A CONVERGENCE FOR BIVARIATE FUNCTIONS AIMED AT THE CONVERGENCE OF SADDLE VALUES

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A CONVERGENCE FOR BIVARIATE FUNCTIONS AIMED AT THE CONVERGENCE OF SADDLE VALUES

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ABSTRACT

Epi/hypo-convergence is introduced from a variational viewpoint. The known topological properties are reviewed and extended. Finally, it is shown that the (partial) Legendre-Fenchel transform is bicontinuous with respect to the topology induced by epi/hypo-convergence on the space of convex-concave bivariate functions.

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1. INTRODUCTION.

One of our motivation is to introduce a notion of convergence well adapted to the study of extremal problems that can not be reduced to minimization problems. For example, let us consider a sequence of variational inequalities

$$(1.1_\varepsilon) \qquad \begin{cases} < A^\varepsilon u_\varepsilon - f, v - u_\varepsilon > \ \geq \ 0 \qquad \forall\, v \in K^\varepsilon \\ u_\varepsilon \in K^\varepsilon \end{cases}$$

where ε is a parameter describing an approximation, or a perturbation, homogenization... procedure. The operators $(A^\varepsilon)_{\varepsilon>0}$, the contraints K^ε are varying with ε, and the problem is to determine the behaviour, as ε goes to zero, of the solutions $(u_\varepsilon)_{\varepsilon>0}$ of the corresponding problems (1_ε). When the operators A^ε are subdifferentials of convex functionals and K^ε is convex, the problems (1_ε) can be viewed as minimization ones; but in general (take A^ε general operators of the calculus of variations, for example non symmetric second order elliptic operators, parabolic operators...) (1_ε) does not come from a minimization problem. However, it can always be expressed as a saddle value problem, under rather general assumptions, as already noticed by Glowinski, Lions and Tremolières [1], see also Rockafellar [13].

1.2 PROPOSITION. *Let* V *be a vector space and denote by* V' *its dual space. Given* $A : V \longrightarrow V'$, *a monotone operator, i.e. for all* $x, y \in V$, $< Ax - Ay, x - y > \ \geq \ 0$, *and* $\phi : V \longrightarrow \,]-\infty, +\infty]$ *a real-valued function defined on* V, $\phi \not\equiv \infty$, *for any* $f \in V'$, *the following statements are equivalent :*

(i) u *is a solution of the variational inequality*

$$(1.3) \qquad < Au - f, v - u > + \phi(v) - \phi(u) \geq 0 \qquad \forall\, v \in V$$

(ii) (u,u) *is a saddle point of the function* $H : V \times V \longrightarrow \bar{\mathbb{R}}$

$$H(u,v) = \langle Au - f, u - v \rangle + \phi(u) - \phi(v).$$

PROOF. By definition of H, u is a solution of the variational inequality (1.3), if and only if

(1.4) $\quad H(u,v) \leq 0, \quad$ for all $v \in V$.

Note that $H(u,v) = 0$ whenever u is a solution of (1.3). Thus it necessarily satisfies

$$H(u,v) \leq H(u,u) \quad \text{for all } v \in V.$$

On the other hand, for all $w \in V$

$$\begin{aligned} H(w,u) &= \langle Aw - f, w - u \rangle + \phi(w) - \phi(u) \\ &= \langle Aw - Au, w - u \rangle + \langle Au - f, w - u \rangle + \phi(w) - \phi(u) \\ &= \langle Aw - Au, w - u \rangle - H(u,w) \\ &\geq 0. \end{aligned}$$

This last inequality following from the monotonicity of A and (1.4). So, for all $v \in V$ and $w \in V$, $H(u,v) \leq H(u,u) \leq H(w,u)$ which means that (u,u) is a saddle point of H.

Conversely if u is a saddle point of H, for all $v \in V$

$$H(u,v) \leq H(u,u) = 0,$$

which from (1.4) implies that u is a solution of the variational inequality (1.3).□

Let us now examine an important example : take $V = H^1_o(\Omega)$, Ω a bounded regular open set in $\mathbb{R}^N$, $V' = H^{-1}(\Omega)$.

$$A^\varepsilon(u) = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a^\varepsilon_{ij}(x) \frac{\partial u}{\partial x_j}\right)$$

where the $a^\varepsilon_{ij} \in L^\infty(\Omega)$ satisfy :

$$|a^\varepsilon_{ij}| \leq M$$

$$\sum a^\varepsilon_{ij}\ \xi_i\ \xi_j \geq \lambda_o|\xi|^2$$

with $\lambda_o > 0$ and M independent of x and ε. We do *not* require that the matrix (a^ε_{ij}) be symmetric, i.e. a^ε_{ij} is not necessarily equal to a^ε_{ji}. This class of problems is being studied by A. Brillard. For simplicity, we only consider the case with no constraints on u, i.e. $K^\varepsilon = V$ or equivalently $\phi^\varepsilon \equiv 0$. So, the variational inequalities (1.1_ε) reduce to the linear partial differential equations $A^\varepsilon u = f$. The natural notion of convergence $A^\varepsilon \xrightarrow{G} A$, as introduced by De Giorgi and Spagnolo [2] and Murat and Tartar [3], is

(1.5) $$\text{for all } f \in H^{-1}(\Omega) : u_\varepsilon = (A^\varepsilon)^{-1} f \underset{w-V}{\longrightarrow} u = (A)^{-1} f,$$

i.e. for the weak topology of $H^1_o(\Omega)$. Let us examine what is the corresponding notion of convergence for the saddle-functions

(1.6) $$H^\varepsilon(u,v) = \langle A^\varepsilon u,\ u - v \rangle.$$

1.7. PROPOSITION. *The following statements are equivalent :*

(i) $A^\varepsilon \xrightarrow{G} A$

(ii) $H^\varepsilon \longrightarrow H$ *in the following sense : for every* $u,v \in V$

(1.8) $$\left|\begin{array}{l} \forall\, u_\varepsilon \longrightarrow u\ \exists v_\varepsilon \longrightarrow v \text{ such that } \liminf_{\varepsilon\to 0} H^\varepsilon(u_\varepsilon,v_\varepsilon) \geq H(u,v), \\ \forall\, v_\varepsilon \longrightarrow v\ \exists u_\varepsilon \longrightarrow u \text{ such that } H(u,v) \geq \limsup_{\varepsilon\to 0} H^\varepsilon(u_\varepsilon,v_\varepsilon). \end{array}\right.$$

where $\longrightarrow$ *denotes weak-convergence.*

PROOF. Let us first note that $A^\varepsilon \xrightarrow{G} A$ if and only if $(A^\varepsilon)^t \xrightarrow{G} A^t$ where $(A^\varepsilon)^t$ and A^t are the elliptic operators with the transposed matrix $(a^\varepsilon_{ij})^t = a^\varepsilon_{ji}$ and $(a_{ij})^t = a_{ji}$.

Let us first verify that (i) $\Rightarrow$ (ii). Fix $u_\varepsilon \longrightarrow u$ and $v \in V$. We are looking for a sequence $v_\varepsilon \longrightarrow v$ such that

$$\liminf_{\varepsilon\to 0} \langle A^\varepsilon u_\varepsilon, u_\varepsilon - v_\varepsilon \rangle \geq \langle Au, u - v \rangle$$

Let w_ε be the solution of

$$(A^\varepsilon)^t w_\varepsilon = A^t(u - v). \tag{1.9}$$

By the definition of G-convergence for the sequence of operators $(A^\varepsilon)^t$ to A^t, as $\varepsilon \downarrow 0$ we have

$$w_\varepsilon \longrightarrow u - v$$

in the weak topology of V. Set

$$v_\varepsilon = u_\varepsilon - w_\varepsilon .$$

Then $v_\varepsilon \longrightarrow u - (u - v) = v$ and $u_\varepsilon - v_\varepsilon = w_\varepsilon$. Hence

$$\begin{aligned} \langle A^\varepsilon u_\varepsilon, u_\varepsilon - v_\varepsilon \rangle &= \langle A^\varepsilon u_\varepsilon, w_\varepsilon \rangle \\ &= \langle u_\varepsilon, (A^\varepsilon)^t w_\varepsilon \rangle \\ &= \langle u_\varepsilon, A^t(u - v) \rangle \end{aligned}$$

as follows from (1.9). Letting ε tend to 0, we get

$$\begin{aligned} \lim_{\varepsilon\to 0} \langle A^\varepsilon u_\varepsilon, u_\varepsilon - v_\varepsilon \rangle &= \langle u, A^t(u - v) \rangle \\ &= \langle Au, u - v \rangle. \end{aligned}$$

This completes the proof of the first part of (1.8). Next, fix $v_\varepsilon \longrightarrow v$ and $\bar{u} \in V$. This time we search for a sequence $u_\varepsilon \longrightarrow \bar{u}$ such that

$$\langle A\bar{u}, \bar{u} - v \rangle \geq \limsup_{\varepsilon\to 0} \langle A^\varepsilon u_\varepsilon, u_\varepsilon - v_\varepsilon \rangle.$$

Let u_ε be the solution of the equation $A^\varepsilon u = A\bar{u}$. Then

$$\langle A^\varepsilon u_\varepsilon, u_\varepsilon - v_\varepsilon \rangle = \langle A\bar{u}, u_\varepsilon - v_\varepsilon \rangle$$

and since $u_\varepsilon \longrightarrow \bar{u}$ and $v_\varepsilon \longrightarrow v$ we get

$$\lim_{\varepsilon\to 0} \langle A^\varepsilon u_\varepsilon, u_\varepsilon - v_\varepsilon \rangle = \langle A\bar{u}, \bar{u} - v \rangle$$

