FUZZY GAMES: THE STATIC AND DYNAMICAL POINTS OF VIEW

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SUMMARY

A locally Lipschitz cooperative generalized game is described by its coalition worth function v defined on the set $[0,1]^n$ of generalized (or fuzzy) coalitions of n players. We assume that v is positively homogeneous and locally Lipschitz. We propose the Clarke's generalized gradient $\partial v(c^N)$ of v at the coalition $c^N = (1,\ldots,1)$ of all players as a set of solutions, and we study its property. We point out that it coincides with the core when v is super-additive and to the Shapley value when v is smooth. We also represent cooperative fuzzy games as "action games", for which we define and prove a concept of equilibrium.

Introduction

We show in this paper how concepts of fuzzy sets and generalized gradients as well as viability theory allow to treat, in a unified way, several competing concepts of cooperative game theory and how to devise new models (called action games) which are dynamical and explain the formation of coalitions.

Many concepts of solutions to a game with side-payments have been proposed: among them, the core and the Shapley value, which yield different outcomes. Many efforts have been made to obtain situations where some of these concepts coincide. Let us mention for instance the replicating procedure introduced by Debreu-Scarf (1963) and Shapley (1953) and the use of continuum of players introduced by Aumann (1969). [See the books of Aumann-Shapley (1979) and Hildenbrand (1974) for further references.]

In Aubin (1974a,b), we proposed the framework of 'fuzzy games' (games defined on a 'continuum of coalitions') for defining and comparing these concepts [see for instance Aubin (1979b, chs. 10, 11, 12)]. This paper deals with the same framework of fuzzy games.

For games with side-payments we propose the generalized gradient ∂v of the coalition worth function v as a set of solutions to a locally Lipschitz game. It can be regarded as the subset of 'marginal gains' that the players receive when they join the coalition of all players.

We do not claim that this is a 'good' concept of solution: we only point out that it 'unifies' competing concepts of solutions.

We characterize this set of solutions in several instances: it is the core when the game is super-additive, the generalized Shapley value when the game is smooth. We characterize (some) solutions when v arises from a game described in 'strategic' (or 'normal') form.

What about the usual games? We proposed a single concept of solution. Still, there are several ways to extend a usual game w into a generalized game πw. Each extension procedure π yields a set of solutions $\partial(\pi v)(c^N)$ that depends upon the choice of π. So, the diversity of these solution concepts results from the different ways by which a usual game is transformed to a generalized game.

We shall observe that this concept of solution does not explain the formation of coalitions. Then, the second purpose of the paper is devoted to a radically different way for modeling cooperative fuzzy games as "action games".

We assume that players act on the environment by transforming it and that we know the law of transformation of each fuzzy coalition of players. We are looking for equilibria, i.e., a fuzzy coalition $\bar{c}$ and states $\bar{x}$ of the environment which are invariant by the action of the fuzzy coalition $\bar{c}$. We prove a theorem of existence of an equilibrium of such a game, an equilibrium which is the stationary set of a dynamical system in which coalitions of players can be regarded as regulation controls.

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Fuzzy Coalitions

We denote by N the set of the n players. Cooperative games are those games that involve the behavior of "coalitions", regarded as subsets S of the "grand coalition" N. Cooperative fuzzy games, consequently, do involue fuzzy coalitions, regarded as fuzzy subsets of N.

Besides the usual benefits gained at using fuzzy subsets, we have to mention that it is also technically advantageous since we "convexify" in some sense the discrete set $\mathcal{P}(N)$ (of subsets of N) and thus, use the results of analysis.

We recall that we identify the set $\mathcal{P}(N)$ with the subset $\{0,1\}^n$ of characteristic functions c_S of subsets S defined by

(1) $c_S(i) = 1$ when $i \in S$ and $c_S(i) = 0$ when $i \notin S$.

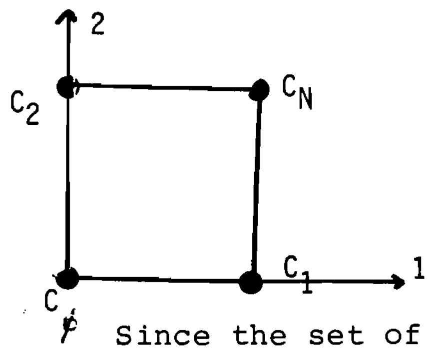
Since $\{0,1\}^n$ is a subset of R^n, we can take its convex hull, which is the cube $[0,1]^n$.

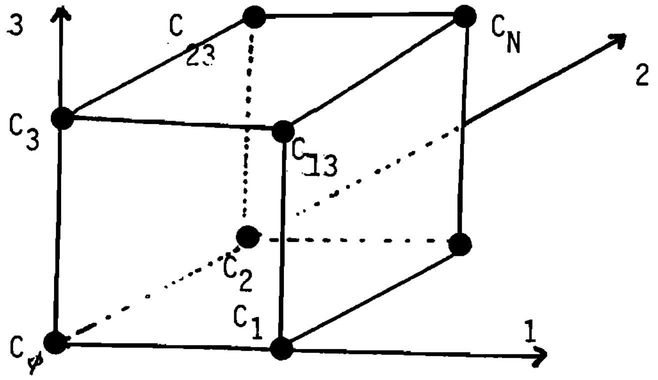
The elements $c \in [0,1]^n$ are called fuzzy coalitions. They associate with any player $i \in N$ its rates of participation $c(i) \in [0,1]$ in the fuzzy coalition c.

A player participates wholly to c when $c(i) = 1$, he does not participate at all when $c(i) = 0$, and he participates in a fuzzy manner when $c(i) \in]0,1[$.

<u>Examples</u>

$N = \{1,2\}$ $N = \{1,2,3\}$





Since the set of fuzzy coalitions is the convex hull of the set of coalitions, we can write any fuzzy coalition in the form

(2) $c = \sum_{S \in \mathcal{P}(N)} m(S) c_S$ where $m(S) \geq 0$, $\sum_{S \in \mathcal{P}(N)} m(S) = 1$

The rates of participation are therefore defined by

(3) $c(i) = \sum_{S \ni i} m(S) \quad i = 1, \ldots, n.$

In other words, if $m(S)$ denotes the probability of coalition S forming, the associated rate of participation of player i is the sum of probabilities of the formation of coalitions S to which i belongs.

