On the Consistency and Large Deviations of the Method of Empirical Means in Stochastic Programming Problems

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1 Introduction

The stochastic programming problem arises when one needs to make decisions in conditions of uncertainty and risk [1, 2]. The mean value of a factor of quality of control depending on a random parameter is optimized.

Indirect methods for solving stochastic programming problems involve approximating a stochastic problem with an approximate deterministic problem. One of the main indirect methods of stochastic programming is the so-called method of empirical means [3–7]. In this method, factors of quality of control are approximated by their empirical estimates. One of the main problems is estimation accuracy and convergence when the area of observations increases.

Large deviations theory (see, for example, [8–12]) is a part of probability theory that considers cases in which empiric estimates deviate from true values of parameters more than from "normal" values, i.e., more than a value that is effectively described by the central limit theorem. A more accurate calculation of the probability of such events demands a more accurate study of the integrals of exponential functionals.

This problem arises in many different contexts. Large deviation theory is applied in probability theory, mathematical statistics, operations research, informatics, statistical physics, financial mathematics, and other spheres.

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Large deviation theory describes rare events. Nevertheless, the necessity of studying rare events is not in doubt because their occurrence can cause many different problems and demands much energy for their liquidation.

As written in many sources (for example, in [8, 9]), "theory" of large deviations is absent. In addition to basic definitions, which are standard, many methods and approaches exist that allow the analysis of such rare events. Often, identical results can be achieved in different manners. Common probabilistic estimates are transferred to partial situations that are under study.

2 The Method of Empirical Means for Discrete and Continuous Models with Dependent Observations

Let us consider stochastic programming problems where the empirical functions are constructed by observations of stationary random processes with a discrete or continuous parameter.

Let $(Y, L(Y))$ be some measurable space, where Y is a metric space, $L(Y)$ is a minimal σ-algebra on Y, and $\|\cdot\|$ is a norm set in Y. Let ξ_i, $i \in N$ be independent identically distributed observations of a random variable or a stationary in a strict sense ergodic random sequence, defined on a probability space $(\Omega, \Im, P)$ with values in measurable space $(Y, L(Y))$, where Ω is a space of elementary events, $\Im$ is a σ-algebra of elementary events on Ω, and P is a probability measure on $\Im$ such that $P(\Omega) = 1$. We assume that I is a closed subset in $\Re^l$, $l \geq 1$, possibly $I = \Re^l$, and that $f : I \times Y \to \Re$ is a nonnegative function satisfying the following conditions:

1. $f(\mathbf{u}, z)$, $\mathbf{u} \in I$, is continuous for all $z \in Y$;
2. for any $\mathbf{u} \in I$, the mapping $f(\mathbf{u}, z)$, $z \in Y$ is $L(Y)$ measurable.

The problem consists of finding the minimum point of the function and its minimal value (the stochastic optimization problem).

$$\min F(u) = Ef\,(u, \xi_1)\,, \quad u \in I.$$

This problem is approximated by the following problem: find the minimum points of the function

$$F_n\,(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} f\,(\mathbf{u}, \xi_i)\,,$$

and its minimal value.

We present some examples of regression models, which are widely known to specialists in the field of theoretical and applied statistics.

1. $y_i = \sum_{t=1}^{p} x_{it}\alpha_t^0 + \varepsilon_i, \quad i = 1, \ldots, n.$

   Here, ε_i and $i = 1, \ldots, n$ are independent or stationary dependent random variables, and $\mathbf{x}_i = \left\{x_{it}, t = \overline{1, p}\right\}$ and $i = 1, \ldots, n$ are independent identically distributed random vectors that are independent of ε_i and $i = 1, \ldots, n$.

   The vector $\boldsymbol{\alpha}^0 = \left(\alpha_1^0, \ldots, \alpha_p^0\right)$ is unknown and is estimated.

2. $y_i = g(\mathbf{x}(i), \boldsymbol{\alpha}^0) + \varepsilon_i, \mathbf{x}(i) \in R^p,$

   where the p-dimensional vectors $\mathbf{x}(i)$ and ε_i are mutually independent, and each of the sequences $\{\mathbf{x}(i)\}$ and $\{\varepsilon_i\}, i = 1, \ldots, n$, is the sequence of independent or stationary random vectors or variables.

Some cost functions characterizing the accuracy of the estimate are as follows:

1. $F_n(\alpha) = \frac{1}{n}\sum_{i=1}^{n}\left[y_i - \sum_{t=1}^{p} x_{it}\alpha_t\right]^2;$
2. $F_n(\alpha) = \frac{1}{n}\sum_{i=1}^{n}\left[y_i - g\left(\mathbf{x}(i), \boldsymbol{\alpha}\right)\right]^2;$
3. $F_n(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \mid y_i - \sum_{t=1}^{p} x_{it}\alpha_t \mid;$
4. $F_n(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \mid y_i - g\left(\mathbf{x}(i), \boldsymbol{\alpha}\right) \mid.$

Theorem 1 [13]. *Let the following conditions be satisfied:*

1. *for any $c > 0$, $E\left(\max_{\|\mathbf{u}\| \le c} f(\mathbf{u}, \xi_1)\right) < \infty$, where $\|\cdot\|$ is a norm in $\Re^l$*
2. *for all $z \in Y'$, $P\{\xi_1 \in Y'\} = 1, f(\mathbf{u}, z) \to \infty$ as $\|u\| \to \infty$;*
3. *there is a unique point $\mathbf{u}_0$ at which the function $F(\mathbf{u})$ attains its minimum.*

Then, for any n and $\omega \in \Omega'$, $P(\Omega') = 1$, there exists at least one vector $\mathbf{u}_n = \mathbf{u}_n(\omega) \in I$ for which the minimum value of $F_n(\mathbf{u})$ is attained, and for any $n \ge 1$, the vector $\mathbf{u}_n$ can be chosen to be G_n'-measurable, where $G_n' = G_n \cap \Omega'$ and $G_n = \sigma\left\{\xi_i, \ i = \overline{1, n}\right\}$. In this case, with probability 1, $\mathbf{u}_n \to \mathbf{u}_0$ and $F_n(\mathbf{u}_n) \to F(\mathbf{u}_0)$, $n \to \infty$.