Next we prove that (ii) $\Rightarrow$ (i), that is to say, we verify if the convergence of the saddle functions H_n has the desired variational properties. Fix $f \in V$ and for $\varepsilon > 0$, let u_ε denote the solution of the equation $A^\varepsilon u = f$. The uniform coerciveness of the operators A^ε yields the boundedness of the u^ε in V. Passing to a subsequence if necessary, we have that

$$u_\varepsilon \longrightarrow \bar{u},$$

for some $\bar{u}$. To complete the proof we need to show that $A\bar{u} = f$. This will follow from the uniqueness of the solution of the equation $Au = f$. From (1.8), for any $v \in V$ there exists $v_\varepsilon \longrightarrow v$ such that

$$\liminf_{\varepsilon \to 0} < A^\varepsilon u_\varepsilon, u_\varepsilon - v_\varepsilon > \geq < A\bar{u}, \bar{u} - v >$$

which means that

$$\liminf_{\varepsilon \to 0} < f, u_\varepsilon - v_\varepsilon > \geq < A\bar{u}, \bar{u} - v >$$

or still

$$< f, \bar{u} - v > \geq < A\bar{u}, \bar{u} - v >,$$

and thus for all $v \in V$

$$< A\bar{u} - f, \bar{u} - v > \leq 0$$

and $A\bar{u} = f$. □

In the preceeding example, we like to stress the fact that the saddle functions H^ε are not convex-concave. The lack of convexity comes from the non-symmetry of the monotone operators A^ε. Note also that in this example is not quite necessary to require both parts of (1.8), since the first part implies the second. This will not be the case in general, both conditions of (1.8) are usually necessary to obtain the desired variational properties.

Our next example is intented to illustrate the problems that arise in connection with Lagrangians and Hamiltonians. Let us consider the following class of optimization problems, for $\nu = 1, 2, \ldots$

(1.10_ν) Minimize $f_o^\nu(x)$

subject to $f_i^\nu(x) \leq 0 \quad i = 1,\ldots,m$

$x \in C \subset X$

with X a reflexive Banach space and C a closed subset. The associated Lagrangian function is

$$(1.11) \qquad L_\nu(x,y) = \begin{cases} f_o^\nu(x) + \sum_{i=1}^m y_i\, f_i^\nu(x) & \text{if } x \in C \text{ and } y \geq 0 \\ +\infty \text{ if } x \notin C \text{ and } y \geq 0 \\ -\infty \text{ otherwise.} \end{cases}$$

We think of the problems (1.10_ν) and their Lagrangians as the approximates of some limit problem :

(1.12) Minimize $f_o(x)$

subject to $f_i(x) \leq 0 \quad i = 1,\ldots,m$

$x \in C \subset X$

with associated Lagrangian

$$(1.13) \qquad L(x,y) = \begin{cases} f_o(x) + \sum_{i=1}^m y_i\, f_i(x) & \text{if } x \in C \text{ and } y \geq 0 \\ +\infty \text{ if } x \notin C \text{ and } y \geq 0 \\ -\infty \text{ otherwise.} \end{cases}$$

A typical situation is when the problems (1.10_ν) are obtained from (1.12) as the result of penalization or barrier terms being added to the objective, or when the (1.10_ν) are the restrictions of (1.12) to finite dimensional subspaces of X, and so on. In particular, when dealing with numerical procedures, one is naturally interested in the convergence of the solutions, but also in the convergence of the multipliers, for reason of stability [4] or to be able to calculate rates of convergence such as in augmented Lagrangian methods. From the convergence of the $\{f_i^\nu,\ \nu = 1,\ldots\}$ to the f_i

one cannot conclude in general that the feasible sets

$$S_\nu = \{x \in C | f_i^\nu(x) \leq 0, \quad i = 1,\dots,m\}$$

converge to the feasible set of the limit problem,

$$S = \{x \in C | f_i(x) \leq 0, \quad i = 1,\dots,m\}.$$

A fortiori, it is not possible to obtain the convergence of the infima or of the optimal solutions. However, there are some relatively weak conditions that can be imposed on the convergence of the objectives and of the constraints that will guarantee the convergence of the Lagrangians L_ν to L in a sense similar to that induced by G-convergence on the saddle functions (1.6) associated with the partial differential equations $A^\varepsilon u = f$. The sought for, convergence of the solutions *and* multipliers will ensue.

Given $\{f; f^\nu : X \longrightarrow \overline{R}, \nu = 1,\dots\}$ a collection of functions, we say that the f^ν *epi-convergence* to f if for all x

(1.14) $$\text{for all } x_\nu \longrightarrow x, \quad \liminf_{\nu\to\infty} f^\nu(x \ ) \geq f(x),$$

and

(1.15) $$\text{there exists } x_\nu \longrightarrow x \text{ with } \limsup_{\nu\to\infty} f^\nu(x_\nu) \leq f(x).$$

As is well-known, epi-convergence is neither implied nor does it imply pointwise convergence, but they coincide, for example, if the sequence of functions is monotone, either increasing or decreasing (provided f is lower semicontinuous). We have so-called *continuous convergence* if condition (1.15) is replaced by the stronger requirement

(1.16) $$\text{for all } x_\nu \longrightarrow x, \ \limsup_{\nu\to\infty} f^\nu(x_\nu) \leq f(x).$$

Continuous convergence is much stronger that both epi- and pointwise-convergence.

1.17 PROPOSITION. *Suppose the* $\{f_o^\nu, \nu = 1,\ldots\}$ *epi-converge* f_o, *and for all* $i = 1,\ldots,m$, *the* $\{f_i^\nu, \nu = 1,\ldots\}$ *continuously converge to* f_i. *Then, the associated Lagrangian functions* L_ν *converge to the Lagrangian* L *in the following sense : for all* $x \in X$ *and* $y \in Y$

$$(1.18)\qquad \left|\begin{array}{l} \textit{for any } x_\nu \longrightarrow x\text{ , there exists } y_\nu \longrightarrow y \textit{ such that} \\ \qquad\qquad \liminf_{\nu\to\infty} L_\nu(x_\nu,y_\nu) \geq L(x,y) \\ \textit{for any } y_\nu \longrightarrow y\text{ , there exists } x_\nu \longrightarrow x \textit{ such that} \\ \qquad\qquad \limsup_{\nu\to\infty} L_\nu(x_\nu,y_\nu) \leq L(x,y). \end{array}\right.$$

Moreover, suppose that the Lagrangians L_ν *converge to* L *in the above sense, and for some subsequence* $\{\nu_k, k = 1,\ldots\}$ *the sequence* $\{(\bar{x}^k,\bar{y}^k), k = 1,\ldots\}$, *which converge to* $(\bar{x},\bar{y})$ *is such that* $\bar{x}^k$ *solves problem* (1.10_{ν_k}) *and* $\bar{y}^k$ *is a (Lagrange) multiplier. Then* $\bar{x}$ *solves* (1.12) *and* $\bar{y}$ *is an associated multiplier.*

PROOF. We start by showing that the conditions imposed on the f_o^ν and $\{f_i^\nu, i = 1,\ldots,m\}$ yield (1.18). Let x^ν be any sequence converging to x and set $y^\nu = y$ for all ν. We have to verify that when $x \in C$ and $y \geq 0$

$$\liminf_{\nu\to\infty} \left(f_o^\nu(x_\nu) + \sum_{i=1}^m y_i\, f_i(x_\nu)\right) \geq f_o(x) + \sum_{i=1}^m y_i\, f_i(x),$$

the cases when $y \ngeq 0$ and/or $x \notin C$ are automatically satisfied. Since C is closed, any sequence that converges to $x \notin C$ is such that $x^\nu \in X\setminus C$ for ν sufficiently large. The inequality in fact follows directly from (1.14) which is satisfied by both the epi-convergence of the f_o^ν and the continuous convergence of the f_i^ν, $i = 1,\ldots,m$.

Next we have to verify that for any sequence $y_\nu \longrightarrow y$, there exists $x^\nu \longrightarrow x$ such that when $x \in C$ and $y \geq 0$

$$\limsup_{\nu\to\infty} \left(f_o^\nu(x_\nu) + \sum_{i=1}^m y_i^\nu f_i^\nu(x_\nu)\right) \leq f_o(x) + \sum_{i=1}^m y_i f_i(x).$$

When $x \notin C$ or/and $y \ngeq 0$ the desired relation between $\limsup_{\nu\to+\infty} L_\nu$ and L is automatically satisfied. The preceeding inequality then follows from (1.15) and (1.16).

If $\bar{x}^k$ solves (1.10_{ν_k}) and $\bar{y}^k$ is an associated multiplier, we have that for i = 1,...,m

$$\bar{y}^k \geq 0 \ , \ f_i^k(\bar{x}^k) \leq 0 \quad \text{and } \bar{y}_i^k f_i^{\nu_k}(\bar{x}^k) = 0,$$

and

$$\bar{x}^k \in \operatorname{argmin}_{x \in C} \left(f_o^{\nu_k}(x) + \sum_{i=1}^m y_i^k f_i^{\nu_k}(x)\right) .$$

This is equivalent to : for all x and y

$$L_{\nu_k}(\bar{x}^k,y) \leq L_{\nu_k}(\bar{x}^k,\bar{y}^k) \leq L_{\nu_k}(x,\bar{y}^k),$$

with the first inequality equivalent to the first part of the optimality conditions and the second inequality is just a restatement of the second part of the optimality conditions.