Remark

We can also introduce more generally generalized coalitions $c \in [-1,+1]^n$ where a negative rate of participation $c(i)$ describes an agressive behavior of player i in the generalized coalition c.

Remark

We can use also a more adequate description of a player i by describing him as a vector $a^i = (a_1^i, \ldots, a_\ell^i)$ of $\mathbb{R}^\ell$, where the indexes $h = 1, \ldots, \ell$ denote "qualities" and the components a^i_k of the player a describe the amount of quality k that player is possesses. Then a generalized coalition C is a matrix of rates of participation c_h^i of the ℓ qualities h of the n players i. See J.P. Aubin, Ch. Louis-Guérin and M. Zavalloni [19 79] .

Remark

We can define as well fuzzy coalitions of an infinite subset of players. In game theory, it is customary to represent a <u>continuum of players</u> as a measure space N supplied with a σ-algebra $\mathcal{A}$ and a <u>non-atomic</u> measure μ (for exemple, $N:=[0,1]$ and the lebesgue measure, which is non-atomic).

The set $\mathcal{A}$ of (measurable) subsets is identified with the subset $L^\infty(N,\{0,1\})$ of (classes of) measurable functions with values in $\{0,1\}$. The set of (measurable) fuzzy coalitions is equal, by definition, to $L^\infty(N, [0,1])$, the unit ball of $L^\infty(N,\mathbb{R})$.

When we supply $L^{\infty}(N,\mathbb{R})$ with the weak star topology, we can prove that the set $L^{\infty}(N,\{0,1\})$ of coalitions is dense in the set $L^{\infty}(N,[0,1])$ of fuzzy coalitions, which is compact and convex. This is a consequence of the Lyapunov convexity Theorem. (See J.P. Aubin, [1979b], Proposition 10-4-1, p. 319)

Cooperative fuzzy games with side-payments

Cooperative fuzzy games with side-payments are described by a coalition loss function V from $[0,1]^n$ to $\mathbb{R}$, associating to every fuzzy coalition c its loss V(c). The problem at hand is to allocate the loss $V(c_N)$ of the grand coalition among the n players, i.e.,

(4) find $s := (s_1,\ldots,s_n) \in \mathbb{R}^n$ such that $\Sigma_{i\in N}\, s_i = V(c_N)$.

We regard elements $s \in \mathbb{R}^n$ as "multilosses". The aim of game theoricists was to find equitable allocations of the loss $V(c_N)$ by taking in account the consequences of the cooperation among players described a priori by the coalition loss function V.

In this framework, the rates of participations are only relative. So we can assume that V is positively homogeneous, and thus, extend it to $\mathbb{R}^n{}_+$.

Definition 1

A cooperative fuzzy game with side payments is described by

$$(5)\begin{cases}\text{a positively homogeneous function } V \text{ from } \mathbb{R}^n_+ \text{ to } \mathbb{R} \text{ which} \\ \text{is locally lipschitz on the interior } \mathring{\mathbb{R}}^n_+ \text{ of } \mathbb{R}^n_+\end{cases}$$

This function V is called the coalition loss function.

The subset

$$(6)\quad M:=\{s \in \mathbb{R}^n_+ \mid \forall\, c \in \mathbb{R}^n_+,\ \sum_{i=1}^{n} c_i\, s_i \leq V(c)\}$$

is called the subset of accepted multilosses. ▲

This is motivated by the fact that, for each coalition $c \in [0,1]^n$, the loss allocated a posteriori to the fuzzy coalition c according to the rates of participation of the players, which is equal to $\sum_{i=1}^{n} c_i\, s_i$, is at most equal to the loss V(c) yielded a priori to this fuzzy coalition according to the rules of the game described by the coalition loss function V.

We observe that the conjugate function V* defined by

$$(7)\quad V^*(s) = \sup_{c \in \mathbb{R}^n_+} (\langle c,s \rangle - V(c))$$

is the indicator of the subset M of accepted multilosses (see J.P. Aubin [1979 a], chapter 10).

Now, we describe several axioms that any allocation rule of the loss $V(c_N)$ should respect. An allocation rule is by definition a set-valued map S that associates with any coalition loss function V a subset S(V) of multilosses $s \in \mathbb{R}^n$ satisfying the condition (4).

This condition is also known under the name of "efficiency axiom" or "Pareto optimality axiom". We define as well other axioms.

Symmetry axiom

Let us consider a permutation θ : $N \rightarrow N$ of the set of n players, which describes the order in which the players are called. We define the action of θ on the function V by

(8) $(\theta * V)(c) := V(c_{\theta^{-1}(1)}, \ldots, c_{\theta^{-1}(n)})$

and the action of θ on the multiloss $s \in \mathbb{R}^n$ by

(9) $(\theta * s)_i = s_{\theta(i)}$ for all $i = 1, \ldots, n$.

The symmetry axiom states that an allocation rule does not depend upon " how the players are named", in the sens that

(10) for all permutation θ, $S(\theta * V) = \theta * S(V)$

Atomicity axiom

When $P := (S_1, \ldots, S_m)$ is a partition of the set N in m nomenpty coalitions S_j $(1 \leq j \leq m)$ we associate with any coalition loss function V of a n-person game the coalition loss function $P \square V$ of a m-person game defined by

(11) $\begin{cases} (P \square V)(d_1, \ldots, d_m) := V(c_1, \ldots c_n) \text{ where } c_k = d_j \\ \text{when } k \text{ belongs to } S_j. \end{cases}$

We associate also to any n-loss $s \in \mathbb{R}^n$ the m-loss $P \square s \in \mathbb{R}^n$ defined by

(12) $(P \square s)_j := \sum_{k \in S_j} s_j \quad , \quad j = 1, \ldots, m$

The atomicity axiom states that

(13) $S(P \square V) = P \square S(V)$

Dummy axiom

Let us consider a subset N of a subset M of m players and a coalition loss function $V : N \to \mathbb{R}$ of a n-person game. Let π_N denote the projector from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by

(14) $(\pi_N s)_j := \begin{cases} s_j & \text{when } j \in N \\ 0 & \text{when } j \notin N \end{cases}$

We associate with V the coalition loss function $\pi_N \Delta V$ of the m-person game defined by

(15) $(\pi_N \Delta V)(d) := V(\pi_N d)$

The dummy axiom states that the dummy players (players who do not belong to N) receive nothing :

(16) $S(\pi_N \Delta V) = \pi_N S(V)$.

