{\text{empirical}}(H) = \sum_{h=1}^{H} \hat{\rho}(h), \quad H = 1, \ldots, T-1. \tag{13}$$

They are derived from 5000 simulated AR(1) processes. The median value is represented by the blue dashed line with triangular markers and the interquartile area is filled in light blue. The dark red-dotted horizontal line represents $S_{\mathrm{ACF}}^{\mathrm{theory}}$, as expressed in Eq. (11), the theoretical target value. The black-dotted horizontal line represents $-1/2$, the empirical target value, as expressed in Eq. (5). As expected form the short-memory property given in Eq. (8), we have $S_{\mathrm{ACF}}^{\mathrm{theory}} > -\frac{1}{2}$. Moreover, we have $S_{\mathrm{ACF}}^{\mathrm{theory,\ finite}} \neq -\frac{1}{2}$ since the equation given in Theorem 2

$$1 + \phi - 2\phi^T = 0,$$

has no solution when $\phi = \pm 0.7$ and $T = 20$ or 100.

In the left panel, ϕ is positive. Initially, when T is sufficiently large, the empirical cumulative sum $S_{\mathrm{ACF}}^{\mathrm{empirical}}(H)$ closely follows the theoretical values $S_{\mathrm{ACF}}^{\mathrm{theory\ finite}}(H)$, rising due to the strong positive autocorrelation at smaller lags. However, deviations appear as the lag increases. We can see that the partial cumulative sums quickly reach their respective target values, which are very different. The same phenomenon is observed in the sub-figures on the right, but the respective target values are closer. Moreover, as $\phi = -0.7$, both curves exhibit fluctuations due to the alternating signs of the ACF values caused by the negative ϕ.

Figure 3 displays the theoretical cumulative sums $S_{\mathrm{ACF}}^{\mathrm{theory,\ finite}}$ for an AR(1) process with various ϕ values, from -0.95 to 0.95, and with varying values of n $(20, 50, 100$ and $500)$. The theoretical sums are calculated using the geometric series formula given in Eq. (10). First, let us note that all the curves $S_{\mathrm{ACF}}^{\mathrm{theory,\ finite}}$ are superimposed, except when the sample size is very small ($T = 20$) and $|\phi| \simeq 1$. This highlights that the convergence of $S_{\mathrm{ACF}}^{\mathrm{theory,\ finite}}$ toward $S_{\mathrm{ACF}}^{\mathrm{theory}}$ is reached very quickly. On the other hand, $S_{\mathrm{ACF}}^{\mathrm{empirical}}$, the $-1/2$ empirical target value, is represented by a black-dotted horizontal line. It is striking to note that, while the theoretical $S_{\mathrm{ACF}}^{\mathrm{theory,\ finite}}$ depends on the underlying model, parameterized by ϕ, and to a lesser extent by T, the empirical results $S_{\mathrm{ACF}}^{\mathrm{empirical}}$ remains invariant. Moreover, as expected

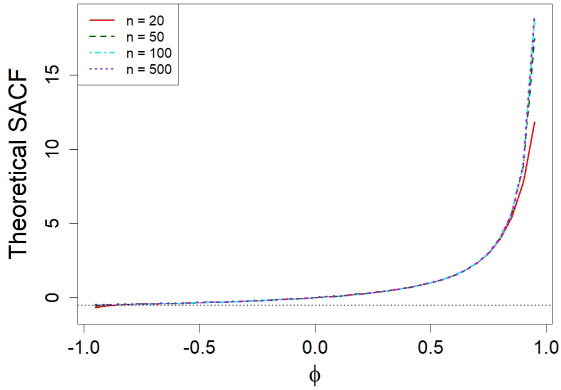


Fig. 3. Cumulative sums $S_{\mathrm{ACF}}^{\mathrm{theory,\ finite}}$, as expressed in Eq. (10), for AR(1) process, with parameter ϕ varying from -0.95 to 0.95. We take several sample sizes, as $T = 20, 50, 100$ and $T = 500$. The black-dotted horizontal line represents $-1/2$, the empirical target value, as expressed in Eq. (5).