Thus the assertion will be complete if we show that $(\bar{x},\bar{y}) = \lim_{k\to\infty}(\bar{x}_k,\bar{y}_k)$ is a saddle point of L, i.e.

$$L(\bar{x},y) \leq L(\bar{x},\bar{y}) \leq L(x,\bar{y}).$$

First note that if the sequence L_ν converges to L in the sense of (1.8) so does the subsequence $\{L_{\nu_k}, k = 1,...\}$. Since the $(\bar{x}_k,\bar{y}_k)$ are saddle points, for any pair of sequences $\{x^k, k = 1...\}$ and $\{y^k, k = 1...\}$ converging to x and y respectively, we have

$$\liminf_{k\to\infty} L_{\nu_k}(\bar{x}^k,y^k) \leq \liminf_{k\to\infty} L_{\nu_k}(\bar{x}^k,\bar{y}^k)$$

$$\leq \limsup_{k\to\infty} L_{\nu_k}(\bar{x}^k,\bar{y}^k) \leq \limsup_{k\to\infty} L_{\nu_k}(x^k,\bar{y}^k)$$

In particular the $\{x^k, k = 1...\}$ and $\{y^k, k = 1...\}$ could have been those satisfying (1.18), and hence

$$L(\bar{x},y) \leq L(x,\bar{y})$$

which yields the saddle point property of $(\bar{x},\bar{y})$. This in turn yields the final assertions of the Proposition. □

Proposition 1.17 extends the results of T. Zolezzi [5, Theorem 4] about stability in mathematical programming. Many assumptions, such as compactness conditions on the feasible regions, can be ignored when one use this type of convergence rather than convergence notions that only involve the x variables.

2. EPI/HYPO-CONVERGENCE FROM A VARIATIONAL VIEWPOINT.

Let $\{F^\nu : X\times Y \longrightarrow \bar{R} = [-\infty, +\infty], \nu = 1,\dots\}$ be a sequence of bivariate functions, and for each ν, let (x_ν, y_ν) denote a saddle point of F^ν, i.e.

(2.1) $$F^\nu(x_\nu, y) \leq F^\nu(x_\nu, y_\nu) \leq F^\nu(x, y_\nu) \quad \text{for all } x \in X \text{ and } y \in Y.$$

We show that the convergence of saddle points and saddle values implicitly subsumes certain topological properties for the sequence $\{F^\nu, \nu = 1,\dots\}$ which lead naturally to the definition of epi/hypo-convergence.

Relation (2.1) yields estimates for x_ν and y_ν and hence also relative compactness properties for the sequence $\{(x_\nu, y_\nu), \nu = 1\dots\}$. Let us assume that for some topologies τ and σ, a subsequence $\{x_{\nu_k}, k = 1\dots\}$ τ-converge to $\bar{x}$ and $\{y_{\nu_k}, k = 1\dots\}$ σ-converge to $\bar{y} \in Y$. Neither τ nor σ need be given a priori, they could for example, be the result of some uniform coerciveness properties of the F^ν and compact embeddings. For any pair $(x,y) \in X\times Y$, not only does (2.1) hold but also

$$\sup_{v \in V} F^\nu(x_\nu, v) \leq \inf_{u \in U} F^\nu(u, y_\nu)$$

for all $U \in \mathcal{N}_\tau(x)$ and $V \in \mathcal{N}_\sigma(x)$ where $\mathcal{N}_\tau(x)$ and $\mathcal{N}_\sigma(y)$ are the τ- and σ-neighborhood systems of x and y respectively. Since $x_{\nu_k} \underset{\tau}{\longrightarrow} \bar{x}$ and $y_{\nu_k} \underset{\sigma}{\longrightarrow} \bar{y}$, for any pair $(U_{\bar{x}}, V_{\bar{y}}) \in \mathcal{N}_\tau(\bar{x}) \times \mathcal{N}_\sigma(\bar{y})$ and k large enough

$$x_{\nu_k} \in U_{\bar{x}} \quad \text{and} \quad y_{\nu_k} \in V_{\bar{y}}$$

and hence

(2.2) $$\inf_{u \in U_{\bar{x}}} \sup_{v \in V} F_{\nu_k}(u,v) \leq \sup_{v \in V_{\bar{y}}} \inf_{u \in U} F_{\nu_k}(u,v).$$

This holds for any convergent subsequence of the $\{(x_\nu, y_\nu), \nu = 1,\dots\}$ and since for any sequence of extended real-numbers $\{a_\nu, \nu = 1,\dots\}$

$$\inf_{\{\nu_k\}\subset\{1,\dots\}} \liminf_{k\to\infty} a_{\nu_k} = \liminf_{\nu\to\infty} a_\nu$$

and

$$\sup_{\{\nu_k\}\subset\{1,\dots\}} \limsup_{k\to\infty} a_{\nu_k} = \limsup_{\nu\to\infty} a_\nu$$

it follows that

$$\liminf_{\nu\to\infty} \inf_{u\in U_{\bar{x}}} \sup_{v\in V} F^\nu(u,v) \leq \limsup_{\nu\to\infty} \sup_{v\in V_{\bar{y}}} \inf_{u\in U} F^\nu(u,v), \tag{2.3}$$

which must hold for any pair $(\bar{x},\bar{y})$.

To extract as much information from (2.3) at the (local) pointwise level, we use the fact that the above holds for all $U\in\mathcal{N}_\tau(x)$, $U_{\bar{x}}\in\mathcal{N}_\tau(\bar{x})$, $V\in\mathcal{N}_\sigma(y)$ and $V_{\bar{y}}\in\mathcal{N}_\sigma(\bar{y})$ to take infs and sups with respect to these neighborhood systems. Since inf sup $\geq$ sup inf, and because the lim inf and lim sup that appear in (2.3) are monotone with respect to U and V as they decrease to x and y respectively, the sharpest inequality one can obtain at x and y is

$$\inf_{V\in\mathcal{N}_\sigma(y)} \sup_{U\in\mathcal{N}_\tau(\bar{x})} \liminf_{\nu\to\infty} \inf_{u\in U} \sup_{v\in V} F^\nu(u,v) \leq \sup_{U\in\mathcal{N}_\tau(x)} \inf_{V\in\mathcal{N}_\sigma(\bar{y})} \limsup_{\nu\to\infty} \sup_{v\in V} \inf_{u\in U} F^\nu(u,v). \tag{2.4}$$

The expression which appears on the left of the inequality is a function of $\bar{x}$ and y, the one on the right depends on x and $\bar{y}$. Let us denote them by h/e-li F^ν and e/h-ls F^ν respectively ; this notation to be justified later on. Rewriting (2.4), we see that whenever $\bar{x}$ and $\bar{y}$ are limit points of saddle points, then

$$\text{h/e-li } F^\nu(\bar{x},y) \leq \text{e/h-ls } F^\nu(x,\bar{y}) \tag{2.5}$$

for all $x \in X$ and $y \in Y$. In particular this implies that

$$\text{h/e-li } F^{\nu}(\bar{x},y) \leq \text{e/h-ls } F^{\nu}(\bar{x},\bar{y}) \text{ for all } y$$

and

$$\text{h/e-li } F^{\nu}(\bar{x},\bar{y}) \leq \text{e/h-ls } F^{\nu}(x,\bar{y}) \text{ for all } x.$$

Suppose $F' = \text{h/e-li } F^{\nu} = \text{e/h-ls } F^{\nu}$, then the preceeding inequalities imply that $(\bar{x},\bar{y})$ is a saddle point of F'. Since admittedly we seek a notion of convergence for bivariate functions that will yield the convergence of the saddle points to a saddle point of the limit function, the function F', if it exists, is a natural candidate. This is somewhat too restrictive and would exclude a large class of interesting applications. In fact any function F with the property that

(2.6) $$\text{e/h-ls } F^{\nu} \leq F \leq \text{h/e-li } F^{\nu}$$

will have the desired property, since then

$$F(\bar{x},y) \leq \text{h/e-li } F^{\nu}(\bar{x},y) \leq \text{e/h-ls } F^{\nu}(\bar{x},\bar{y}) \leq F(\bar{x},\bar{y})$$

and

$$F(\bar{x},\bar{y}) \leq \text{h/e-li } F^{\nu}(\bar{x},\bar{y}) \leq \text{e/h-ls } F^{\nu}(x,\bar{y}) \leq F(x,\bar{y})$$

for all $x \in X$ and $y \in Y$, i.e. $(\bar{x},\bar{y})$ is a saddle point of F.

We started with a collection of bivariate functions whose only property was to possess a (sub)sequence of convergent saddle points. If the limit of such a sequence is to be a saddle point of the limit function, we are led to certain conditions that must be satisfied by the limit function(s), and it is precisely these conditions that we shall use for the definition of epi/hypo-convergence.

We now review this at a somewhat more formal level. As we have seen, we need the two functions associated to the sequence $\{F^{\nu}, \nu = 1,\ldots\}$

$$\text{h/e-li } F^{\nu} = h_{\sigma}/e_{\tau}\text{-li } F^{\nu} = \text{hypo}_{\sigma}/\text{epi}_{\tau}\text{-}\liminf_{\nu\to\infty} F^{\nu}$$

$$\text{e/h - ls } F^{\nu} = e_{\tau}/h_{\sigma} - \text{ls } F^{\nu} = \text{epi}_{\tau}/\text{hypo}_{\sigma} - \limsup_{\nu\to\infty} F^{\nu}$$

with

$$h_\sigma/e_\tau\text{-li } F^\nu(x,y) = \tag{2.7}$$
$$\inf_{V \in \mathcal{N}_\sigma(y)} \sup_{U \in \mathcal{N}_\tau(x)} \liminf_{\nu\to\infty} \inf_{u \in U} \sup_{v \in V} F^\nu(u,v)$$

called the *hypo/epi-limit inferior*, and

$$e_\tau/h_\sigma\text{-ls } F^\nu(x,y) = \tag{2.8}$$
$$\sup_{U \in \mathcal{N}_\tau(x)} \inf_{V \in \mathcal{N}_\sigma(y)} \limsup_{\nu\to\infty} \sup_{v \in V} \inf_{u \in U} F^\nu(u,v)$$

called the *epi/hypo-limit superior*. The properties of these limit functions will be reviewed in the next Section.