Clarke generalized gradients

We refer to Aubin [1978] , Clarke [1975] and Rockafellar [1978] for further details.

When V is lipschitz around $c_0 \in \mathbb{R}^n$, we can define the folowing limit (called the upper Clarke derivative of V at c_0 in the direction d)

$$C_+ V(c_0)(d) := \limsup_{\substack{c \to c_0 \\ h \to 0+}} \frac{V(c+hd)-V(c)}{h} \in \mathbb{R}$$

We thus can prove that

$d \to C_+V(c_0)(d)$ is convex, positively homogeneous, continuous.

and that

$$\begin{cases} (c,d) \to C_+V(c)(d) \text{ is upper semicontinuous} \\ \text{at } (c_0,d_0) \text{ for all } d_0 \in \mathbb{R}^n \end{cases}$$

Therefore, $d \to C_+V(c_0)(d)$ is the support function of the bounded closed convex subset

$$\partial V(c_0) := \{s \in \mathbb{R}^n \mid \forall\, d \in \mathbb{R}^n,\ \langle s,d\rangle \leq C_+V(c_0)(d)\}$$

which is called the Clarke generalized gradient.

We observe that

i/ When V is continuously differentiable at c_0, then

$$\partial V(c_0) = \{\nabla V(c_0)\}$$

ii/ When V is convex and continuous at c_0, then

$$\partial V(c_0) = \{s \in \mathbb{R}^n \mid V(c_0)-V(c) \leq \langle s, c_0-c\rangle\ \forall\, c \in \mathbb{R}^n\}$$

is the subdifferential of V at C_0 of convex analysis (see Aubin [1979] a, chap. 10, Rockafellar [1970] for further details).

This is the reason why $\partial V(C_0)$ is called a generalized gradient.

We also define the upper contingent derivative defined by

$$D_{+} V(C_{\circ})\ (d_{\circ}) := \liminf_{\substack{d \to d_{\circ} \\ h \to o+}} \frac{V(c_{\circ} + hd) - V(C_{\circ})}{h}$$

We always have

$$D_{+} V(c_{\circ})(d_{\circ}) \leq C_{+} V(c_{\circ})(d_{\circ})$$

We say that V is <u>regular</u> at $c_{\circ}$ if

$$\forall\ d \in \mathbb{R}^{n},\ D_{+}V(c_{\circ})(d_{\circ}) = C_{+}V(c_{\circ})(d_{\circ}).$$

Continuously differentiable functions at $c_{\circ}$ and convex continuous functions at $c_{\circ}$ are regular at $c_{\circ}$.

We have the folowing properties

$$\begin{cases} \partial(-V)(c_{\circ}) = -\partial V(c_{\circ}) \\ \partial(V+W)(c_{\circ}) \subset \partial V(c_{\circ}) + \partial W(c_{\circ}) \\ \partial(\lambda V)(c_{\circ}) = \lambda \partial V(c_{\circ}) \\ \partial(V \circ A)(c_{\circ}) \subset A^{*}\, \partial V(Ac_{\circ}) \end{cases}$$

Equality holds when A is surjective or when V is regular

If V is non decreasing, $\partial V(c_{\circ}) \subset \mathbb{R}^{n}_{+}$

If V is positively homogeneous,

$$\forall\, s \in \partial V(c_{\circ}),\quad \langle s, c_{\circ} \rangle = V(c_{\circ})$$

If $V := \sup_{i=1,\ldots,n} V_{i}$, if $I(c_{\circ}) = \{i \mid V(c_{\circ}) = V_{i}(c_{\circ})\}$,

then

$$\partial V(c_{\circ}) \subset \overline{co} \bigcup_{i \in I(c_{\circ})} \partial V_{i}(c_{\circ})$$

Equality holds when the functions V_{i} are regular.

Definition of the set of solutions to a cooperative fuzzy game

So, let us consider a game whose coalition loss function V is locally Lipschitz on $\mathring{\mathbb{R}}^n_+$.

We propose the folowing definition of a solution concept to the game.

Definition 2.

Let $c_N := (1,\dots,1)$ denote the whole set of players. We shall say that the generalized gradient $\partial V(c_N)$ of V at c_N is the set of solutions to the game. We set

(17) $S(V) := \partial V(c_N)$

▲

A multi-utility $s \in \partial V(c_N)$ can be interpreted as the sequence of marginal losses s_i of players i when they join the whole set of players. The ith component s_i allocated to the ith player satisfies

$$(18) \quad s_i \leq \limsup_{\substack{d \to c_N \\ h \to 0+}} \frac{V(d+he^i)-V(d)}{h}$$

Theorem 1

Let V be a locally Lipschitz game. The set S(V) of solutions to the game is non-empty, convex and compact. It satisfies the Pareto optimality, symmetry and dummy properties, as well as :

(19) i/ $S(\lambda V) = \lambda S(V)$ for all $\lambda \in \mathbb{R}$

ii/ $S(V+W) \subset S(V) + S(W)$.

iii/ If V is increasing, then $S(V) \subset \mathbb{R}^n_+$.

iv/ If $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ satisfies $Ac_M = c_N$ then

$S(VA) \subset A^*S(V)$ where A^* is the transpose of A.

If A is surjective or if V is regular at c_N, we have $S(V\ A) = A^*S(V)$. When V is regular, the solution set satisfies the atomicity property.