from the short-memory property satisfied by a stationary causal AR(1) process, we have $S_{\mathrm{ACF}}^{\mathrm{theory}} > -\frac{1}{2}$. Furthermore, this inequality also holds for $S_{\mathrm{ACF}}^{\mathrm{theory,\ finite}}$, except when the sample size is very small ($T = 20$) and $\phi \simeq -1$. Indeed, the equation $1 + \phi - 2\phi^T = 0$ given in Theorem 2 has a unique solution in $]-1, 1[$ for any even sample size T: $\phi \simeq -0.872$ for $T = 20$, $\phi \simeq 0.934$ for $T = 50$, and the solution tends to 1 when T (even) increases. Thus, the theoretical and empirical SACFs are close for negative values of ϕ, but differ significantly as ϕ increases toward 1.

As a conclusion, we obtain that the empirical ACFs are relatively close to the theoretical ones when $\phi < 0$ and T is an even number, even with small sample sizes. On the other hand, we observe discrepancies between finite sample observations $S_{\mathrm{ACF}}^{\mathrm{empirical}}$ and theoretical assumptions, reflected by $S_{\mathrm{ACF}}^{\mathrm{theory}}$ otherwise. This points to fundamental limitations in using current empirical methods for accurately detecting short-memory processes. These findings are consistent across a broader range of T and ϕ values, emphasizing the need for more robust diagnostic tools.

4.2. *Real example of short memory in meteorology*

Meteorological data, such as temperature or precipitation, often exhibit stochastic behavior due to the complex interactions of atmospheric processes. These data are generally well-suited for modeling with short-memory processes because their autocorrelations tend to decay rapidly, indicating limited dependence on distant past observations. For example, precipitation levels often show temporal correlations mainly over short periods. The `ice.river` dataset from the `tseries` R-package includes precipitation data (`prec`) that can be used to illustrate these properties. It contains daily precipitation measurements recorded in Hveravellir, Iceland, from January 1st, 1972, to December 31st, 1974. Hveravellir is a geothermal area situated in the central highlands of Iceland, known for its unique climate and weather patterns. Figure 4 (left) plots the $T = 1096$ observations of the amount of precipitation (in millimeters) for a specific day within this three-year period. An AR(1) model is a suitable candidate for modeling such data. This choice is supported by the fact that AR(1) models effectively capture short-term dependencies and rapid autocorrelation decay, which are characteristic of precipitation time series. Such a candidate model is confirmed by the empirical partial ACFs (pacf), given in Fig. 4 (right). The lag h associated with the last value of pacf that does not lie between the 2 thresholds $\pm\frac{1.96}{\sqrt{T}}$, gives the order p of the appropriate AR(p) model.[4] Moreover, from Fig. 5, we can valid the AR(1)-model by checking the usual white noise conditions on the associated residuals.

Figure 6 (left) displays the ACF for the amount of precipitation, from lags $h = 1$ to $T - 1$. The blue vertical segments represent the empirical ACF for the amount of precipitation, while the red line represents the theoretical ACF for an AR(1)-process, as given in Eq. (3), where ϕ is estimated by maximum likelihood as 0.249. Both empirical and theoretical ACF are very close. They show a rapid decay and next fluctuate around zero, reflecting a weak autocorrelation. Figure 6 (right)

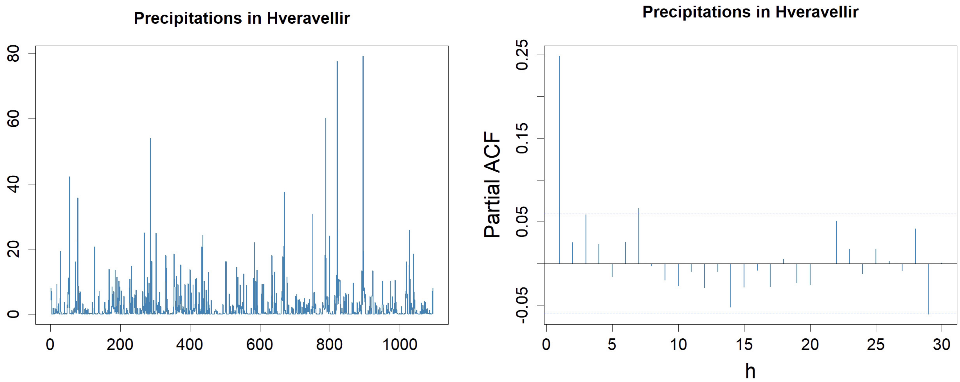


Fig. 4. (left) Daily precipitation measurements recorded in Hveravellir, Iceland, from January 1st, 1972, to December 31st, 1974. (right) Partial ACFs for precipitations data. The usual thresholds $\pm\frac{1.96}{\sqrt{T}}$, represented as horizontal dashed lines, are used to identify an AR(p) model.