A (bivariate) function F is said to be an *epi/hypo-limit* of the sequence $\{F^\nu, \nu = 1,\ldots\}$ if

$$e_\tau/h_\sigma\text{-ls } F^\nu \le F \le h_\sigma/e_\tau\text{-li } F^\nu. \tag{2.9}$$

Thus in general epi/hypo-limits are not unique, i.e. the topology induced by epi/hypo-convergence on the space of (bivariate) functions is not Hausdorff. This is intimately connected to the nature of saddle functions, as is again exemplified in Section 7.

As already suggested by our discussion, this is not the only type of convergence of bivariate functions that could be defined. In fact our two limit functions are just two among many possible limit functions introduced by De Giorgi [6] in a very general setting and called Γ-limits. In his notation

$$h_\sigma/e_\tau\text{-li } F^\nu(x,y) = \Gamma(N^-, \tau^-, \sigma^+) \lim_{\substack{\nu\to\infty \\ u\to x \\ v\to y}} F^\nu(u,v)$$

and

$$e_\tau/h_\sigma\text{-ls } F^\nu(x,y) = \Gamma(N^+, \sigma^+, \tau^-) \lim_{\substack{\nu\to\infty \\ v\to y \\ u\to x}} F^\nu(u,v)$$

(We have adopted a simplified notation because it carries important geometric information, cf. Section 3, that gets lost with the Γ-notation). It is however important to choose these two functions since, not only do they arise naturally from the convergence of saddle points, but in some sense they are the "minimal" pair, as made clear in Section 4 of [7] . Other definitions have been proposed by Cavazutti [8], [9], see also Sonntag [10], that imply epi/hypo-convergence, but unfortunately restrict somewhat the domain of applications.

Finally, observe that when the F^ν do not depend on y, then the definition of epi/hypo-convergence specializes to the classical definition of epi-convergence (with respect to the variable x). On the other hand if the F^ν do not depend on x, then epi/hypo-convergence is simply hypo-convergence. Thus, the theory contains both the theory of epi- and hypo-convergence.

The variational properties of epi/hypo-convergence, that motivated the definition, are formalized by the next Theorem.

2.10 THEOREM [7] . *Suppose* (X,τ) *and* (Y,σ) *are two topological spaces and* $\{F^\nu, \nu = 1,\dots\}$ *a sequence of bivariate functions, defined on* $X\times Y$ *and with values in the extended reals, that* $epi_\tau/hypo_\sigma$-*converge to a function* F. *Suppose that for some subsequence of functions* $\{F_{\nu_k}, k = 1,\dots\}$ *with saddle points* (x_k,y_k) i.e. *for all* $k = 1,\dots$

$$F_{\nu_k}(x_k,y) \le F_{\nu_k}(x_k,y_k) \le F_{\nu_k}(x,y_k),$$

the saddle points converge with $\bar{x} = \tau\text{-}\lim_{k\to\infty} x_k$ *and* $\bar{y} = \sigma\text{-}\lim_{k\to\infty} y_k$. *Then* $(\bar{x},\bar{y})$ *is a saddle point of* F *and*

$$F(\bar{x},\bar{y}) = \lim_{k\to\infty} F_{\nu_k}(x_k,y_k)$$

The second property which gives to this notion of convergence a great flexibility and renders it significant, when applied to variational problems, is its stability properties with respect to a large class of perturbations.

2.11 THEOREM. *Suppose* X,Y *and the* $\{F^\nu, \nu = 1,\ldots\}$ *are as in Theorem 2.10 with*

$$F = \text{epi}_\tau/\text{hypo}_\sigma\text{-lim}_{\nu\to\infty} F^\nu .$$

Then, for any continuous function $G : (X,\tau)\times(Y,\sigma) \longrightarrow R$,

$$F + G = \text{epi}_\tau/\text{hypo}_\sigma\text{-lim}_{\nu\to\infty}(F^\nu + G).$$

PROOF. Since G is continuous at (x,y), for every $\varepsilon > 0$ there exists $U_\varepsilon \in \mathcal{N}_\tau(x)$ and $V_\varepsilon \in \mathcal{N}_\sigma(y)$ such that for all $u \in U_\varepsilon$, $v \in V_\varepsilon$

$$G(x,y) - \varepsilon \leq G(u,v) \leq G(x,y) + \varepsilon$$

From this, it follows that

$$\text{e/h-ls}(F^\nu + G)(x,y)$$

$$= \sup_{U\subset U_\varepsilon} \inf_{V\subset V_\varepsilon} \limsup_{\nu\to\infty} \sup_{v\in V} \inf_{u\in U}(F^\nu + G)(u,v)$$

$$\geq \sup_{U\subset U_\varepsilon} \inf_{V\subset V_\varepsilon} \limsup_{\nu\to\infty} [\sup_{v\in V} \inf_{u\in U}(F_\nu(u,v) + G(x,y) - \varepsilon)]$$

$$\geq (\text{e/h-ls } F^\nu)(x,y) + G(x,y) - \varepsilon.$$

This holds for every $\varepsilon > 0$ and thus

$$\text{e/h-ls}(F^\nu + G) \geq (\text{e/h-ls } F^\nu) + G.$$

Again using the continuity of G, one shows similarly the converse inequality which thus yields

$$\text{e/h-ls}(F^\nu + G) = G + \text{e/h-ls } F^\nu .$$

The same arguments can be used to obtain the identity involving $\text{e/h-li}(F^\nu + G)$ and $\text{e/h-li } F^\nu$. Thus, if

$$\text{e/h-ls } F^\nu \leq F \leq \text{h/e-li } F^\nu$$

it implies that

$$\text{e/h-ls}(F^\nu + G) \leq F + G \leq \text{h/e-li}(F^\nu + G)$$

which is precisely what is meant by $F+G = \text{e/h-lim}(F^\nu + G)$. □

3. PROPERTIES OF EPI/HYPO-LIMITS. GEOMETRICAL INTERPRETATION.

In general, an arbitrary collection of saddle functions does not have an epi/hypo-limit, and when it does the limit is not necessarily unique. This all comes from the fact that, *in general, the two limit functions are not comparable*. For example, let $X = Y = R$ and for ν odd

$$F^{\nu}(x,y) = \begin{cases} y\,x^{-1} \text{ on } [0,1]\times[0,1]\setminus\{(0,0)\}\ , \\ \text{arbitrary when } (x,y) = (0,0), \\ -\infty \text{ if } x\in[0,1] \text{ and } y \notin [0,1], \\ +\infty \text{ otherwise,} \end{cases}$$

and for ν even, $F^{\nu} = 2\,F_1$. Then

$$\text{h/e-li } F^{\nu}(x,y) = y\,x^{-1} < 2\,y\,x^{-1} = \text{e/h-ls } F^{\nu}(x,y) \quad \text{on }]0,1]\times]0,1]$$

but

$$\text{h/e-li } F^{\nu}(0,0) = +\infty > \text{e/h-ls } F^{\nu}(0,0) = 0.$$

When a sequence of bivariate functions $\{F^{\nu},\ \nu = 1,\ldots\}$ epi-hypo-converges, its epi/hypo-limits form an interval

$$(3.1) \qquad [\text{e/h-ls } F^{\nu},\ \text{h/e-li } F^{\nu}] = \{F: X\times Y \to \bar{R} \mid \text{e/h-ls } F^{\nu} \le F \le \text{h/e-li } F^{\nu}\}$$

These two limit functions have semicontinuity properties that follow directly from the definition and the following general lemma [7, Lemma 4.30] .

3.2 LEMMA. Suppose (X,τ) is a topological space and q an extended real valued function defined on the subsets of X. Then the function

$$x \longmapsto \sup_{U\in\mathcal{N}_{\tau}(x)} q(U)$$

is τ-lower semicontinuous, and the function

$$x \longmapsto \inf_{U\in\mathcal{N}_{\tau}(x)} q(U)$$

is τ-upper semicontinuous.

PROOF. Simply note that for every x

$$g(x) = \sup_{U\in \mathcal{N}_\tau(x)} q(U) \leq \mathrm{cl}_\tau g(x) = \sup_{U\in \mathcal{N}_\tau(x)} \inf_{u\in U} g(u),$$

as follows from the definition of g, since

$$q(U) \leq \inf_{u\in U} g(u). \quad \square$$

3.3. PROPOSITION. *Suppose* $\{F^\nu : (X,\sigma)\times(Y,\tau) \to \overline{\mathbb{R}},\ \nu = 1,\ldots\}$ *is a sequence of bivariate functions. Then for all* y,

$$x \longmapsto e_\tau/h_\sigma\text{-ls } F^\nu(x,y)$$

is τ-*l.sc. in* x , *and for all* x

$$y \longmapsto h_\sigma/e_\tau\text{-li } F^\nu(x,y)$$

is σ-*u.sc. in* y.