▲

Proof

The properties of the Clarke generalized gradient imply at once that S(V) is non-empty, convex and compact and that properties (19) hold true.

The fact that S(V) is an allocation rule follows from the fact that V is positively homogeneous, because

(20) $\forall\ s \in \partial V(c_N)$, $\langle s, c_N \rangle = V(c_N)$

The others axioms are satisfied thank to property (19) iv/:

Symmetry Property

We apply property (19)iv/ for the matrix $A = (a_i^j)_{i,j=1,\dots,n}$ defined by $a_i^j := 1$ if $j = \theta^{-1}(i)$ and $a_i^j := 0$ if $j \neq \theta^{-1}(i)$, which is an isomorphism satisfying $Ac_N = c_N$.

Atomicity Property

We apply property (19)iv/ for the matrix P from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by

$$(Pd)_i := d_j \text{ whenever } i \in A_j.$$

which is an injective map satisfying $P\, c_M = c_N$.

Dummy Property

We apply property (19)iv/ for the matrix π_N from $\mathbb{R}^m$ to $\mathbb{R}^n$, which is a surjective map satisfying $\pi_N c_M = c_N$.

■

The concept of solution $S(V) := \partial V(c_N)$ requires that the grand coalition c_N plays a privileged role. We observe that for every coalition $c \in \operatorname{int} \mathbb{R}^n_+$, the generalized gradient $\partial V(c)$ provides a subset of allocations of $V(c)$ since

$$(21) \quad \forall\, s \in \partial V(c), \quad \langle c,s \rangle = V(c)$$

for V is positively homogeneous. We can prove a partial converse to his remark.

Proposition 1

Assume that V is locally Lipschitz on a neighborhood of $\mathbb{R}^n_+$. Then we can associate to any accepted multiloss $s \in M$ a fuzzy coalition $\bar{c}$ such that

(22) $\quad s \in \partial V(\bar{c}) - \mathbb{R}^n_+$ ▲

<u>Proof</u>

We apply Ky Fan's inequality (See Fan [1968]) to the function ψ defined on the n-simplex S^n by

$$\psi\ (c,d) := \langle s,d \rangle - C_+\ V(c)(d)$$

which is concave whith respect to d, lower semi-continuous with respect to c. Il also satisfies

$$\psi(c,c) = \langle c,s \rangle - C_+\ V(c)(c) = \langle c,s \rangle - V(c) \leq 0$$

when s belongs to the subset M of accepted multilosses, since V is positively homogeneous.

Since S^n is convex and compact, Ky Fan's inequality implies the existence of $\bar{c} \in S^n$ such that

$$\forall\ d \in S^n, \quad \langle d,s \rangle \leq C_+\ V(\bar{c})(d) = \sigma(\partial V(\bar{c}),d)$$

We infer that s belongs to $\partial V(\bar{c}) - \mathbb{R}^n_+$.

<u>Remark</u>

The properties of the generalized gradient imply the corresponding properties of the solution sets S(V). We mention for instance the following one.

Let J be a finite set, $V := \sup_{j \in J} V_j$ be the pointwise supremum of the functions V_j and $J(c_N) = \{j \in J$ such that $V(c_N) = V_j(c_N)\}$. Then

$$(23) \quad S(V) \subset co\left(\bigcup_{j \in J(c_N)} S(V_j) \right)$$

If the functions V_j are regular at c_N, then

$$(24) \quad S(V) = co\left(\bigcup_{j \in J(c_N)} S(V_j) \right)$$

<u>Core of sub-additive games</u>

We shall say that the fuzzy game described by a coalition loss function V is sub-additive if

$$(25) \quad \forall\, c_1, c_2 \quad , \quad V(c_1 + c_2) \leq V(c_1) + V(c_2)$$

Since V is positively homogeneous, this is equivalent to say that V is convex. Such games capture the idea that "l'union fait la force".

Indeed, if S and T are two disjoint usual coalitions, then $c_{S \cup T}$ is the characteristic function of $S \cup T$ and inequality (25) implies that

$$(26) \quad V(c_{S \cup T}) \leq V(c_S) \cup V(c_T).$$

When V is convex and finite on $\mathbb{R}^n_+$, it is continuous on $\mathring{\mathbb{R}}^n_+$. We shall extend it to $\mathbb{R}^n$ by setting $V(c) = +\infty$ when $c \notin \mathbb{R}^n_+$ and assume that

(27) V is lower semicontinuous from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$

Then we know that $S(V) = \partial V(c_N)$ is the subdifferential of V at c_N. We have therefore :

<u>Proposition 2</u>

We assume that the coalition loss function is sub-additive and lower semicontinuous. Then

(28) $$S(V) = \{s \in M \mid \sum_{i=1}^{n} s_i = V(c_N)\}$$

▲

<u>Proof</u>

Let s belong to S(V).

Since

(29) $$V(c_N) - V(c) \leq \langle s, c_N - c \rangle \quad \text{for all } c \in \mathbb{R}^n_+ ,$$

we deduce that, by taking $c := \lambda c_N$,

$$(1-\lambda)\ (V(c_N) - \langle s, c_N \rangle) \leq 0$$

Therefore, by choosing $\lambda = 1 \pm \varepsilon$, we obtain

(30) $$\sum_{i=1}^{n} s_i = \langle s, c_N \rangle = V(c_N)$$

and thus

$$(31) \quad \forall\, c \in \mathbb{R}^n_+ \;,\quad \langle s,c\rangle \leq V(c) \quad (\text{i.e., } s \in M).$$

Conversely, inequalities (30) and (31) imply (29).

Definition 3

The subset of accepted multilosses satisfying the Pareto optimality axiom is called the core of the game.