Theorem 2 [13]. *Let $\{\xi(t), \ t \in \Re\}$ be a random ergodic process stationary in the strict sense and defined on the probability space $(\Omega, \Im, P)$ with values in $(Y, L(Y))$. Suppose that the trajectories of the process are continuous and that the function f, described above, is continuous. Let the following conditions be satisfied:*

1. *for any $c > 0$, $E\left\{\max_{\|\mathbf{u}\| \le c} f(\mathbf{u}, \xi(0))\right\} < \infty$;*
2. *if I is an unbounded set then for any $z \in Y'$ and $P\{\xi(t) \in Y' \ \ \forall t \ge 0\} = 1$, one has $f(\mathbf{u}, z) \to \infty$ as $\|\mathbf{u}\| \to \infty$;*
3. *there is a unique element $\mathbf{u}_0 \in I$ for which the minimal value of the function $F(\mathbf{u}) = Ef(\mathbf{u}, \xi(0))$ is attained.*

Then, for all $T > 0$ and $\omega \in \Omega'$, $P(\Omega') = 1$, there is at least one vector $\mathbf{u}(T) \in I$ for which the minimal value of the function

$$F_T(\mathbf{u}) = \frac{1}{T} \int_0^T f(\mathbf{u}, \xi(t))\, dt$$

is attained and measurable, and we have

$$P\left\{\lim_{T\to\infty} \mathbf{u}(T) = \mathbf{u}\right\} = 1, \qquad P\left\{\lim_{T\to\infty} F_T(\mathbf{u}_T) = F(\mathbf{u}_0)\right\} = 1.$$

Remark The condition of the ergodic in the Theorems 1 and 2 will be fulfilled if the random stationary sequence or the random stationary process satisfies a condition of strong intermixing.

3 The Method of Empirical Means Is Applied to Nonstationary Models

Consider a more general case of a nonstationary model. We assume that the criterion function also depends on the temporal parameter, i.e., it is a function of three variables. For example, in discrete time, the criterion function has the form

$$F_n(u) = \frac{1}{n} \sum_{i=1}^{n} f(i, u, \xi_i).$$

For example, one can take

$$F_n(u) = \frac{1}{n} \sum_{i=1}^{n} [y_i - g(i, u)]^2$$

$$F_n(u) = \frac{1}{n} \sum_{i=1}^{n} | y_i - g(i, u) |$$

for the model of the observations

$$y_i = g(i, u_0) + \xi_i.$$

The unknown parameter is also assumed to be an element of some functional space. For example, one can consider the problem of the estimation of the unknown function $u(t) \in K$, where K is the compact set of functions defined on [0,1], by observations

$$y_i = u\left(\frac{i}{n}\right) + \xi_i, i = 0, \dots, n$$

with some criterion function.

Theorem 3 [13]. *Let a stochastic function $f\left(i, \overrightarrow{u}, \xi_i\right)$ satisfy the following conditions:*

1. *for any $\overrightarrow{u} \in I$, there exists a function $F\left(\overrightarrow{u}\right)$ such that $F\left(\overrightarrow{u}\right) = \lim_{n\to\infty} F_n\left(\overrightarrow{u}\right)$ and a point $\overrightarrow{u}_0 \in I$ such that we have $F\left(\overrightarrow{u}_0\right) < F\left(\overrightarrow{u}\right)$ when $\overrightarrow{u} \neq \overrightarrow{u}_0$;*
2. *the function $f\left(i, \overrightarrow{u}, z\right)$ is continuous with respect to the second argument uniformly relative to i and z;*
3. *if the set I is unlimited, then $f\left(i, \overrightarrow{u}, z\right) \to \infty$ as $\|u\| \to \infty$ for fixed i and z;*
4. *there is a function $c(\gamma) \to 0$ as $\gamma \to 0$, and for any $\delta > 0$, there exists γ_0 such that, for any element $\overrightarrow{u}' \in I$ and $0 < \gamma < \gamma_0$, the following relationship is true:*
   $$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} E \sup_{\substack{\left\|\overrightarrow{u} - \overrightarrow{u}'\right\| < \gamma \\ \left\|\overrightarrow{u} - \overrightarrow{u}_0\right\| < \delta}} \left| f\left(i, \overrightarrow{u}, \xi_i\right) - f\left(i, \overrightarrow{u}', \xi_i\right)\right| < c\left(\gamma\right);$$
5. *the function $f\left(i, \overrightarrow{u}, \xi_i\right)$ satisfies the strong mixing condition.*
   $$\alpha(j) = \sup_i \sup_{\substack{A \in \sigma_{-\infty}^{i} \\ B \in \sigma_{i+j}^{\infty}}} |P(AB) - P(A)P(B)| \le \frac{c}{1+j^{1+\varepsilon}},$$
   $$\varepsilon > 0, \sigma_n^m = \sigma\left\{f\left(i, \overrightarrow{u}, \xi_i\right), \quad n \le i \le m, \quad \overrightarrow{u} \in I\right\};$$
6. $E\left(f\left(i, \overrightarrow{u}, \xi_i\right)\right)^{2+\delta} < \infty, \varepsilon\delta > 2.$

Let $\overrightarrow{u}_n = \arg\min_{\overrightarrow{u} \in I} F_n\left(\overrightarrow{u}\right)$

Then, we have

$$P\left\{\lim_{n\to\infty} \left\|\overrightarrow{u}_n - \overrightarrow{u}_0\right\| = 0\right\} = 1,$$

$$P\left\{\lim_{n\to\infty} F_n\left(\overrightarrow{u}_n\right) = F\left(\overrightarrow{u}_0\right)\right\} = 1.$$

A similar statement is also true for the case in which a continuous stochastic function $f\left(t, \overrightarrow{u}, \xi(t)\right)$ is considered in an interval $[0, T]$, i.e., the following functional is considered:

$$F_T\left(\overrightarrow{u}\right) = \frac{1}{T}\int_0^T f\left(t, \overrightarrow{u}, \xi(t)\right) dt,$$

where $\xi(t)$ *is a random process that is stationary in a strong sense. It is necessary to find* $\min\limits_{\vec{u}\in I} F_T\left(\vec{u}\right)$ *and investigate the asymptotic behaviors of* $\vec{u}_T = \arg\min\limits_{\vec{u}\in I} F_T\left(\vec{u}\right)$ *and* $F_T\left(\vec{u}_T\right)$ *as* $T \to \infty$.