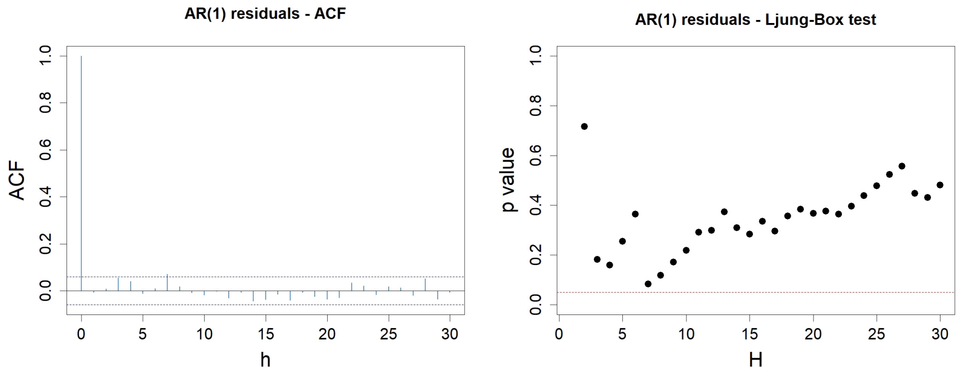
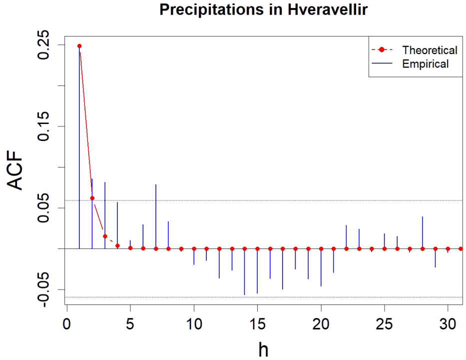


Fig. 5. (left) Sample ACF for the residuals of the AR(1) model computed on the rainfall data. The usual thresholds $\pm\frac{1.96}{\sqrt{T}}$, represented as horizontal dashed lines, are used to confirm that the residuals can be considered as a white noise. (right) P-values when using Ljung–Box's test on the residuals associated to the AR(1)-model. The red-dotted horizontal line represents 5%.

displays the successive $\mathrm{SACF}(H)$ for the amount of precipitation, from lags $H = 1$ to $T - 1$. $S_{\mathrm{ACF}}^{\mathrm{empirical}}(H)$ tends to $-\frac{1}{2}$, the empirical target, as expected by Eq. (5). Moreover, since $\phi > 0$, it rapidly moves away from $S_{\mathrm{ACF}}^{\mathrm{theory,\ finite}}(H)$, computed from Eq. (12).

With the precipitation example, we illustrate that the empirical sum of ACF consistently equals $-1/2$. This observation diverges from the theoretical results as mentioned earlier, highlighting a discrepancy between empirical and theoretical expectations. Importantly, this phenomenon is not limited to hypothetical scenarios but applies to real-world cases as well. It is worth mentioning that this example does not try to identify the best model for analysis and prediction. It illustrates that any definition used for detection or modeling of a series as a short-memory