Therefore, when V is sub-additive, we have proved that the solution set coincides with the core of the game. ▲

Smooth games and their Shapley values

Let us consider the class of games whose coalition worth function V is continuously differentiable at c_N. Then

$$(32) \quad S(V) = \{\nabla V(c_N)\}$$

contains only one element, which is the gradient of V at c_N. For instance, if we take $V := \gamma_k$ where

$$(33) \quad \gamma_k(c) := \left(\prod_{i=1}^{n} c_i^{k_i}\right)^{1/|k|}, \quad k=(k_1,\ldots,k_n), |k|=k_1+k_2+\ldots+k_n,$$

we see that

$$S(\gamma_k)_i = k_i/|k|.$$

Definition 4

We shall say that the map $v \to \nabla V(c_N)$ is the generalized Shapley value of the game. ▲

Proposition 2

Let $\mathcal{V}$ be the vector space of games spanned by the functions γ_k when k ranges over N^n. Then S is the unique linear map from $\mathcal{V}$ to $\mathbb{R}^n$ that satisfies the Pareto optimality, symmetry and atomicity properties. ▲

Let Φ be a map satisfying those three properties. Let $\mathbb{1} := (1,\dots,1)$. The Pareto optimality and symmetry properties imply that $\Phi(\gamma_{\mathbb{1}})_i = (1/n)\mathbb{1}$, for all $i=1,\dots,n$.

Let $k=(k_1,\dots k_n)$ belong to N^n. If we consider the partition P of the set of $|k|$ players in n subsets A_1 of k_1 players,...,A_γ of k_n players, we can write that

$$\gamma_k = P \circ \gamma^{|k|} \quad \text{where } \gamma^{|k|}(c) = \left(\prod_{j=1}^{|k|} c_j\right)^{1/|k|} .$$

Hence the atomicity axiom implies that

$$\Phi(\gamma_k)_i = \sum_{j \in A_i} \Phi(\gamma^{|k|}) = \sum_{j \in A_i} \frac{1}{|k|} = \frac{k_i}{|k|} .$$

So, $\Phi(\gamma_k) = S(\gamma_k)$ for all $k \in \mathbb{N}^n$. Since the maps S and Φ are both linear, they coincide on $\mathcal{V}$.

Strategic Games

We shall associate a fuzzy cooperative game with a strategy space Y, a loss function f defined on $Y \times [0,1]^n$ and a set-valued map F from Y to $\mathbb{R}^n_+$ describing either the fuzzy coalitions that form when a strategy x is implemented and/or the strategies implemented by a generalized coalition.

Namely, we introduce

(34)
- i/ a Banach space Y and a closed convex cone $K \subset Y$, regarded as the cone of feasible strategies.
- ii/ a positively homogeneous locally Lipschitz function defined on a neighborhood of $K \times \mathbb{R}^n_+$; (for any fuzzy coalition c, $y \to f(y,c)$ is regarded as the loss function of c).
- iii/ a set-valued map F from K to $\mathbb{R}^n_+$, whose graph is a closed convex cone (such set valued maps are called closed convex processes (see Rockafellar [1967] and [1970] section 39).

It is clear that V is positively homogeneous.

For studying the properties of the solution to this game, we introduce the adjoint process F* of F defined by

$$\text{graph } F^* := \{(x,p) \in \mathbb{R}^n \times Y^* \mid \langle p,y\rangle - \langle x,c\rangle \leqq 0, \forall (y,c) \in \text{graph}(F)\}.$$

It is another closed convex process mapping $\mathbb{R}^n$ to Y*.

Theorem 2

We posit assumptions (34). We assume also that

$$(35)\begin{cases} \text{i/ } F(K) = \mathbb{R}^n_+ \\ \text{ii/ } \forall c \in \overset{\circ}{\mathbb{R}}{}^n_+, \ \exists\, \eta > 0 \ \text{ such that } F^{-1}(c+\eta B) \text{ is bounded.} \end{cases}$$

Then we can associate with any optimal strategy $\bar{y} \in F^{-1}(c_N)$, achieving the maximum of $f(y,c_N)$ on $F^{-1}(c_N)$, a solution $s \in S(V)$ to the game, and $\bar{p} \in X^*$, $\bar{\zeta} \in \mathbb{R}^n$ satisfying

$$(36)\begin{cases} \text{i/ } (\bar{p}, \bar{\zeta}) \in \partial f(\bar{y}, c_N) \text{ and } \bar{p} \in F^*(\bar{s} - \bar{\zeta}), \\ \text{ii/ } \langle \bar{p}, \bar{y} \rangle + \sum_{i=1}^{n} \bar{\zeta} = \sum_{i=1}^{n} \bar{s}_i \ [= f(\bar{y}, c_N) = V(c_N)]. \end{cases}$$

Remark

If f is continuously differentiable, condition (36) becomes

$$\nabla_y f(\bar{y}, c_N) \in F^*(\bar{s} - \nabla_c f(\bar{y}, c_N)).$$

¤

Proof

Assumptions (34) and (35) imply that v is locally Lipschitz on $\mathbb{R}^n_+$: this is a direct consequence of the Robinson-Ursescu theorem ; (see Robinson [1976], Ursescu [1977].

Let $\bar{y} \in F^{-1}(c_N)$ satisfying $V(c_N)=f(\bar{y},\bar{c})$. Let any (y,c) be chosen in the graph of F. Since it is convex, $(1-\theta)(\bar{y},c_N)+\theta(y,c)$ belongs to the graph of F, i.e., $\bar{y}+\theta(y-\bar{y})\in F^{-1}(c_N+\theta(c-c_N))$.