4 Method of Empirical Means for the Models with Random Functions Depending on Several Variables or Random Fields

Let us consider a stochastic programming problem where the empirical function is constructed on the basis of observations of a homogeneous random field.

Let $\left\{\xi\left(\vec{t}\right), \vec{t} \in \mathbb{R}^m\right\}$ be an ergodic homogeneous in a strict sense random field with continuous trajectories, defined on a probabilistic space (Ω, G, P), with values in some metric space $(Y, L(Y))$. We assume that I is a closed subset in $\Re^l$, $l \geq 1$, possibly $I = \Re^l$, and that $f: I \times Y \to \Re$ is a nonnegative continuous function.

Theorem 4 [13]. *Suppose that for the random field* $\xi(t) \in Y,\ t \in R^m$, *the conditions below are fulfilled:*

1. *for any* $c > 0$
$$E\left\{\max_{\|u\|<c}\left(f\left(\mathrm{u}, \xi(0)\right)^2\right\} < \infty;$$
2. *if I is unbounded then for each* $z \in Y$
$$f(\mathrm{u}, z) \to \infty, \|u\| \to \infty;$$
3. *there is a unique element* $\vec{u_0} \in I$ *for which the minimal value of the function* $F(u) = Ef(u, \xi(0))$ *is attained.*

Then, for all $T_i > 0,\ i = 1, \ldots, m$ *and* $\omega \in \Omega',\ P(\Omega') = 1$, *there is at least one vector* $\mathbf{u}\left(\vec{T}\right) \in I,\ \ T = (T_1, \ldots, T_m)$ *for which the minimal value of the function*

$$F_T(\mathbf{u}) = \frac{1}{\prod\limits_{i=1}^{m} T_i} \int\limits_{t\in[0,T_1]\times\cdots\times[0,T_m]} f(\mathbf{u}, \xi(t))\, dt$$

is attained and

$$P\left\{u(T) \to u_0, F_T(u(T)) \to F(u_0), T_i \to \infty, i = 1, \ldots, m\right\} = 1.$$

Remark The ergodic condition will be fulfilled if the homogeneous field satisfies the condition of strong intermixing.

Now, we consider the case in which we observe the random field in the ball. Let $\left\{\xi\left(\vec{t}\right) = \xi\left(\vec{t}, \omega\right), \vec{t} \in \mathcal{R}^m\right\}, m \geq 1$ be a homogeneous in a strict sense

152 random field on a complete probabilistic space (Ω, G, P) with values in some metric
153 space $(Y, \mathfrak{B}(Y))$. Suppose that the realizations of $\xi\left(\overrightarrow{t}\right)$ are continuous on $\mathscr{R}^m$
154 with probability 1. We have a continuous nonnegative function $f : J \times Y \to R$,
155 where J is a closed subset of $\mathscr{R}^l$, $l \geq 1$.

156 One has the observations $\left\{\xi\left(\overrightarrow{t}\right) : \left\|\overrightarrow{t}\right\| < T\right\}$, $T > 0$. The problem is to find
157 the minimum points and the minimal value of the function

$$F\left(\overrightarrow{u}\right) = Ef\left(\overrightarrow{u}, \xi\left(\overrightarrow{0}\right)\right), \overrightarrow{u} \in J. \tag{1}$$

158 Let us investigate problem (1). We approximate it via minimization of the function

$$F_T\left(\overrightarrow{u}\right) = \frac{\int\limits_{\left\|\overrightarrow{t}\right\|<T} f\left(\overrightarrow{u}, \xi\left(\overrightarrow{t}\right)\right) d\overrightarrow{t}}{\int\limits_{\left\|\overrightarrow{t}\right\|<T} d\overrightarrow{t}}, \overrightarrow{u} \in J. \tag{2}$$

159 Denote

$$b_1\left(\overrightarrow{t}\right) = b_1\left(\overrightarrow{t}, c\right) = \frac{E\left(\inf\limits_{\left\|\overrightarrow{u}\right\|>c} f\left(\overrightarrow{u}, \xi\left(\overrightarrow{t}\right)\right) - E\inf\limits_{\left\|\overrightarrow{u}\right\|>c} f\left(\overrightarrow{u}, \xi\left(\overrightarrow{0}\right)\right)\right)\left(\inf\limits_{\left\|\overrightarrow{u}\right\|>c} f\left(\overrightarrow{u}, \xi\left(\overrightarrow{0}\right)\right) - E\inf\limits_{\left\|\overrightarrow{u}\right\|>c} f\left(\overrightarrow{u}, \xi\left(\overrightarrow{0}\right)\right)\right)}{E\left(\inf\limits_{\left\|\overrightarrow{u}\right\|>c} f\left(\overrightarrow{u}, \xi\left(\overrightarrow{0}\right)\right) - E\inf\limits_{\left\|\overrightarrow{u}\right\|>c} f\left(\overrightarrow{u}, \xi\left(\overrightarrow{0}\right)\right)\right)^2};$$

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$$b_2\left(\overrightarrow{t}\right) = b_2\left(\overrightarrow{t}, \overrightarrow{u}\right) = \frac{E\left(f\left(\overrightarrow{u}, \xi\left(\overrightarrow{t}\right)\right) - F\left(\overrightarrow{u}\right)\right)\left(f\left(\overrightarrow{u}, \xi\left(\overrightarrow{0}\right)\right) - F\left(\overrightarrow{u}\right)\right)}{E\left(f\left(\overrightarrow{u}, \xi\left(\overrightarrow{0}\right)\right) - F\left(\overrightarrow{u}\right)\right)^2};$$

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$$b_3\left(\overrightarrow{t}\right) = b_3\left(\overrightarrow{t}, K, \gamma\right) = \frac{E\left(\Psi\left(K, \gamma, \xi\left(\overrightarrow{t}\right)\right) - E\Psi\left(K, \gamma, \xi\left(\overrightarrow{0}\right)\right)\right)\left(\Psi\left(K, \gamma, \xi\left(\overrightarrow{0}\right)\right) - E\Psi\left(K, \gamma, \xi\left(\overrightarrow{0}\right)\right)\right)}{E\left(\Psi\left(K, \gamma, \xi\left(\overrightarrow{0}\right)\right) - E\Psi\left(K, \gamma, \xi\left(\overrightarrow{0}\right)\right)\right)^2},$$

162 where

$$\Psi\left(K, \gamma, z\right) = \sup_{\overrightarrow{u}, \overrightarrow{v} \in K : \left\|\overrightarrow{u} - \overrightarrow{v}\right\| < \gamma} \left| f\left(\overrightarrow{u}, z\right) - f\left(\overrightarrow{v}, z\right)\right|.$$

The following theorem takes place.