One can also derive the semicontinuity properties of the limit functions from their geometrical interpretation as done in [7]

3.4. THEOREM. *Suppose* $\{F^\nu : (X,\sigma)\times(Y,\tau) \to \overline{\mathbb{R}},\ \nu = 1,\ldots\}$ *is a sequence of bivariate functions. Then for every* $y\in Y$ *and* $x\in X$

$$\mathrm{epi}(e/h\text{-ls } F^\nu)(\cdot,y) = \operatorname*{Lim\,inf}_{\substack{\nu\to\infty \\ y'\to y}} \mathrm{epi}\, F^\nu(\cdot,y'),$$

and

$$\mathrm{hypo}(h/e\text{-li } F^\nu)(x,\cdot) = \operatorname*{Lim\,inf}_{\substack{\nu\to\infty \\ x'\to x}} \mathrm{hypo}\, F^\nu(x',\cdot).$$

Thus the epi-hypo-convergence of a sequence of bivariate functions is a limit concept that involves both epi- and hypo-convergence. That is clearly at the origin of our terminology. However note that both formulas require that limits be taken with respect to both ν and either x or y, and can not be equated with the epi- or hypo-convergence of the univariate functions $F^{\nu}(.,y)$ and $F^{\nu}(x,.)$ respectively. It is a much weaker notion, more sophisticated, which does not allow the two variables x and y to be handled independently.

4. EPI/HYPO-CONVERGENCE : THE METRIZABLE CASE.

In the metric case, or more generally when (X,τ) and (Y,σ) are metrizable, it is possible to give a representation of the limit functions in terms of sequences that turn out to be very useful in verifying epi/hypo-convergence, cf. [7, Corollary 4.4] . The formulas that we give here in terms of sequence--rather than subsequence-- are new and thus complement those given earlier in [7, Theorem 4.10 and Corollary 4.14] .

4.1 THEOREM. *Suppose* (X,τ) *and* (Y,σ) *are two metrizable spaces, and* $\{F^\nu : X\times Y \to \bar{R},\ \nu = 1,\dots\}$ *a sequence of functions. Then for every* $(x,y) \in X\times Y$

$$(4.2)\qquad \text{e/h-ls } F^\nu(x,y) = \sup_{y_\nu \underset{\sigma}{\to} y} \min_{x_\nu \underset{\tau}{\to} x} \limsup_{\nu\to\infty} F^\nu(x_\nu,y_\nu),$$

$$= \sup_{\substack{\{\nu_k\}\subset N \\ y_k \underset{\sigma}{\to} y}} \min_{x_k \underset{\tau}{\to} x} \limsup_{k\to\infty} F^{\nu_k}(x_k,y_k),$$

and

$$(4.3)\qquad \text{h/e-li } F^\nu(x,y) = \inf_{x_\nu \underset{\tau}{\to} x} \max_{y_\nu \underset{\sigma}{\to} y} \liminf_{\nu\to\infty} F^\nu(x_\nu,y_\nu),$$

$$= \inf_{\substack{\{\nu_k\}\subset N \\ x_k \underset{\tau}{\to} x}} \max_{y_k \underset{\sigma}{\to} y} \liminf_{k\to\infty} F^{\nu_k}(x_k,y_k)$$

These characterizations of the limit functions yield directly the following criteria for epi/hypo-convergence.

4.4 COROLLARY. *Suppose* (X,τ) *and* (Y,σ) *are metrizable, and* $\{F^\nu : X\times Y \to \bar{R},\ \nu = 1,\dots\}$ *a sequence of functions. Then the following assertions are equivalent*

(4.5) $F = e_\tau/h_\sigma\text{-lim } F^\nu$

(4.6) (i) *For all* $y_\nu \xrightarrow[\sigma]{} y$, *there exists* $x_\nu \xrightarrow[\tau]{} x$ *such that*

$$\limsup_{\nu\to\infty} F^\nu(x_\nu,y_\nu) \le F(x,y),$$

and

(ii) *for all* $x_\nu \xrightarrow[\tau]{} x$, *there exists* $y_\nu \xrightarrow[\sigma]{} y$ *such that*

$$F(x,y) \le \liminf_{\nu\to\infty} F^\nu(x_\nu,y_\nu),$$

hold for all $(x,y) \in X\times Y$,

(4.7) (i) *for all* $\{\nu_k, k=1,\dots\} \subset N$, $y_k \xrightarrow[\sigma]{} y$ *there exists* $x_k \xrightarrow[\tau]{} x$ *such that* $\limsup_{k\to\infty} F^{\nu_k}(x_k,y_k) \le F(x,y)$

and

(ii) *for all* $\{\nu_k\} \subset N$, $x_k \xrightarrow[\tau]{} x$ *there exists* $y_k \xrightarrow[\sigma]{} y$ *such that* $F(x,y) \le \liminf_{k\to\infty} F^{\nu_k}(x_k,y_k)$

hold for all $(x,y) \in X\times Y$.

PROOF OF THEOREM 4.1. Since $e/h\text{-ls } F^\nu = -(h/e\text{-li}(-F^\nu))$ it clearly suffices to prove one of the identities (4.2) or (4.3), say (4.3). We denote by G and H the following functions

$$G(x,y) = \inf_{x_\nu \xrightarrow[\tau]{} x} \sup_{y_\nu \xrightarrow[\sigma]{} y} \liminf_{\nu\to\infty} F^\nu(x_\nu,y_\nu),$$

and

$$H(x,y) = \inf_{\substack{\{\nu_k\}\subset N\\ x_k \xrightarrow[\tau]{} x}} \sup_{y_k \xrightarrow[\sigma]{} y} \liminf_{k\to\infty} F^{\nu_k}(x_k,y_k)$$

Obviously $G \geq H$, thus to obtain (4.3) we only need to prove that

$$G \le h/e\text{-li } F^\nu \le H.$$

First, we show that $G \le h/e\text{-li } F^\nu$. There is nothing to prove if $h/e\text{-li } F^\nu \equiv +\infty$, so let us assume that for some pair (x,y), $h/e\text{-li } F^\nu(x,y) < \infty$. Given any $\beta > h/e\text{-li } F^\nu(x,y)$, the definition of h/e-li F^ν yields a neighborhood $V_\beta \in \mathcal{N}_\sigma(y)$ such that for all $U \in \mathcal{N}_\tau(x)$

$$\beta \geq \liminf_{\nu\to\infty} \inf_{u\in U} \sup_{v\in V_\beta} F^\nu(u,v).$$

Let $\{U_\mu,\ \mu=1,\ldots\}$ be a countable base of open neighborhoods of x, decreasing with μ to $\{x\}$. The preceeding inequality with U replaced by U_μ, implies the existence of a sequence $\{X_{\nu_\mu} \in U_\mu,\ \nu = 1,\ldots\}$ such that

$$\beta \geq \liminf_{\nu\to\infty} \sup_{v\in V_\beta} F^\nu(x_{\nu_\mu},v).$$

Since this holds for all μ, we get that

$$\beta \geq \limsup_{\mu\to\infty} \liminf_{\nu\to\infty} \sup_{v\in V_\beta} F^\nu(x_{\nu_\mu},v).$$

We now rely on the Diagonalization Lemma, proved in the Appendix, to obtain a sequence $\{x_\nu = x_{\nu,\mu(\nu)},\ \nu = 1,\ldots\}$ with $\nu \mapsto \mu(\nu)$ increasing (which implies that $x_\nu \xrightarrow[\tau]{} x$) such that

$$\beta \geq \liminf_{\nu\to\infty} \sup_{v\in V_\beta} F^\nu(x_\nu,v).$$

Now, for any sequence $y_\nu \xrightarrow[\sigma]{} y$, for ν sufficiently large $y_\nu \in V_\beta$ and hence

$$\beta \geq \liminf_{\nu\to\infty} F^\nu(x_\nu,y_\nu).$$

The above holds for any sequence $\{y_\nu,\ \nu = 1,\ldots\}$ σ-converging to y. Using this and the fact that the x_ν τ-converge to x we have that

$$\beta \geq \sup_{y_\nu \xrightarrow[\sigma]{} y} \liminf_{\nu\to\infty} F^\nu(x_\nu,y_\nu)$$

and also

$$\beta \geq \inf_{x_\nu \xrightarrow[\tau]{} x} \sup_{y_\nu \xrightarrow[\sigma]{} y} \liminf_{\nu\to\infty} F^\nu(x_\nu,y_\nu) = G(x,y).$$

Since this holds for every $\beta <$ h/e-li $F^\nu(x,y)$ we get that h/e-li $F^\nu \geq G$.

Next we show that $H \geq$ h/e-li F^ν. Again there is nothing to prove if h/e-li $F^\nu \equiv -\infty$, so let us assume that for some (x,y), h/e-li $F^\nu(x,y) > -\infty$. The definition of h/e-li F^ν implies that given any $\alpha < h/e\text{-}li\ F^{\nu}(x,y)$ and any $V \in \mathcal{N}_{\sigma}(y)$ there corresponds a neighborhood $U = U_{\alpha,V}$ of x such that

$$\alpha < \liminf_{\nu\to\infty} \inf_{u\in U} \sup_{v\in V} F^{\nu}(u,v).$$

Let $\{V_{\mu},\ \mu=1,\dots\}$ be a countable base of open neighborhoods of y, decreasing with μ to $\{y\}$. To any such V_{μ}, there corresponds U_{μ} with

$$\alpha < \liminf_{\nu\to\infty} \inf_{u\in U_{\mu}} \sup_{v\in V_{\mu}} F_{\nu}(u,v)$$

For any subsequence $\{\nu_k,\ k=1,\dots\}$ and any $x_k \xrightarrow[\tau]{} x$

$$\alpha < \liminf_{k\to\infty} \sup_{v\in V_{\mu}} F_{\nu_k}(x_k,v)$$

because for k sufficiently large $x_k \in U_{\mu}$ and $\liminf_{\nu\to\infty} \leq \liminf_{\nu_k\to\infty}$.