Thus $f(\bar{y}+\theta(y-\bar{y}),\ c_N + \theta(c - c_N)) \geqq V(c_N+\theta(c-c_N))$

and we deduce that $\forall\ (y,c) \in \text{graph}\ (F)$,

$$0 \leqq \{V(c_N + \theta(c-c_N)) - V(c_N)\}/\theta$$

$$+ \{(-f)(\bar{y} + \theta(y-\bar{y}),c_N+\theta(c-c_N))-(-f)(\bar{y},c_N)\}/\theta.$$

By taking the lim sup when $\theta \to 0$, we deduce that

$$0 \leqq \inf_{(y,c)\in\text{graph}(F)} C_+(V)(c_N)(c-c_N)+C_+(-f)(\bar{y},c_N)(y-\bar{y},c-c_N).$$

We recall that the upper Clarke derivative is the support function of the generalized gradient. Therefore,

$$0 \leqq \inf_{(y,c)\in\text{graph}(F)} \sup_{s\in S(V)} \sup_{(p,\zeta)\in\partial f(\bar{y},c^N)} [\langle s-\zeta,c-c_N\rangle+\langle -p,y-\bar{y}\rangle]$$

The graph of F is convex, the subset $S(V)\times\partial f(\bar{y},c_N)$ is convex and compact and the function $((y,c),(s,p,\zeta))\longmapsto\langle s-\zeta,c-c_N\rangle+\langle -p,y-\bar{y}\rangle$ is separately affine.

Hence the lop-sided minimax Theorem (see Aubin [1979] a, chap. 2) implies the existence of $\bar{s} \in S(V)$ and $(\bar{p},\bar{y})\in\partial f(\bar{y},c_N)$ such that

$$0 \leqq \inf_{(y,c)\in\text{graph}(F)} [\langle\bar{s} - \bar{\zeta},c-c_N\rangle+\langle -\bar{p},y-\bar{y}\rangle] .$$

Since the graph of F is a cone, this implies that

$$\sum_{i=1}^{n} \bar{s}_i = \langle\bar{p},\bar{y}\rangle + \sum_{i=1}^{n} \bar{\zeta}_i,$$

and that $\langle\bar{p},y\rangle - \langle\bar{s}-\bar{\zeta},c\rangle \leqq 0$ for all $(y,c)\in\text{graph}(F)$. Hence $\bar{p} \in F^*(\bar{s}-\bar{\zeta})$.

■

Core and Shapley values of usual cooperative games

Usual cooperative games are defined by coalition loss functions v from the subset $\mathcal{P}(N)$ of usual coalitions to $\mathbb{R}$, associating to each coalition S its loss $v(S)\in\mathbb{R}$.

We shall be able to associate a concept of solution whenever we can associate with a function $v:\mathcal{P}(N)\to\mathbb{R}$ a positively homogeneous $V=\pi v$ from $\mathbb{R}^n_+$ to $\mathbb{R}$, locally Lipschitz on the interior of $\mathbb{R}^n_+$, by taking $S(\pi v)$. So, we may devise as many concepts of solutions as extension maps π from usual cooperative games to fuzzy cooperative games.

Let v be a coalition loss function from $\mathcal{P}(N)$ to $\mathbb{R}$. We define the set of accepted multilosses as the

(37) $M := \{s\in\mathbb{R}^n \mid \forall\, S \subset N, \sum_{i\in S} s_i \leq v(S)\}$

Definition 5

We shall say that the core $C(v)$ of the usual cooperative game described by $v:\mathcal{P}(N)\to\mathbb{R}$ is the set of accepted multilosses $s \in M$ such that $\sum_{i=1}^{n} s_i = v(N)$. ▲

This suggests to associate with v a sub-additive cooperative fuzzy game whose set of accepted multilosses is equal to M.

This can be done by defining $\pi v: \mathbb{R}^n \to \mathbb{R}$ in the following way

$$(38) \qquad \pi v(c) := \sup_{s\in M} \langle c,s\rangle$$

which is called the "convex cover" of the game v.
The fuzzy coalition loss function πv is the support function of the set M of accepted multilosses.
We always have inequalities

$$\forall S \subset N,\ \pi v(c_S) \leq v(S)$$

We shall say that the game is balanced if

$$(39) \qquad \pi v(c_N) = v(N).$$

It is an exercise to verify the following statement.

Proposition 3

The core $C(v)$ is nonempty if and only if the game is balanced. In this case,

$$(40) \qquad C(v) = S(\pi v).$$ ▲

This extension map π sends usual coalition loss functions to sub-additive fuzzy coalition loss functions. Now, we introduce another extension map χ associating smooth fuzzy coalition loss functions. We introduce the functions

γ_S ($S \in N$) defined by

$$(41) \quad \gamma_S(c) := \left(\prod_{i \in S} c_i\right)^{1/|S|} \quad \text{where } |S| = \text{card}(S)$$

(We observe that $\gamma_S = \gamma_{c_S}$ defined in (33) with $k = c_S$

We associate with any coalition S the functionals α_S defined by

$$(42) \quad \alpha_S(v) := \sum_{T \subset S} (-1)^{|S|-|T|} \; v(T)$$

We define χ by

$$(43) \quad \forall\, c \in \mathbb{R}^n_+, \quad \chi\, v(c) = \sum_{S \subset N} \alpha_S(v) \gamma_S(c)$$

We check the following statement.

<u>Proposition 4</u>

The Shapley value of the fuzzy game defined by χv is equal to

$$(44) \ \forall\, i=1,\dots,n, \ S(\chi v)_i = \sum_{S \ni i} \frac{1}{|S|} \alpha_S(v)$$

▲

We recognize the Shapley value of usual games (See Aubin [1979] b, chap. 11, Shapley [1953]).

The map $v \to S(\chi v)$ is the unique linear operator from the space of functions $v:\mathcal{P}(N)\to\mathbb{R}$ to $\mathbb{R}^n$ satisfying the Pareto symmetry and dummy axioms.

Hence, the difference between the concepts of core and Shapley values of usual games does result only from the two different ways by which a usual game is transformed to a fuzzy game, but does not follow from a conflict between two antagonist views over what a solution concept should be, because this difference is resolved in the framework of fuzzy games.

Games without side-payments

We associate to any fuzzy coalition c the map c• from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

(45) $(c\cdot x)_i = c_i x_i$.

A fuzzy game without side-payments is defined by its coalition loss set-valued map $\underset{\sim}{V}$ from $[0,1]^n$ to $\mathbb{R}^n$ satisfying.