Theorem 5 [13]. *Let the next conditions be fulfilled:*

1. *for any* $c > 0$
$$E\left\{\max_{\|\overrightarrow{u}\|<c}\left(f\left(\overrightarrow{u},\xi\left(\overrightarrow{0}\right)\right)\right)^2\right\} < \infty;$$
2. *if* J *is unbounded then for each* $z \in Y$
$$f\left(\overrightarrow{u}, z\right) \to \infty, \text{ as } \left\|\overrightarrow{u}\right\| \to \infty;$$
3. *function (1) has a unique minimum point* u_0;
4. *for all* $c > 0$, $\overrightarrow{u} \in J$, *and any compact* $K \subset J$, $\gamma > 0$
$$\int_0^1 \frac{(\ln\rho)^2}{\rho^m}\left(\int_0^\rho \frac{1}{\tau^2}\left|B_i(\tau)\right| d\tau\right) d\rho < \infty, i = \overline{1,3},$$
where $B_i(\tau) = \int\limits_{\left\|\overrightarrow{t}\right\|<\tau} b_i\left(\overrightarrow{t}\right) d\overrightarrow{t}, i = \overline{1,3}.$

Then for all $T > 0$, $\omega \in \Omega'$, $P(\Omega') = 1$, there exists at least one minimum point $\overrightarrow{u}(T) = \overrightarrow{u}(T,\omega)$ of function (2). For any $T > 0$, the map $\overrightarrow{u}(T,\omega)$ can be chosen G-measurable, where $G = \{A \in G : A \subset \Omega'\}$.

For any minimum point $\overrightarrow{u}(T)$

$$P\left\{\overrightarrow{u}(T) \to \overrightarrow{u}_0, F_T\left(\overrightarrow{u}(T)\right) \to F\left(\overrightarrow{u}_0\right), T \to \infty\right\} = 1.$$

Let us consider a model with nonhomogeneous observations for a random field.

Let $\{\xi(t_1,t_2),(t_1,t_2) \in \mathbb{R}^2\}$ be a homogeneous in a strict sense real random field with continuous trajectories, defined on a complete probabilistic space (Ω, G, P), $X = [a;b] \subset \mathbb{R}$; $h : \mathbb{R}^2 \times X \times \mathbb{R} \to \mathbb{R}$ be a continuous function, convex on $x \in X$.

Investigating the problem

$$F_{T_1T_2}(x) = \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} h\left(t_1,t_2,x,\xi\left(t_1,t_2\right)\right) dt_1dt_2 \to \min, x \in X. \qquad (3)$$

Let the next conditions be fulfilled:

$$\sup\left\{E\left[\max\left|h\left(t_1,t_2,x,\xi\left(t_1,t_2\right)\right)\right|, x \in X\right], t_1,t_2 \geq 0\right\} < \infty;$$

For any $x \in X$,

$$F(x) = \lim EF_{T_1T_2}(x), T_1, T_2 \to \infty;$$

there exist such $x_0 \in X$, $c > 0$, that

$$F(x) \geq F\left(x_0\right) + c\left|x - x_0\right|, x \in X. \qquad (4)$$

Condition (3) implies that x_0 is a unique solution of the problem

$$F(x) \to \min, x \in X. \tag{5}$$

Convexity on $x \in X$ of a function h implies convexity of $F_{T_1 T_2}(x)$ for any T_1, T_2, ω; convexity $EF_{T_1 T_2}(x)$ for all T_1, T_2 and convexity $F(x)$.

For an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, denote

$$g_+{}'(x) = \lim \frac{g\,(x + \triangle) - g(x)}{\triangle}, \triangle \to +0, \tag{6}$$

$$g_-{}'(x) = \lim \frac{g\,(x - \triangle) - g(x)}{\triangle}, \triangle \to +0, \tag{7}$$

if the limits exist.

The next lemma is evidently implied by properties of expectation.

Lemma 1 *Let the function $u : X \times \Omega \to \mathbb{R}$, be convex on the first argument and measurable on the second one, and*

$$E\,|u\,(x, \omega)| < \infty, x \in X.$$

Denote $v(x) = Eu(x, \omega)$. Then,

$$v_+{}'(x) = Eu_+{}'\,(x, \omega)\,, v_-{}'(x) = Eu_-{}'\,(x, \omega)\,.$$

Denote

$$g_{T_1 T_2}(x) = EF_{T_1 T_2}(x), x \in X.$$

The convexity of a function implies the existence of limits (6) and (7) for the function. Therefore, such limits exist:

- *for all t_1, t_2, y for $h(t_1, t_2, \cdot, y)$;*
- *for any t_1, t_2 for $Eh(t_1, t_2, \cdot, \xi(t_1, t_2))$;*
- *for all T_1, T_2, ω for $F_{T_1 T_2}(\cdot)$;*
- *for all T_1, T_2 for $g_{T_1 T_2}(\cdot)$;*
- *for $F(\cdot)$.*

By Lemma 1 for all $t_1, t_2 \in \mathbb{R}, T_1, T_2 > 0, x \in X$

$$(Eh)_+{}'\,(t_1, t_2, x, \xi\,(t_1, t_2)) = E\left\{h_+{}'\,(t_1, t_2, x, \xi\,(t_1, t_2))\right\},$$

$$(Eh)_-{}'\,(t_1, t_2, x, \xi\,(t_1, t_2)) = E\left\{h_-{}'\,(t_1, t_2, x, \xi\,(t_1, t_2))\right\},$$

$$\left(g_{T_1T_2}\right)_+{}'(x) = E\left\{\left(F_{T_1T_2}\right)_+{}'(x)\right\}, \left(g_{T_1T_2}\right)_-{}'(x) = E\left\{\left(F_{T_1T_2}\right)_-{}'(x)\right\}.$$