This implies the existence of a sequence $\{y_{k\mu},\ k=1,\dots\}$ such that

$$\alpha < \liminf_{k\to\infty} F_{\nu_k}(x_k,y_{k,\mu}).$$

This being true for any μ, we get

$$\alpha \leq \liminf_{\mu\to\infty} \liminf_{k\to\infty} F_{\nu_k}(x_k,y_{k\mu})$$

This and the Diagonalization Lemma A.1 of [7, Appendix] yields a sequence $\{y_k = y_{k,\mu(k)} \in V_k,\ k=1\dots\}$ such that

$$\alpha \leq \liminf_{k\to\infty} F_{\nu_k}(x_k,y_k)$$

and hence

$$\alpha \leq \sup_{y_k \xrightarrow[\sigma]{} y} \liminf_{k\to\infty} F_{\nu_k}(x_k,y_k)$$

Since this holds for any subsequence $\{\nu_k,\ k=1,\dots\}$ and $x_k \xrightarrow[\tau]{} x$, $\alpha \leq H(x,y)$. This being true for any $\alpha < h/e\text{-}li\ F^{\nu}$, we finally get $h/e\text{-}li\ F^{\nu} \leq H$. □

In the metrizable setting it is also possible to characterize the epi/hypo-convergence in terms of the *Moreau-Yosida approximates* [7, Section 5] . Here we review briefly the main results

4.8 DEFINITION. *Let* (X,τ) *and* (Y,σ) *be metrizable, and* d_τ *and* d_σ *metrics compatible with* τ *and* σ *respectively ; and* $F: X\times Y \to \overline{R}$ *a bivariate function. For* $\lambda > 0$ *and* $\mu > 0$, *the lower Moreau-Yosida approximate (with parameters* λ *and* μ*) is*

$$F^{\downarrow}(\lambda,\mu,x,y) = \sup_{v\in Y} \inf_{u\in X}\left[F(u,v) + \frac{1}{2\lambda}d_\tau^2\ (u,x) - \frac{1}{2\mu}d_\sigma^2\ (v,y)\right]$$

and the upper Moreau-Yosida approximate (with parameters λ *and* μ*) is*

$$F^{\uparrow}(\lambda,\mu,x,y) = \inf_{u\in X} \sup_{v\in Y}\left[F(u,v) + \frac{1}{2\lambda}d_\tau^2\ (u,x) - \frac{1}{2\mu}d_\sigma^2\ (v,y)\right]$$

4.9 THEOREM. *Suppose* $\{F^\nu,\ \nu = 1,\ldots\}$ *is a sequence of extended-real valued bivariate functions defined on the product of the metric spaces* (X,d_τ) *and* (Y,d_σ) . *Suppose there exists* $r > 0$ *and some pair* $(u_o,v_o) \in X\times Y$ *such that* $F^\nu(u_o,v) \le r[d_\sigma^2(v,v_o)+1]$ *and* $F^\nu(u,v) \ge -\ r[d_\tau^2(u,u_o) + d_\sigma^2(v,v_o)+1]$ *for all* $\nu = 1,\ldots$. *Then*

$$\text{e/h-ls } F^\nu(x,y) = \sup_{\lambda>0} \inf_{\mu>0} \limsup_{\nu\to\infty} F_\nu^{\downarrow}(\lambda,\mu,x,y).$$

If there exist r *and* (u_o,v_o) *such that for all* $\nu = 1,\ldots$
$F^\nu(u,v_o) \ge -\ r[d_\tau^2(u,u_o)+1]$ *and* $F^\nu(u,v) \le +\ r[d_\tau^2(u,u_o)+ d_\sigma^2(v,v_o)+1]$,
then

$$\text{h/e-li } F^\nu(x,y) = \inf_{\mu>0} \sup_{\lambda>0} \liminf_{\nu\to\infty} F_\nu^{\uparrow}(\lambda,\mu,x,y).$$

The Moreau-Yosida approximates [7, Theorem 5.8] are locally equi-Lipschitz, at least when the bivariate functions F^ν can be minorized/majorized as in Theorem 4.9. This is a very useful property ; it allows us to work with well-behaved functions. Moreover, when expressed in terms of the Moreau-Yosida approximates, the epi/hypo-convergence reduces to pointwise limit operations.

5. SEQUENTIAL COMPACTNESS.

The fact that any sequence of bivariate functions, at least in the metrizable case, possesses an epi/hypo-convergent subsequence plays an important role in many applications. One relies on this compactness result to assert the existence of an epi/hypo-limit of a subsequence, then use the specific properties of the elements of the sequence to identify the limit function and finally obtain the epi/hypo-convergence of the whole sequence. In [7], the proof of this compactness theorem is obtain with the help of the Moreau-Yosida approximates and the identities that appear in Theorem 4.9. The proof given here follows the more standard techniques of De Giorgi and Franzoni [11], that such an argument might work was suggested to us by Cavazzuti.

5.1. THEOREM. *Suppose* (X,τ) *and* (Y,σ) *are topological spaces with countable base. Then any sequence of bivariate functions* $\{F^\nu : X \times Y \to \bar{R},\ \nu = 1,\ldots\}$ *contains a subsequence which is* epi_τ/ $hypo_\sigma$*-convergent.*

PROOF. We have to find a subsequence $\{\nu_k,\ k=1,\ldots\}$ such that

$$\text{e/h-ls } F^{\nu_k} \leq \text{h/e-li } F^{\nu_k}$$

Let $\{U_\mu | \mu=1,\ldots\}$ and $\{V_{\mu'} | \mu'=1,\ldots\}$ a countable sequence of open sets in X and Y resp.. From the compactness of $\bar{R} = [-\infty, +\infty]$ and the classical diagonalization lemma, follows the existence of a subsequence $\{\nu_k | k=1,\ldots\}$ such that for every μ and μ'

$$\lim_{k\to 0} \inf_{u \in U_\mu} \sup_{v \in V_{\mu'}} F^{\nu_k}(u,v)$$

and

$$\lim_{k\to 0} \sup_{v \in V_{\mu'}} \inf_{u \in U_\mu} F^{\nu_k}(u,v)$$

exist. It follows that for every μ and μ'

$$\limsup_{k\to\infty} \sup_{v\in V_{\mu'}} \inf_{u\in U_\mu} F^{\nu_k}(u,v) \leq \liminf_{k\to\infty} \inf_{u\in U_\mu} \sup_{v\in V_{\mu'}} F^{\nu_k}(u,v).$$

Hence, for every x and y,

$$\sup_{U_\mu\in \mathcal{N}_\tau(x)} \inf_{V_{\mu'}\in \mathcal{N}_\sigma(y)} \limsup_{k\to\infty} \sup_{v\in V_{\mu'}} \inf_{u\in U_\mu} F^{\nu_k}(u,v)$$

$$\leq \inf_{V_{\mu'}\in \mathcal{N}_\sigma(y)} \sup_{U_\mu\in \mathcal{N}_\tau(x)} \liminf_{k\to\infty} \inf_{u\in U_\mu} \sup_{v\in V_{\mu'}} F^{\nu_k}(u,v)$$

which is the desired inequality. □

6. RELATED NOTIONS TO EPI/HYPO-CONVERGENCE.

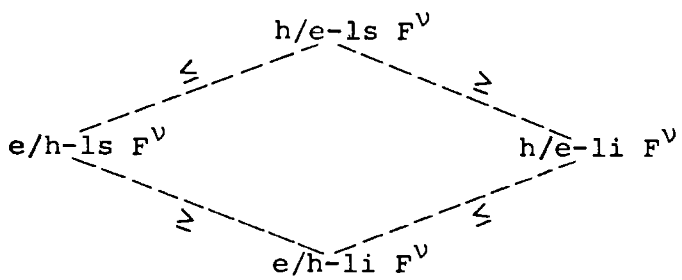
Up to now, we have been motivated by the search for a *minimal convergence concept* that allows us to obtain the convergence of saddle points and saddle values, cf. Theorem 2.10. This has led us to a notion of convergence whose limit is not necessarily unique. This is not unexpected, since bivariate functions are not completely determined by their saddle value properties, as already observed by Rockafellar [12] in his work on duality. In the convex-concave setting we formalize this by introducing equivalence classes. The definition of epi/hypo-convergence makes the two variables x and y play a symmetric role, it specializes to epi- or hypo-convergence when the functions are univariate. However, in some applications the F^ν enjoy some continuity properties and it is possible to work with stronger notions of convergence. We exemplify this by giving one such possibility. We proceed as before and start with the definition of two limit functions :

(6.1) $$h_\sigma/e_\tau\text{-ls } F^\nu(x,y) = \inf_{V\in\mathcal{N}_\sigma(y)} \sup_{U\in\mathcal{N}_\tau(x)} \limsup_{\nu\to\infty} \inf_{u\in U} \sup_{v\in V} F^\nu(u,v)$$

and

(6.2) $$e_\tau/h_\sigma\text{-li } F^\nu(x,y) = \sup_{U\in\mathcal{N}_\tau(x)} \inf_{V\in\mathcal{N}_\sigma(y)} \liminf_{\nu\to\infty} \sup_{v\in V} \inf_{u\in U} F^\nu(u,v)$$

We have the following relations :



Thus :

$$e/h\text{-}ls\ F^{\nu} = e/h\text{-}li\ F^{\nu}$$

$$h/e\text{-}li\ F^{\nu} = h/e\text{-}ls\ F^{\nu}$$

and, a fortiori,

$$h/e\text{-}ls\ F^{\nu} = e/h\text{-}li\ F^{\nu}$$

imply each epi/hypo-convergence. The convergence induced by the equality $e/h\text{-}ls\ F^{\nu} = e/h\text{-}li\ F^{\nu}$, now with unique limit (τ-l.sc. with respect to x), has been studied by Cavazzuti [8] [9]. The study of the convergence induced by the last equality $h/e\text{-}ls\ F^{\nu} = e/h\text{-}li\ F^{\nu}$, has also been sketched out in [7]. It is possible, for all of these, to develop a theory similar to that for epi/hypo-convergence, but each one of these notions requires a certain regularity for the limit function which, a priori, cannot be guaranteed in many applications.