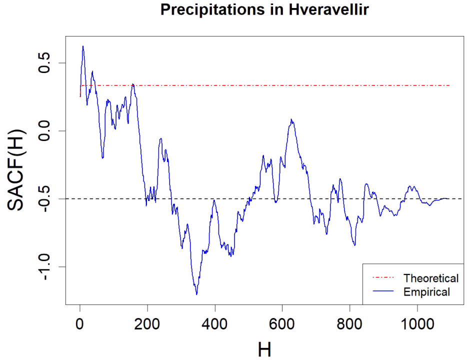


Fig. 6. (left) Empirical and theoretical ACF for the amount of precipitation, from lag $h = 1$ to $T - 1$. The blue vertical segments represent the empirical ACF for the amount of precipitation, while the red line with circular symbols represents the theoretical ACF for an sAR(1)-process, as given in Eq. (3), where ϕ is estimated by 0.249. The usual thresholds $\pm\frac{1.96}{\sqrt{T}}$, represented as horizontal dashed lines, are used to check if empirical ACF $\hat{\rho}(h)$ can be considered null. (right) Empirical and theoretical SACF(H) for the amount of precipitation, from lag $h = 1$ to $T-1$. The blue curve represents $S_{\text{ACF}}^{\text{empirical}}(H)$ whereas the red dashed curve represents $S_{\text{ACF}}^{\text{theory, finite}}(H)$, computed from Eq. (12). The black-dotted horizontal line represents $-1/2$, the empirical target value, as expressed in Eq. (5).

process based on the sum of the ACF is greatly in need of re-evaluation. In this case study, we considered an AR(1) model, but this work can be generalized to incorporate models like MA(q) or even ARIMA for additional examples.

5. Conclusions

In conclusion, the actual weaknesses of existing definitions and methodologies in testing for short-memory processes were brought out by this study. Hassani's $-1/2$ theorem challenged the power of the traditional diagnostic tools to be effective because the empirical ACFs sum turned out to be $-1/2$. It therefore brings to the fore the fact that current definitions were such that short-memory processes could not be distinguished appropriately in a finite sample. As a result, the work calls for the development of frameworks that are more robust and reliable to correct these discrepancies and enhance proper short-memory processes' identification in time series analysis. Future work should be toward theoretical improvements that can better provide necessary practical steps for achieving this end.

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Appendix A. Proof of Theorem 2

(i)

$$\begin{aligned}S_{\mathrm{ACF}}^{\text{theory, finite}} > S_{\mathrm{ACF}}^{\text{empirical}} &\Leftrightarrow \phi\frac{1-\phi^{T-1}}{1-\phi} > -\frac{1}{2}\\ &\Leftrightarrow \phi-\phi^T > -\frac{1}{2}+\frac{1}{2}\phi\\ &\Leftrightarrow 1+\phi-2\phi^T > 0.\end{aligned}$$

(ii)

- If $\phi = 0$, then $(X_t)_t$ is a white noise, so that $S_{\mathrm{ACF}}^{\text{theory, finite}} = 0 > -\frac{1}{2}$.
- If $0 < \phi < 1$, then $0 < \phi^T < \phi$. Consequently,

$$1+\phi-2\phi^T > 1-\phi > 0.$$

  So that $S_{\mathrm{ACF}}^{\text{theory, finite}} > S_{\mathrm{ACF}}^{\text{empirical}}$.
- If $-1 < \phi < 0$ and T is an odd number, then equation $1+\phi-2\phi^T$ can be rewritten $g(y) = 1-y+2y^T$, with $0 < y < 1$. Function g has a derivative vanishing at $y_0 = \sqrt[(T-1)]{\frac{1}{2T}}$, with a negative sign when $0 < y < y_0$ and positive otherwise. As a consequence, equation $1+\phi-2\phi^T$ reaches its minimum positive value when $\phi = -\sqrt[(T-1)]{\frac{1}{2T}}$. Finally, $S_{\mathrm{ACF}}^{\text{theory, finite}} > S_{\mathrm{ACF}}^{\text{empirical}}$.

On the contrary, when T is an even number, equation $1+\phi-2\phi^T$ can be rewritten $f(y) = 1-y-2y^T$, with $0 < y < 1$. Function f has a derivative that remains negative on $]0,1[$. Then f decreases from 1 to -2. As a continuous function, it vanishes for a unique value $y_0 \in\,]0,1[$. Consequently, there exists one value $\phi_0 \in\,]-1,0[$ such that $S_{\mathrm{ACF}}^{\text{theory, finite}} = S_{\mathrm{ACF}}^{\text{empirical}}$. But for any ϕ satisfying $\phi_0 < \phi < 0$, we have $S_{\mathrm{ACF}}^{\text{theory, finite}} > S_{\mathrm{ACF}}^{\text{empirical}}$.

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