Lemma 2 *In addition to the conditions formulated above, the next conditions are fulfilled:*

1. *The field* $\xi(t_1,t_2)$ *satisfies a strong mixing condition, i.e., such a function* $a(d)$, $d \geq 0$; $a(d) \to 0$, $d \to \infty$ *exists such that for any* $H_1, H_2 \subset R^2$, *one has*
   $\sup\{|P(A \cap B) - P(A)P(B)|; A \in \sigma(H_1), B \in \sigma(H_2)\} \leq a(d(H_1H_2))$,
   where
   $\sigma(H) = \sigma\{\xi(t_1,t_2),(t_1,t_2) \in H\}$,
   $d(H_1,H_2) = \inf\{\|(t_1,t_2) - (s_1,s_2)\|; (t_1,t_2) \in H_1, (s_1,s_2) \in H_2\}$;
2. $a(d) \leq \frac{c_0}{1+d^{2+\varepsilon}}$, $\varepsilon > 0$,
3. *There exists* $L > 0$, *such that for all* t_1, t_2, ω
   $|h_+{}'(t_1,t_2,x_0,\xi(t_1,t_2))| \leq L$, $|h_-{}'(t_1,t_2,x_0,\xi(t_1,t_2))| \leq L$;
4. *For some* $\delta > \frac{8}{\varepsilon}$
   $E\{|h(t_1,t_2,x,\xi(t_1,t_2))|^{4+\delta}\} < \infty$, $x \in X$, $t_1, t_2 \in \mathbb{R}$;
5. $\left(g_{T_1T_2}\right)_+{}'(x_0) \to F_+{}'(x_0)$, $\left(g_{T_1T_2}\right)_-{}'(x_0) \to F_-{}'(x_0)$, $T \to \infty$;
6. *It exists* $c'' > 0$, *such that for any* $t_2 \in R^+$ $\int_0^{+\infty}\int_0^{+\infty} E\left|\beta_{t_1t_2}\beta_{s_1t_2}\right| dt_1 ds_1 \leq c''$; *where*
   $\beta_{t_1t_2} = h_+{}'(t_1,t_2,x_0,\xi(t_1,t_2)) - Eh_+{}'(t_1,t_2,x_0,\xi(t_1,t_2))$;
7. *It exists* $c''' > 0$, *such that for all* $t_1 \in \mathbb{R}^+$, *one has* $\int_0^{+\infty}\int_0^{+\infty} E\left|\beta_{t_1t_2}\beta_{t_1s_2}\right| dt_2 ds_2 \leq c'''$;.
8. *Analogous to (6) and (7), the conditions are fulfilled for the left derivative.*

Then, with probability 1,

$$\left(F_{T_1T_2}\right)_+'(x_0) \to F_+{}'(x_0), T_1, T_2 \to \infty; \tag{8}$$

$$\left(F_{T_1T_2}\right)_-{}'(x_0) \to F_-{}'(x_0), T_1, T_2 \to \infty. \tag{9}$$

The lemma is by standard means implied by the Borel–Cantelli lemma.

Theorem 6 *With probability 1, there exist such* $T_{01} = T_{01}(\omega)$, $T_{02} = T_{02}(\omega)$ *that for all* $T_1 > T_{01}$, $T_2 > T_{02}$, *problem (3) has a unique solution* $x(T_1,T_2) = x_0$.

Proof By (4)

$$F_+{}'(x_0) \geq c, F_-{}'(x_0) \geq c.$$

Owing to Lemma 2 with probability 1 beginning from some T_{01}, T_{02}, one has

$$\left(F_{T_1T_2}\right)_+{}'(x_0) > 0, \left(F_{T_1T_2}\right)_-{}'(x_0) > 0. \tag{10}$$

Now, (10) and the convexity of $F_{T_1T_2}$ imply the theorem.

Another important property of estimates is their limit distributions. It is important to know if the true value is an interior point of the domain of admissible values or if it belongs to the boundary of this domain. We will not formulate all the conditions under which one can prove the statement on the limit distribution of the estimate because these conditions are indeed very complicated. However, under some conditions on the smoothness of the criterion function and strong mixing condition on the respective random processes (or random fields), the asymptotic distribution is Gaussian.

For example, if we have the observations of the random field in area $\left\| \vec{t} \right\| < T$, then the normed variable has the form

$$\left(\int_{\left\| \vec{t} \right\| < T} d\,\vec{t} \right)^{\frac{1}{2}} \left(\vec{u}(T) - \vec{u_0} \right)$$

and

$$\left(\int_{\left\| \vec{t} \right\| < T} d\,\vec{t} \right)^{\frac{1}{2}} \left(F_T \left(\vec{u}(T) \right) - F\left(\vec{u_0} \right) \right).$$

5 Method of Empirical Means Under the Restrictions of Unknown Parameters, Described in the Form of Equalities and Inequalities

Furthermore, we consider a case where the restrictions are of the form

$$J = \{u : g(u) = (g_1(u), \ldots, g_n(u)) \leq 0\}.$$

Then, ~~the family of vectors~~ converges weakly to the random vector $\boldsymbol{\eta}$, which is the solution to the problem

$$\frac{1}{2} \vec{u}\, \boldsymbol{\Phi} \left(\vec{u_0} \right) \vec{u} + \varsigma \vec{u} \to \min,$$

$$\nabla g^T \left(\vec{u_0} \right) \vec{u} \leq 0$$

These models have been studied in detail in [14].

6 Large Deviations in the Method of Empirical Means

The following problem consists of obtaining some theorems of large deviations for a method of empirical means for the dependent observations. We formulate them for random fields. The results for random processes are analogous.

Let $\{\xi(t_1,t_2),(t_1,t_2)\in\mathbb{R}^2\}$ be a homogeneous in a strict sense random field with continuous trajectories defined on a full probabilistic space (Ω,G,P), with values in some metric space (Y,ρ).