7. EPI/HYPO-CONVERGENCE OF CONVEX-CONCAVE FUNCTIONS.

This last section is devoted to the continuity properties of the Legendre-Fenchel transform, which establishes a natural correspondence between convex and convex-concave bivariate functions. The argumentation is surprisingly complex, in part this comes from the fact that the functions can take on both the values $+\infty$ and $-\infty$, and that the conjugate operation, or equivalently the Legendre-Fenchel transformation, then looses its local characteristics and it is only the global properties of the operation that are preserved. An elegant study of this phenomena and its implications has been made by Rockafellar [12], [13] and [14] and further analysed by McLinden [15], [16] ; see also Ekeland-Temam [17] and Aubin [18].

Let * denote the *conjugate operation*. For any $F : X \to \bar{R}$ the *conjugate function* is defined by

$$F^*(x^*) = \sup_{x \in X} [\langle x^*;x\rangle - F(x)] \ .$$

Then one can show [12] that for convex functions

$$F^{**} = \underline{\text{cl}}\ F,$$

where $\underline{\text{cl}}$ F is the *extended closure* of F with

$$\underline{\text{cl}}\ F(x) = \begin{cases} \text{cl } F(x) & \text{if cl } F > -\infty \\ -\infty & \text{otherwise} \end{cases}$$

and cl F is the lower semicontinuous closure of F.

Convex-concave bivariate functions can be related to convex bivariate functions through partial conjugation, which means conjugation with respect to one of the variables. We are led to introduce equivalence classes. For the sake of the uninformed reader we review quickly the motivation and the main features of Rockafellar's scheme [13].

Let K_o be convex-concave continuous function on $[-1,1] \times [-1,1]$. We associate to K_o the two functions :

$$K_1(x,y) = \begin{cases} +\infty \text{ if } |x| > 1 \\ K_o(x,y) \text{ on } [-1,1]\times[-1,1] \\ -\infty \text{ if } |x| \leq 1 \text{ and } |y| > 1 \end{cases}$$

and

$$K_2(x,y) = \begin{cases} +\infty \text{ if } |x| > 1 \text{ and } |y| \geq 1 \\ K_o(x,y) \text{ on } [-1,1]\times[-1,1] \\ -\infty \text{ if } |y| > 1. \end{cases}$$

Then both K_1 and K_2 have the same saddle points (and values) as K_o, although they differ on substantial portions of the plane. However, not only do these two functions have the same saddle points but so do all linear perturbations of these two functions. So from a variational viewpoint these two functions appear to be undistinguishable. It is thus natural when studying limits of a variational character that we need to deal with equivalence classes whose members have similar saddle point properties.

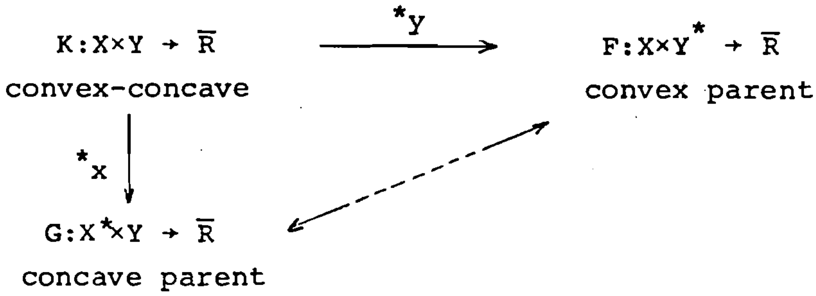
Let $K : X\times Y \to \overline{R}$ be a convex-concave function. We associate to K its convex and concave parents defined by

$$F(x,y^*) = \sup_{y \in Y} [K(x,y) + \langle y^*,y\rangle]$$

and

$$G(x^*,y) = \inf_{x \in X} [K(x,y) - \langle x^*,x\rangle].$$

Thus we have the following relations between these functions



In the example above K_1 and K_2 have the same parents, they cannot be distinguished as coming from different bivariate convex or bivariate concave functions.

Given any pair of convex-concave bivariate functions K_1 and K_2, we say that they are *equivalent* if they have the same parents. A bivariate function K is said to be *closed* if its parents are conjugates of each other, i.e. if the above diagram can be closed through the classical Legendre-Fenchel transform :

$$-G(x^*,y) = \sup_{\substack{x \in X \\ y^* \in Y^*}} \left[\langle x,x^*\rangle + \langle y,y^*\rangle - F(x,y^*)\right]$$

For closed convex-concave functions K, the associated equivalence class is an interval, denoted by $[\underline{K},\overline{K}]$ with

$$\underline{K} = \underline{cl}_x K = \sup_{x^* \in X} \left[G(x^*,y) + \langle x,x^*\rangle\right] ,$$

and

$$\overline{K} = \overline{cl}_y K = \inf_{y^* \in Y} \left[F(x,y^*) - \langle y,y^*\rangle\right],$$

where $\underline{cl}_x$ denotes the extended lower closure with respect to x and $\overline{cl}_y\, K(= - \underline{cl}_y(-K))$ is the extended upper closure with respect to y. A convex function $F:X\times Y^* \to \overline{R}$ is *closed* if $\underline{cl}_{(x,y^*)}F = F$.

7.1 THEOREM. [13] .*The map* $K \xmapsto{*y} F$ *establishes a one-to-one correspondence between closed convex-concave (equivalence) classes and closed convex functions.*

This correspondence $*y$ has continuity properties that are made explicit here below. Given a sequence of convex bivariate functions $\{F^\nu,\ \nu = 1,\ldots\}$ epi_τ-converging to F, we could study the induced convergence for the associated convex-concave (bivariate) functions (through the Legendre-Fenchel transform $^*y$).

In the reflexive Banach case, it would be natural to consider epi_τ-convergence to be epi-convergence induced by the weak and the strong topologies on $X \times Y$. To illustrate these type of results, we consider the situation when the $\{F^\nu,\ \nu = 1,\ldots\}$ epi-converge to F with respect to the Mosco-topology.

We start with a quick review of Mosco-epi-convergence or for short, Mosco-convergence. Suppose X and Y are reflexive Banach spaces whose weak and strong topologies are denoted by w_X, w_Y, s_X, s_Y respectively. A sequence of functions, which for reasons of exposition we take here as bivariate,

$$\{F^\nu: X \times Y \to \bar{R},\ \nu = 1,\ldots\}$$

is said to *Mosco-converge* to the (bivariate) function

$$F : X \times Y \to \bar{R}$$

if

$$e_s\text{-ls } F^\nu \leq F \leq e_w\text{-li } F^\nu$$

where

$$e_s\text{-ls } F^\nu(x,y) = e_{s_X \times s_Y}\text{-ls } F^\nu(x,y)$$

$$= \sup_{(U,V) \in \mathcal{N}_{s_X}(x) \times \mathcal{N}_{s_X}(y)} \limsup_{\nu\to\infty} \inf_{u \in U, v \in V} F^\nu(u,v)$$

and

$$e_w\text{-li } F^\nu(x,y) = e_{w_X \times w_Y}\text{-li } F^\nu(x,y)$$

$$= \sup_{(U,V) \in \mathcal{N}_{w_X}(x) \times \mathcal{N}_{w_Y}(y)} \liminf_{\nu\to\infty} \inf_{u \in U, v \in V} F^\nu(u,v).$$

Because of the natural relations between epi-limits, this means that

$$F = e_w\text{-li } F^\nu = e_s\text{-li } F^\nu = e_w\text{-ls } F^\nu = e_s\text{-ls } F^\nu.$$

This type of convergence has been introduced by Mosco [19] and studied extensively because of the role it plays in many applications, cf. for example [19], [20] and [10]. The basic result for convex functions first proved by Wijsman [21] in the finite dimensional case and extended by Mosco and Joly [22] to the Banach reflexive case is the bicontinuity of the Legendre-Fenchel transform with respect to the Mosco-topology. In our setting, we can express this result through the identity

(7.2) $$(e_w\text{-li } F^\nu)^* = e_s\text{-ls } F^{\nu *}$$

where * denotes conjugation with respect to both variables. This is a special case of the more general relation that we need between convex bivariate functions and classes of convex-concave bivariate functions. We only sketch the proof whose details appear in [23].