(46)

i/ $\forall\ c\in[0,1]^n,\ \underset{\sim}{V}(c)\subset c\cdot\mathbb{R}^n$

ii/ $\forall\ c\in[0,1]^n$, $\underset{\sim}{V}(c)$ is closed, convex, comprehensive in the sense that $\underset{\sim}{V}(c)\subset\underset{\sim}{V}(c)+c\cdot\mathbb{R}^n$] and bounded below [in the sens that $\exists x_0\in\mathbb{R}^n$ such that $\underset{\sim}{V}(c)\subset c\cdot(x_0+\mathbb{R}^n_+)$],

iii/ V is positively homogeneous [in the sens that $\forall\lambda\geq 0,\ \underset{\sim}{V}(\lambda c) = \lambda\underset{\sim}{V}(c)$].

This allows to extend $\underset{\sim}{V}$ to $\mathbb{R}^n_+$ by setting

$$\underset{\sim}{V}(c) := (\sum_{i=1}^{n} c_i)\underset{\sim}{V}(c_i / \sum_{j=1}^{n} c_j).$$

Since the subsets $\underset{\sim}{V}(c)$ are closed and convex, they can be characterized by their lower support function defined by

(47) $$V(c,\lambda)=\sup \{\Sigma\lambda_i s_i \,|\, s\in\underset{\sim}{V}(c)\}.$$

Since $V(c)$ is comprehensive and bounded below, $V(c,\lambda)$ is finite if and only if $\lambda\in c\cdot\mathbb{R}^n_+$.

We shall consider the class of locally Lipschitz games. We say that a game is locally Lipschitz if

(48) $$\begin{cases} \text{i/ the functions } c\to V(c,\lambda) \text{ are univormly locally Lipschitz on } \mathring{\mathbb{R}}^n_+, \\ \text{ii/ } \forall\lambda\in S^n := \{\lambda\in\mathbb{R}^n_+ \,|\, \sum_{i=1}^{n} \lambda_i=1\}, \\ \limsup_{\substack{d\to c_N \\ \mu\to\lambda^N \\ \theta\to 0^+}} (V(d+\theta a,\mu)-V(d,\mu))/\theta=C_+V(c_N,\lambda)(a). \end{cases}$$

We associate with any $\lambda\in S^n$ the subset

(49) $$\begin{aligned} C(\lambda) &:= \{s\in\mathbb{R}^n \,|\, \lambda\cdot s\in\partial V(C_N,\lambda)\} \\ &= \{s\in\mathbb{R}^N \,|\, \lambda\cdot s\in S(V(\cdot,\lambda)\}. \end{aligned}$$

Definition 6

We shall say that the subset

(50) $S(\underset{\sim}{V}) = \underset{\sim}{V}(c_N) \cap \bigcup_{\lambda \in S^n} C(\lambda)$

is the set of solutions to the game.

▲

Theorem 3

Let $\underset{\sim}{V}$ be a locally Lipschitz game without side-payments. Then its set of solutions is non-empty.

▲

Remark

We can regard the game with side payments whose coalition loss fonction is $V(c,\lambda)$ as a tangent game whose set of solutions is $C(\lambda)$. Then $S(\underset{\sim}{V})$ is the set of those multi-utilities $s \in \underset{\sim}{V}(c_N)$ that are solutions to at least one of the tangent games.

■

Proof

We recall that $V(c,\mu) = \sup_{s \in \underset{\sim}{V}(c)} \langle \mu, s \rangle$ is the support function of the closed convex bounded above and comprehensive subset $\underset{\sim}{V}(c) \subset c \cdot \mathbb{R}^n$.

If $\lambda \in \overset{\circ}{\mathbb{R}}{}^n_+$, then

$$\sigma(-C(\lambda),\mu) = \sup_{\lambda \cdot s \in \partial V(c_N,\lambda)} \Sigma \mu_i s_i = C_+(-V)(c_N,\lambda)(\mu/\lambda),$$

where μ/λ is the vector of components μ_i/λ_i.

Assumption (48)ii/ implies that $(\lambda,a) \to C_+V(c_N,\lambda)(a)$ is upper-semicontinuous on $\mathring{\mathbb{R}}^n_+ \times \mathbb{R}^n$.

Indeed, let $(\lambda_o,a_o) \in \mathring{\mathbb{R}}^n_+ \times \mathbb{R}^n$; for all $\varepsilon > 0$, there exists $\alpha > o$ such that

$$\sup_{\substack{\|c-c_N\| \leq \alpha \\ \|\lambda-\lambda_o\| \leq 2\alpha \\ \theta \leq \alpha}} \frac{1}{\theta} (V(c+\theta a_o,\lambda) - V(c,\lambda)) \leq C_+V(c_N,\lambda_o)(a_o)+\varepsilon/2.$$

Let us take $\|\lambda_1 - \lambda_0\| \leq \alpha$ and $\|\lambda_1 - \lambda\| \leq \alpha$, $\|a_1 - a_0\| \leq \varepsilon/2\ell$

(ℓ being the Lipschitz constant).

Hence

$$\sup_{\substack{\|c-c_N\| \leq \alpha \\ \|\lambda-\lambda_1\| \leq \alpha \\ \theta \leq \alpha}} \frac{1}{\theta} (V(c+\theta a_o,\lambda)-V(c,\lambda)) + \ell\|a-a_o\| \leq C_+V(c_N,\lambda_o)(a_o)+\varepsilon.$$

By letting α converge to 0, we deduce that

$$C_+V(c_N,\lambda_1)(a_1) \leq C_+V(c_N,\lambda_o)a_o)+\varepsilon,$$

whenever $\|\lambda_1 - \lambda_o\| \leq \alpha_0$, $\|a_1 - a_0\| \leq \varepsilon/2\ell$.