Considering the problem

$$F(x)=E\,f\,(x,\xi\,(0,0))\to\min,\quad x\in X, \tag{11}$$

where X is a nonempty compact subset of $\mathbb{R}$ and where $f:X\times Y\to\mathbb{R}$ is a continuous nonnegative function.

The problem can be approximated as follows:

$$F_{T_1T_2}(x)=\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}f\,(x,\xi\,(t_1,t_2))\,dt_1dt_2\to\min,\quad x\in X, \tag{12}$$

where $T_1>0$, $T_2>0$.

Owing to the properties of continuous functions at least one solution $x(T_1,T_2)$ of problem (12) exists, which is a measurable function of ω.

Suppose that

$$E\,\max\{|f\,(x,\xi\,(0,0))|\,,x\in X\}<\infty.$$

Then, $F(\cdot)$ is continuous, and at least one solution x_0 of problem (11) exists. Suppose that it is unique.

Let the field $\xi(t_1,t_2)$ satisfy the strong mixing condition, i.e., there exists such a function $a(d)$, $d\geq 0$; $a(d)\searrow 0$, $d\to\infty$ such that for all $H_1,H_2\subset\mathbb{R}^2$, one has

$$\sup\{|P\,(A\cap B)-P(A)P(B)|\,;A\in\sigma\,(H_1)\,,B\in\sigma\,(H_2)\}\leq a\,(d\,(H_1,H_2))\,,$$

where $\sigma(H)=\sigma\{\xi(t_1,t_2),(t_1,t_2)\in H\}$, $d(H_1,H_2)=\inf\{\|(t_1,t_2)-(s_1,s_2)\|;\ (t_1,t_2)\in H_1,(s_1,s_2)\in H_2\}$.

Suppose that $a(d)=O(d^{-2-\varepsilon})$, $d\to\infty$, for some $\varepsilon>0$, and for some $\delta>\frac{8}{\varepsilon}$

$$E\left\{|f\,(x,\xi\,(0,0))|^{4+\delta}\right\}<\infty,x\in X.$$

If the conditions are fulfilled, then by [13] with probability 1

$$x\,(T_1,T_2)\to x_0,\,F_{T_1T_2}\,(x\,(T_1,T_2))\to F\,(x_0)\,;T_1,T_2\to\infty.$$

Consider the probability of large deviations $x(T_1, T_2)$ from x_0 and $F_{T_1T_2}(x(T_1, T_2))$ from $F(x_0)$.

For any fixed y, one can consider $f(\cdot\ , y)$ as an element of the space of continuous functions $C(X)$. Suppose that a convex compact set $K \subset C(X)$ exists such that for all $y \in Y$, one has $f(\cdot\ , y) - F(\cdot) \in K$. Consider $F_{T_1T_2} - F$ as random elements on (Ω, G, P) with values in K.

Let us use some results from functional analysis.

Definition 1 [15]. Let $(V, \|\circ\|)$ be a linear norm space; $B(x, r)$ is a closed ball with a radius r and a center x; $f : V \to [-\infty, +\infty]$ is some function; and x_f is its minimum point on V. Conditioning function ψ for f in x_f is a monotone nondecreasing function $\psi : [0, +\infty) \to [0, +\infty]$, $\psi(0) = 0$, such that $r > 0$, exists for any $x \in B(x_f, r)$ one has

$$f(x) \geq f\left(x_f\right) + \psi\left(\left\|x - x_f\right\|\right).$$

Let $V_0 \subset V$. Denote

$$\delta_{V_0}(x) = 0, x \in V_0,$$

$$\delta_{V_0}(x) = +\infty, x \notin V_0.$$

Theorem 6 [15]. *Let* $(V, \|\circ\|)$ *be a linear normed space, where* $V_0 \subset V$ *is closed and where* $f_0, g_0 : V \to \mathbb{R}$ *are continuous on* V *functions. Suppose*

$$\varepsilon = \sup\left\{\left|f_0(x) - g_0(x)\right|, x \in V_0\right\}.$$

Introduce functions $f, g : V \to (-\infty, +\infty]$:

$$f = f_0 + \delta_{V_0}, g = g_0 + \delta_{V_0}.$$

Then,

$$\left|\inf\left\{f(x), x \in V\right\} - \inf\left\{g(x), x \in V\right\}\right| \leq \varepsilon.$$

Let x_f be a minimum point of f on V, ψ is a conditioning function for f in x_f with a coefficient r. If ε is small enough for

$$\psi\left(\left\|x - x_f\right\|\right) \leq 2\varepsilon \Rightarrow \left\|x - x_f\right\| \leq r,$$

then, for any $x_g \in \arg\min\{g(x), x \in B(x_f, r)\}$, one has $\psi(\|x_f - x_g\|) \leq 2\varepsilon$. When ψ is convex and strictly increasing on $[0, r]$,

$$\psi^{-1}(2\varepsilon) \leq r \Rightarrow \left\|x_f - x_g\right\| \leq \psi^{-1}(2\varepsilon)$$

$$\forall x_g \in \arg\min\left\{g(x),\, x \in B\left(x_f, r\right)\right\}.$$

Let us use some results from large deviation theory.

Definition 2 [8]. Let Σ be a separable Banach space, where $\{\zeta(t_1,t_2),(t_1,t_2)\in\mathbb{R}^2\}$ is a homogeneous in a strict sense random field on (Ω, G, P) with values in Σ. For $\tau > 0$, random values $\eta_1, \ldots, \eta_p$; $p \geq 2$ are called τ-measurably separated if η_j is $\sigma\{\zeta(t_1,t_2),(t_1,t_2)\in H_j\}$—measurable for all $j \in \{1, \ldots, p\}$, where $d(H_i, H_j) \geq \tau$, $i \neq j$; $H_j, j = 1, \ldots, p$ are Borel sets in $\mathbb{R}^2$; and $d(\cdot,\cdot)$ is the distance between the sets.