7.3. THEOREM. X *and* Y *are reflexive Banach spaces, and* $\{F; F^\nu : X \times Y \to \overline{R},\ \nu = 1,\ldots\}$ *is a collection of closed proper convex functions. Let* $[\overline{K}, \underline{K}]$ *and* $\{[\overline{K}^\nu, \underline{K}^\nu],\ \nu = 1,\ldots\}$ *be the corresponding classes of bivariate convex-concave functions. Then, the following statements are equivalent :*

(7.4) *the* F^ν *Mosco-converge to* F

and

(7.5) *for all* K *in* $[\overline{K}, \underline{K}] \neq \emptyset$ *, we have*

$$\underline{\mathrm{cl}}_x(e_s/h_w\text{-ls } K^\nu) \le K \le \overline{\mathrm{cl}}^y(h_s/e_w\text{-li } K^\nu).$$

PROOF (Sketch). The key step consists in extending the result about the bicontinuity of the Legendre-Fenchel transform for convex functions to this setting, i.e. for bivariate convex functions and partial conjugation. In particular one shows that if the F and F^ν are a collection of bivariate closed proper convex functions, and $F = e_{w(X\times Y^*)}\text{-lim } F^\nu$ then

(7.6) $\quad \overline{K} = \overline{cl}_y(h_s/e_w\text{-li } \overline{K}^\nu)$

and

(7.7) $\quad \underline{K} = \underline{cl}_x(h_s/e_w\text{-li } \overline{K}^\nu)$

The proof of the inequality $\overline{K} \geq \overline{cl}_y(h_s/e_w\text{-li } \overline{K}^\nu)$ follows directly from the definitions of epi- and epi/hypo-convergence and the Legendre-Fenchel transform. The converse inequality is much more difficult to obtain. One start with deriving

(7.8) $\quad \underline{K} \leq h_s/e_w\text{-li } \overline{K}^\nu,$

which is first obtained under the additional condition that the F and F^ν , $\nu = 1,\ldots$ are equi-coercive. To bring the general problem in this more restrictive framework, we rely on Moreau-Yosida approximates. The F^ν are replaced by

$$F^{\nu,\lambda}(x,y^*) = F^\nu(x,y^*) + \frac{\lambda}{2}|y^*|^2$$

and F by

$$F^\lambda(x,y^*) = F(x,y^*) + \frac{\lambda}{2}|y^*|^2 .$$

For each $\lambda > 0$, we then have

$$\overline{K}^\lambda(x,y) \leq h_s/e_w\text{-li } \overline{K}^{\nu,\lambda}$$

We then use the monotonocity in λ, and a diagonalization lemma [7, Lemma A.1] to conclude that

$$\overline{K}(x,y) \leq \liminf_{\nu\to\infty} \overline{K}^{\nu,\lambda(\nu)}(x_\nu,y)$$

for all $x_\nu \xrightarrow[w]{} x$ and $\lambda(\nu)$ some subsequence converging to 0 as $\nu \to \infty$. Using the properties of conjugation this allows us to obtain the existence of $\{y_\nu,\ \nu = 1,\ldots\}$ such that

$$\overline{K}(x,y) \leq \liminf \left[\overline{K}^\nu(x_\nu,y_\nu) - \frac{1}{2\lambda(\nu)}|y - y_\nu|^2\right].$$

There remains only to show that the sequence $\{\overline{K}^{\nu}(x_{\nu},y_{\nu}),\ \nu = 1,\ldots\}$ is bounded above. This is equivalent to the assertion that for every weakly convergent sequence $\{x_{\nu},\ \nu = 1,\ldots\}$ there exists a bounded sequence $\{y^{*}_{\nu},\ \nu = 1,\ldots\}$ such that $\sup_{\nu} F^{\nu}(x_{\nu},y^{*}_{\nu}) < +\infty$. This would impose a very strong restriction on the sequence $\{F^{\nu},\ \nu = 1,\ldots\}$. To avoid imposing any such condition, rather than working with the F^{ν} and F, we work with the F^{ν}_{μ} and F_{μ} that are the Moreau-Yosida approximates with respect to x of the F^{ν} and F respectively, i.e.

$$F^{\nu}_{\mu}(x,y^{*}) = \inf_{u\in X}\left[F^{\nu}(u,y^{*}) + \frac{1}{2\mu}|u - x|^{2}\right]$$

and

$$F_{\mu}(x,y^{*}) = \inf_{u\in X}\left[F(u,y^{*}) + \frac{1}{2\mu}|u - x|^{2}\right]$$

The desired inequality is then obtained for the $\overline{K}_{\mu}$ and $\overline{K}^{\nu}_{\mu}$, the (partial) Legendre-Fenchel transforms of the F_{μ} and F^{ν}_{μ}. It is then shown that

$$\sup_{\mu>0} \overline{K}_{\mu} = \underline{cl}_{x}\ \overline{cl}_{y}\ K = \underline{K}$$

which allows us to obtain (7.8) without any coercivity restrictions on the function F^{ν} and F.

We now return to the core of the proof of the Theorem. To say that the $\{F^{\nu},\ \nu = 1,\ldots\}$ Mosco-converge to F means that

$$e_{s}\text{-ls } F^{\nu} \le F \le e_{w}\text{-li } F^{\nu}$$

The second inequality, through (7.6) yields

(7.9) $$\overline{K} \le \overline{cl}_{y}(h_{s}/e_{w}\text{-li } \overline{K}^{\nu}).$$

The first inequality yields

$$\underline{K} \le \overline{cl}_{x}(e_{w}/h_{s}\text{-ls } \underline{K}^{\nu})$$

through (7.7) and using this time the following arguments : the inequality $e_s\text{-ls } F^\nu \le F$ implies

$$e_w\text{-li}(-G^\nu) \ge -G$$

with G the concave conjugate of F as identified in the diagram above ; and then G is used in the construction of $\underline{K}$ (and G^ν for $\underline{K}^\nu$). The reasoning is totally symmetric.

There remains to show that (7.5) implies (7.4). By exploiting duality and the fact that we are working with closed convex-concave functions, one can prove that it really will suffice to show that for all $\{(x_\nu,y^*_\nu),\ \nu = 1,\ldots\}$ that weakly converge to (x,y^*) we have

$$\text{(7.10)} \qquad F(x,y^*) \le \liminf_{\nu\to\infty} F^\nu(x_\nu,y^*_\nu).$$

Since $K \le \overline{\text{cl}}^Y(h_s/e_w\text{-li } \overline{K}^\nu)$, we have that for every (x,y^*)

$$\begin{aligned} F(x,y^*) &= \sup_{y\in Y}\Big[\langle y^*,y\rangle + K(x,y)\Big] \\ &\le \sup_{y\in Y}\Big[\langle y^*,y\rangle + \overline{\text{cl}}^Y(h_s/e_w\text{-li } \overline{K}^\nu)\Big] \\ &\le \sup_{y\in Y}\Big[\langle y^*,y\rangle + h_s/e_w\text{-li } \overline{K}^\nu\Big]. \end{aligned}$$

Thus to prove (7.10) it suffices to show that for any sequence $(x_\nu,y^*_\nu) \xrightarrow[w]{} (x,y^*)$ and any $y\in Y$

$$\langle y^*,y\rangle + h_s/e_w\text{-li } \overline{K}^\nu(x,y) \le \liminf_{\nu\to\infty} F^\nu(x_\nu,y^*_\nu).$$

Using the definition of epi/hypo-convergence, in particular of $h_s/e_w\text{-li } \overline{K}^\nu$, we see that to each $x_\nu \xrightarrow[w]{} x$ we can associate a strongly convergent sequence $y_\nu \xrightarrow[s]{} y$ such that

$$\langle y^*,y\rangle + h_s/e_w\text{-li } \overline{K}^\nu(x,y) \le \langle y^*,y\rangle + \liminf_{\nu\to\infty} \overline{K}^\nu(x_\nu,y_\nu)$$

and thus we only need to show that

$$\langle y^*,y\rangle + \liminf_{\nu\to\infty} \overline{K}^\nu(x_\nu,y_\nu) \le \liminf_{\nu\to\infty} F^\nu(x_\nu,y^*_\nu).$$

But this follows directly from the relation

$$F^{\nu}(x_{\nu},y_{\nu}^{*}) \;=\; \sup_{y \in Y}\{\langle y_{\nu}^{*},y\rangle + \overline{K}^{\nu}(x_{\nu},y)\}$$

$$\geq\; \langle y_{\nu}^{*},y_{\nu}\rangle + \overline{K}^{\nu}(x_{\nu},y_{\nu}).\quad \square$$

APPENDIX.

We give here the proof that we take from [24] , of a diagonalization result used in the proof of Theorem 4.1.

A.1. LEMMA (Diagonalization). Suppose $\{a_{\nu,\mu} | \nu = 1,\dots\}$ is a doubly indexed family of extended-reals numbers. Then there exists a map $\nu \longrightarrow \mu(\nu)$ increasing such that

(A.2) $$\limsup_{\mu\to\infty} \liminf_{\nu\to\infty} a_{\nu,\mu} \geq \liminf_{\nu\to\infty} a_{\nu,\mu(\nu)}$$

PROOF. Let us denote $a_\mu = \liminf_{\nu\to+\infty} a_{\nu,\mu}$ and $a = \limsup_{\mu\to+\infty} a_\mu$

If $a = +\infty$ there is nothing to prove. So, let us assume that $a < +\infty$. By definition of a, there exists an increasing sequence $\{\mu_p ; p = 1,2\dots\}$, $\mu_p \xrightarrow[p\to+\infty]{} +\infty$ such that

$$\sup(-p, a + 2^{-p}) \geq a_\mu \quad \text{for all } \mu \geq \mu_p$$

By definition of a_μ, there exists an increasing sequence $\{\nu_p ; p = 1,2\dots\}$, $\nu_p \xrightarrow[p\to+\infty]{} +\infty$ such that

$$\sup(-p, a_{\mu_p} + 2^{-p}) \geq a_{\nu_p,\mu_p} \quad \text{for all } p \in \mathbb{N}.$$

Let us define $\mu(\nu) = \mu_p$ if $\nu_p \leq \nu \leq \nu_{p+1}$

Then, $\liminf_{\nu\to+\infty} a_{\nu,\mu(\nu)} \leq \liminf_{p\to+\infty} a_{\nu_p,\mu(\nu_p)} = \liminf_{p\to+\infty} a_{\nu_p,\mu_p}$, as follows from the definition of $\mu(\nu_p) = \mu_p$.

From the two above inequalities, we derive

$$\liminf_{\nu\to+\infty} a_{\nu,\mu(\nu)} \leq \liminf_{p\to+\infty} \left[\sup(-p, \sup(-p, a+2^{-p}) + 2^{-p})\right]$$

$$\leq a \quad . \square$$

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