So, the function ψ, defined on $\mathring{\mathbb{R}}^n_+ \times \mathring{\mathbb{R}}^n_+$ by

$$\psi(\lambda,\mu) = -V(c_N,\mu)-C_+(-V)(c_N,\lambda)(\mu/\lambda),$$

is concave with respect to μ, lower semicontinuous with respect to λ, and satisfies

$$\begin{aligned}\psi(\mu,\mu) &= -V(c_N,\mu)-C_+(-V)(c_N,\mu)(\mu/\mu)\\ &= -V(c_N,\mu)-C_+(-V)(c_N,\mu)(c_N)\\ &= -V(c_N,\mu)+V(c_N,\mu)=0,\end{aligned}$$

since the function $c \mapsto V(c,\lambda)$ is positively homogeneous. Let us take $\varepsilon<1/n$ and consider the convex compact subset $S^n_\varepsilon := \{\lambda \in \mathbb{R}^n_+ | \min \lambda_i \geqq \varepsilon \text{ and } \Sigma^n_{i=1}\lambda_i = 1\}$. The Ky Fan inequality (see Fan [1968] or Aubin [1978a,] ch.5, p. 203) implies the existence of $\lambda_\varepsilon \in S^n_\varepsilon$ such that

$$\forall \quad \mu \in \mathbb{R}^n_+,\ 0 \leqq \sigma(-C(\lambda_\varepsilon)+\underset{\sim}{V}(c_N)-(S^n_\varepsilon)^+,\mu).$$

Hence there exists $s_\varepsilon \in C(\lambda_\varepsilon) \cap (\underset{\sim}{V}(c_N)-(S^n_\varepsilon)^+)$.

We can check, as in Aubin [1979b,] ch. 12 , that s_ε is bounded. Thus subsequences (again denoted s_ε and λ_ε) converge to some $s \in \underset{\sim}{V}(c_N)$ and $\lambda \in S^n$. Since

$$\lambda_\varepsilon \cdot s_\varepsilon \in \partial V(c_N,\lambda_\varepsilon),$$

we deduce that for all $a \in \mathbb{R}^n_+$,

$$\langle a,\lambda_\varepsilon \cdot s_\varepsilon\rangle \leqq C_+V(c_N,\lambda_\varepsilon)(a).$$

Since the right-hand side is upper semicontinuous with respect to λ_ε, it follows that

$$\forall\, a \in \mathbb{R}^n_+, \qquad \langle a, \lambda \cdot s\rangle \leqq C_+V(c_N, \lambda)(a),$$

i.e., that $s \in C(\lambda)$. ¤

Action games and formation of coalitions

We change radically our point of view for defining games, by adopting a dynamical point of view.

Let us consider n players i=1,...,n. We suppose that the behavior of the i^{th} player is described by its action on the environment for transforming it. We describe the environment by

(51) $\begin{cases} \text{a closed convex subset } L \text{ of a finite dimensional} \\ \text{space } X = \mathbb{R}^p \end{cases}$

Action of player i is described by a map f_i from L to X associating with each state $x \in L$ of the environment the rate of change $f_i(x)$ that player i forces on the environment

A very important example is the case when $f_i(x) = \nabla U_i(x)$ is the gradient at x of a utility function U_i. In this case, action of player i amounts to the marginal increase of utility.

We suppose that the action of a fuzzy coalition $c \in [0,1]^n$ on the environment is the sum of players i multiplied by their rates of participation (i.e., $\sum_{i=1}^{n} c_i f_i(x)$).

Let $g : L \to X$ describe the endogeneous evolution law of the environment in the absence of players.

This describes an <u>action game</u>, in the sense that the evolution law of the states of the environment is described by the set $C(x)$ of velocities defined by

$$(52) \quad C(x) := \{g(x) + \sum_{i=1}^{n} c_i f_i(x)\}_{c \in [0,1]^n}$$

Now, an equilibrium $\bar{x} \in L$ is a state of the environment that remains invariant under the action of a fuzzy coalition $\bar{c} \in [0,1[^n$.

<u>Definition 7</u>

An equilibrium is a pair $(\bar{x},\bar{c})$ of a state $\bar{x}$ and a fuzzy coalition $\bar{c}$ satisfying

$$(53) \quad g(\bar{x}) + \sum_{i=1}^{n} \bar{c}_i f_i(\bar{x}) = 0.$$

▲

If L is a closed convex subset, we define the tangent cone $T_L(x)$ to L at x by

$$T_L(x) := c\ell\left(\bigcup_{\lambda > o} \lambda(L-x)\right)$$

It is a closed convex cone, which coincides with the tangent space when K is a smooth manifold.

Theorem 4

Assume that L is compact and that

(54) $\forall x \in L, \exists c \in [0, 1]^n$ such that $g(x) + \sum_{i=1}^{n} c_i f_i(x) \in T_L(x)$.

Then there exists an equilibrium $(\bar{x}, \bar{c})$ of the action game.

Proof

We apply Browder-Fan's Theorem (see Aubin [1979a]: chap.15) to the set-valued map C defined on the compact convex subset L by

(55) $C(x) := \{g(x) + \sum_{i=1}^{n} c_i f_i(x)\}_{c \in [0,1]^n}$

which is obviously upper semicontinuous with convex compact values. Assumption (55) implies that the tangential condition

(56) $\forall x \in L, \quad C(x) \cap T_L(x)$

is satisfied. Hence, there exists a state $\bar{x} \in L$ such that 0 belongs to $C(\bar{x})$, and thus, there exists a fuzzy coalition $\bar{c}$ satisfaying (53). ■

Actually, this framework allows a dynamical treatment of action games. We deduce from a theorem of Haddad (see Haddad [1980]) the following result.

Theorem 5

We posit the assumptions of Theorem 4. For any initial state $x_o \in L$, there exists an absolutely continuous function $x(.)$ and a measurable function $c(.)$ such that

$$(57)\begin{cases} \text{i/ for almost all } t \geq o, \; x'(t) = g(x(t)) + \sum_{i=1}^{n} c_i(t) f_i(x(t) \\ \text{ii/ } x(o) = x_o \end{cases}$$

satisfying the viability condition

$$(58) \quad \forall t \geq o \quad x(t) \in L$$

For almost all $t \geq o$, the state $x(t)$ and the fuzzy coalition $c(t)$ are related by the feedback relation :

$$(59) \quad c(t) \in C(x(t)).$$

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