Definition 3 [8]. One says that the random field from Definition 2 satisfies the first hypothesis of hypermixing if such $\tau_0 \in \mathbb{N}\cup\{0\}$ and a nonincreasing function $\alpha : \{\tau > \tau_0\} \to [1, +\infty)$ exist,

$$\lim_{\tau\to\infty}\alpha(\tau) = 1;\ \left\|\eta_1\ldots\eta_p\right\|_{L^1} \leq \prod_{j=1}^{p}\left\|\eta_j\right\|_{L^{\alpha(\tau)}}$$

for all $p \geq 2$, $\tau > \tau_0$; $\eta_1, \ldots, \eta_p$ τ-measurably separated, where

$$\|\eta\|_{L^r} = \left(E\left\{|\eta|^r\right\}\right)^{\frac{1}{r}}.$$

Suppose that $X = [a, b] \subset \mathbb{R}$. As is known [16], one has $(C(X))^* = M(X)$—a space of bounded signed measures on X, and

$$\langle g, Q\rangle = \int_X g(x)Q(dx);\ g \in C(X),\ Q \in M(X).$$

The next assertion takes place.

Theorem 7 *Let $\zeta(t_1,t_2)$, $(t_1,t_2)\in R^2$ be a homogeneous in a strict sense random field satisfying the hypermixing condition on (Ω, G, P), with values in a compact convex set $K \subset C(X)$. Then for all $Q \in M(X)$, there exists a finite limit*

$$\Lambda(Q) = \lim_{T_1,T_2\to\infty}\frac{1}{T_1T_2}\ln\left(E\exp\int_X\int_0^{T_1}\int_0^{T_2}\zeta\left(t_1,t_2\right)(x)dt_1dt_2Q(dx)\right),$$

and for any closed $A \subset K$,

$$\limsup_{T_1,T_2\to\infty}\frac{1}{T_1T_2}\ln P\left\{\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\zeta\left(t_1,t_2\right)dt_1dt_2\in A\right\} \leq -\inf\left\{\Lambda^*(g), g\in A\right\},$$

where $\Lambda^*(g) = \sup\left\{\int\limits_X g(x)Q(dx) - \Lambda(Q),\ Q \in M(X)\right\}$ *is a nonnegative convex lower semicontinuous function.*

Proof By the partition method analogous to that of [6], the following exists:

$$\Lambda(Q) = \lim_{T_1,T_2\to\infty} \frac{f_{T_1T_2}}{T_1T_2} \in [-\infty, +\infty]\,.$$

The function under the limit is bounded, so the limit is finite.

Let us use the theorem from large deviations theory [6]. One has

$$H = K,\ J = C(X),\ J^* = M(X),\ \langle Q, g\rangle = \int\limits_X g(x)Q(dx),\ \varepsilon_1 = \frac{1}{T_1},\ \varepsilon_2 = \frac{1}{T_2}.$$

Furthermore, $\mu\left(\frac{1}{T_1}, \frac{1}{T_2}\right)$ is a probability measure on $C(X)$, which is defined by the distribution

$$\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} \zeta\,(t_1, t_2)\,dt_1dt_2.$$

Then,

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\varepsilon_2\Lambda_{\mu(\varepsilon_1,\varepsilon_2)}\left(\frac{Q}{\varepsilon_1\varepsilon_2}\right) =$$

$$= \lim_{T_1,T_2\to\infty}\frac{1}{T_1T_2}\ln\left(\int\limits_{C(X)} \exp\left\{\int\limits_X g(x)T_1T_2Q(dx)\right\}\mu\left(\frac{1}{T_1}, \frac{1}{T_2}\right)(dg)\right) =$$

$$= \lim_{T_1,T_2\to\infty}\frac{1}{T_1T_2}\ln\left(E\exp\left\{\int\limits_X \left(\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\zeta\,(t_1, t_2)\,dt_1dt_2\right)(x)T_1T_2Q(dx)\right\}\right)$$

$$= \lim_{T_1,T_2\to\infty}\frac{f_{T_1T_2}}{T_1T_2} = \Lambda(Q).$$

The theorem is proved.

Return to problems (11) and (12).

Theorem 8 Let $\xi(t_1, t_2)$ *satisfy the hypermixing condition. Then, for any* $\varepsilon > 0$

$$\limsup_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \ln P\left\{\left\| F_{T_1 T_2} - F \right\| \geq \varepsilon\right\} \leq -\inf\left\{I(z), z \in A_\varepsilon\right\},$$

where $I(z) = \Lambda^*(z) = \sup\left\{\int_X z(x) Q(dx) - \Lambda(Q),\ Q \in M(X)\right\}$ *is a nonnegative lower semicontinuous convex function,*

$$\Lambda(Q) = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \ln\left(E \exp\left\{\int_X \int_0^{T_1} \int_0^{T_2} \left[f(x, \xi(t_1, t_2)) - F(x)\right] dt_1 dt_2 Q(dx)\right\}\right),$$

$$A_\varepsilon = \left\{z \in K : \|z\| \geq \varepsilon\right\}.$$

Proof A_ε is a closed subset of K. The field

$$\zeta(t_1, t_2) = f(\cdot, \xi(t_1, t_2)) - F(\cdot),$$

with values in K, is a continuous function of $\xi(t_1, t_2)$, and then it satisfies the conditions of Theorem 7. Therefore, the theorem is a consequence of Theorem 7.

Theorem 9 *Let the conditions of Theorem 8 be satisfied. Then,*

$$\begin{aligned} &\limsup_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \ln P\left\{\left|\min\left\{F(x), x \in X\right\} - \min\left\{F_{T_1 T_2}(x), x \in X\right\}\right| \geq \varepsilon\right\} \\ &\qquad \leq -\inf\left\{I(z), z \in A_\varepsilon\right\}, \end{aligned} \tag{13}$$

where $I(\cdot)$ *and* A_ε *are defined in Theorem 8.*

Suppose that a conditioning function ψ exists for F in x_0 with some constant r. Let $x(T_1, T_2)$ be a minimum point of (2) on $B(x_0, r)$. If ε is sufficiently small, then

$$\psi(|x - x_0|) \leq 2\varepsilon \Rightarrow |x - x_0| \leq r,$$

Then, one has

$$\limsup_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \ln P\left\{\psi(|x(T_1, T_2) - x_0|) \geq 2\varepsilon\right\} \leq -\inf\left\{I(z), z \in A_\varepsilon\right\}. \tag{14}$$

Proof By Theorem 6 for all ω

$$\left|\min\left\{F(x), x \in X\right\} - \min\left\{F_{T_1 T_2}(x), x \in X\right\}\right| \leq \left\| F_{T_1 T_2} - F \right\|.$$

Then, Theorem 8 implies (13).

Furthermore, by Theorem 6 for allω

$$\psi\left(|x_0 - x\left(T_1, T_2\right)|\right) \le 2\left\| F_{T_1T_2} - F\right\|,$$

In addition, Theorem 8 implies (14).

Remark If, in addition to the conditions of Theorem 9, ψ is convex and strictly increasing on $[0, r]$,, then

$$\limsup_{T_1,T_2\to\infty}\frac{1}{T_1T_2}\ln P\left\{|x\left(T_1, T_2\right) - x_0| \ge \psi^{-1}\left(2\varepsilon\right)\right\} \le -\inf\left\{I\left(z\right), z \in A_\varepsilon\right\}. \tag{15}$$

Proof By Theorem 6 for allω

$$|x_0 - x\left(T_1, T_2\right)| \le \psi^{-1}\left(2\left\| F_{T_1T_2} - F\right\|\right).$$

Then,

$$P\left\{|x\left(T_1, T_2\right) - x_0| \ge \psi^{-1}\left(2\varepsilon\right)\right\} \le P\left\{\psi^{-1}\left(2\left\| F_{T_1T_2} - F\right\|\right) \ge \psi^{-1}\left(2\varepsilon\right)\right\} =$$

$$= P\left\{\left\| F_{T_1T_2} - F\right\| \ge \varepsilon\right\},$$

In addition, Theorem 8 implies (15).

Theorem 10 *Let the field $\{\xi(t_1,t_2),(t_1,t_2) \in R^2\}$ has hypermixing. Suppose that the function h does not depend on t_1, t_2. Let the function*

$$\min\left[h_+{}'\left(x_0, y\right), h_-{}'\left(x_0, y\right)\right], y \in R$$

be continuous.

Then,

$$\limsup_{T_1,T_2\to\infty}\frac{1}{T_1T_2}\ln P\left(A^c_{T_1T_2}\right) \le -\inf\left\{V^*(z), z \in [-L; 0]\right\}, \tag{16}$$

where $V^*(z) = \sup\{zQ(X) - V(Q), Q \in M(X)\}$,

$$V\left(Q\right) = \lim\frac{1}{T_1T_2}\ln E\exp\left\{Q\left(X\right)\times\right.$$

$$\times \int_0^{T_1} \int_0^{T_2} \min\left[h_+{}'\left(x_0, \xi\left(t_1, t_2\right)\right), h_-{}'\left(x_0, \xi\left(t_1, t_2\right)\right)\right] dt_1 dt_2 \right\}, T_1, T_2 \to \infty,$$

$$A_{T_1T_2} = \left\{\omega : \arg\min F_{T_1T_2}(x) = \{x_0\}, x \in X\right\}, A^c_{T_1T_2} = \Omega - A_{T_1T_2}.$$

Proof One has

$$P\left(A^c_{T_1T_2}\right) = P\left\{\min\left[\left(F_{T_1T_2}\right)_+{}'(x_0), \left(F_{T_1T_2}\right)_-{}'(x_0)\right] \in [-L; 0]\right\} \leq$$

$$\leq P\left\{\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} \min\left[h_+{}'\left(x_0, \xi\left(t_1, t_2\right)\right), h_-{}'\left(x_0, \xi\left(t_1, t_2\right)\right)\right] dt_1 dt_2 \in [-L; 0]\right\}. \tag{17}$$

Denote

$$K = \{\alpha(x) = \alpha, x \in X; \alpha \in [-L; L]\}.$$

Evidently, K is a compact convex subset of $C(X)$.

Investigating the function

$$a_{t_1t_2} = a_{t_1t_2}(x) = \min\left[h_+{}'\left(x_0, \xi\left(t_1, t_2\right)\right), h_-{}'\left(x_0, \xi\left(t_1, t_2\right)\right)\right], x \in X.$$

For all t_1, t_2, ω, one has $a_{t_1t_2}(\cdot) \in K$. Let

$$K_1 = \{\alpha(x) = \alpha, x \in X; \alpha \in [-L; 0]\}.$$

Then, K_1 is a closed subset of K. Furthermore,

$$P\left\{\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} \min\left[h_+{}'\left(x_0, \xi\left(t_1, t_2\right)\right), h_-{}'\left(x_0, \xi\left(t_1, t_2\right)\right)\right] dt_1 dt_2 \in [-L; 0]\right\} =$$

$$= P\left\{\left(\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} a_{t_1t_2} dt_1 dt_2 = \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} a_{t_1t_2} dt_1 dt_2(x), x \in X\right) \in K_1\right\}. \tag{18}$$

Let us use the theorem from large deviations theory. One has

$$\limsup_{T_1, T_2 \to \infty} \frac{1}{T_1T_2} \ln P\left\{\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} a_{t_1t_2} dt_1 dt_2 \in K_1\right\} \leq -\inf\left\{V^*(z), z \in K_1\right\}, \tag{19}$$

where $V^*(z) = \sup\{zQ(X) - V(Q), Q \in M(X)\}$,

$$V(Q) = \lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2} \ln E \exp\left\{ \int_0^{T_1} \int_0^{T_2} \int_X a_{t_1t_2} Q(dx) dt_1 dt_2 \right\} =$$

$$= \lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2} \ln Eexp \left\{ Q(X) \int_0^{T_1} \int_0^{T_2} a_{t_1t_2} dt_1 dt_2 \right\}.$$

Now, (17), (18), and (19) imply (16). The proof is complete.

7 Consequences

Large deviation theory is used in many spheres of science and practice, such as mathematics, physics, informatics, and economics. It continues to develop prospectively, and many new works on large deviations have appeared.

The results can be used for solving different stochastic optimization problems, which appear in recognition theory and regressive analysis, when one needs to find optimal solutions via observations of a stationary random process or a homogeneous random field [17–25].

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