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THE CARBON CYCLE OF THE EARTH--
A MATERIAL BALANCE APPROACH*

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PREFACE

For several years there has been growing concern among scientists that the increasing rate of burning fossil fuels (coal, oil and gas) and the associated release of carbon dioxide into the atmosphere might increase the temperature of the earth (the so-called "greenhouse effect") and thereby affect the climate. Efforts have, therefore, been made by several groups to analyze the behavior of carbon dioxide in the atmosphere, oceans and biosphere (the "carbon cycle" of the earth) in the preindustrial undisturbed as well as in the present disturbed state; and to find a way in which the increasing release of carbon dioxide into the atmosphere can be avoided. With respect to the latter question, members of the IIASA study project on energy systems have made a specific proposal, namely to bury the CO_2 from burnt fossil fuels directly in the deep sea, which represents the ultimate sink of all carbon dioxide.

In this paper, the carbon dioxide cycle of the earth has been analyzed from a material-accountability point of view. For this purpose a four-box model for the cycle is considered, and equilibrium as well as disturbed states are evaluated; in particular, the time constants of the disturbed state are discussed. The theoretical results are illustrated with the help of numerical data, and the idea of deep-sea burial of carbon dioxide is discussed in the light of these results.

ABSTRACT

In this paper an idealized model of the carbon cycle of the earth has been analyzed. The model consists of four boxes (atmosphere, biosphere, surface layer of the oceans, and deep sea) and the carbon cycle is described as a flow of the carbon through the four boxes. The mathematical analysis, using a discrete time formalism, is easily generalized to models with more boxes.

The results of the analysis are applied to several practical problems. Consistency checks of data on inventories and transition coefficients reported in the literature are made, and the influence of disturbances of the cycle (especially the addition of carbon dioxide to the atmosphere by burning fossil fuels) with respect to their sizes and time scales is discussed.

1. INTRODUCTION

The carbon cycle of the earth[†] has been a subject of major interest in recent years. There is considerable concern that the continuously increasing burning of fossil fuels and the related release of carbon dioxide into the atmosphere might have a major impact on the climate as well as on the ecological system of the oceans. It is argued that the increased carbon dioxide in the atmosphere might lead to an increase in the temperature on the earth (the so-called greenhouse effect; see, e.g., [2]); and that, because of the carbon dioxide exchange between the atmosphere and the oceans, a critical situation might arise for calcareous organisms living in the sea and for the food chains of which they are a part ([3], [4]). In any case, according to Sawyer [5] "there is little doubt that in assessing the future level of carbon dioxide in the atmosphere, it is important to understand fully the balance between the carbon dioxide in the atmosphere and the ocean".

In this paper, the carbon cycle of the earth has been considered from a material balance point of view. The work was stimulated by earlier work by Machta [6] and Zimen and Altenheim [3], [4], as well as by related methodological work in various fields [7]. The idea was to establish a careful material balance for the carbon flowing through the different media of interest, which in the following will be called "boxes". The mathematical formalism developed in this paper is based on a four-box model including the atmosphere, biosphere, upper layer of the sea above the thermocline ("mixed sea") and deep sea. However, it lends itself easily to the treatment of different models, such as the seven-box model as used by Machta [6].

The main results of the theory are relations of transition coefficients between boxes and inventories of boxes which have to be satisfied in the equilibrium state, as well as, relations

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[†] For a complete description see, e.g., [1].

describing the influence of external inputs (such as carbon dioxide released into the atmosphere as a result of the burning of fossil fuels) and the way in which the system "digests" these inputs.

The results are used for the analysis of carbon cycle data reported in the literature, for checking some measurement data, and finally, for giving an indication of the long term carbon dioxide content of the atmosphere.

2. THEORETICAL CONSIDERATIONS

The four-box model for the carbon cycle of the earth as presented by Sawyer [5] and Zimen and Altenheim [4] may be described as follows (see Figure 1). Given are the four boxes atmosphere (a), biosphere (b), upper mixed layer of the sea (m) and deep sea (d). At time t_i these boxes contain the CO_2 inventories I_i^a, I_i^b, I_i^m and I_i^d (measured in mol). In the time interval (t_i, t_{i+1}) parts of the inventories are exchanged; the transition from box x to box y is determined by the exchange coefficient k^{xy} (measured in reciprocal years). In addition, during time (t_i, t_{i+1}) we have the CO_2 input $n_{i,i+1}$ into the atmosphere that results from the burning of fossil fuels. Therefore, according to Figure 1, we have the following relations for the CO_2 inventories in the different boxes at time t_{i+1}:

$$
\begin{aligned}
I_{i+1}^a &= I_i^a - k^{ab} \cdot I_i^a - k^{am} \cdot I_i^a + k^{ba} \cdot I_i^b + k_i^{ma} \cdot I_i^m + n_{i,i+1} \\
I_{i+1}^b &= I_i^b + k^{ab} \cdot I_i^a - k^{ba} \cdot I_i^b \\
I_{i+1}^m &= I_i^m + k^{am} \cdot I_i^a + k^{dm} \cdot I_i^d - k^{ma} \cdot I_i^m - k^{md} \cdot I_i^m \qquad (1) \\
I_{i+1}^d &= I_i^d + k^{md} \cdot I_i^m - k^{dm} \cdot I_i^d \quad .
\end{aligned}
$$

In the following we will consider one-year time intervals. This assumption will be discussed in Section 2.3.

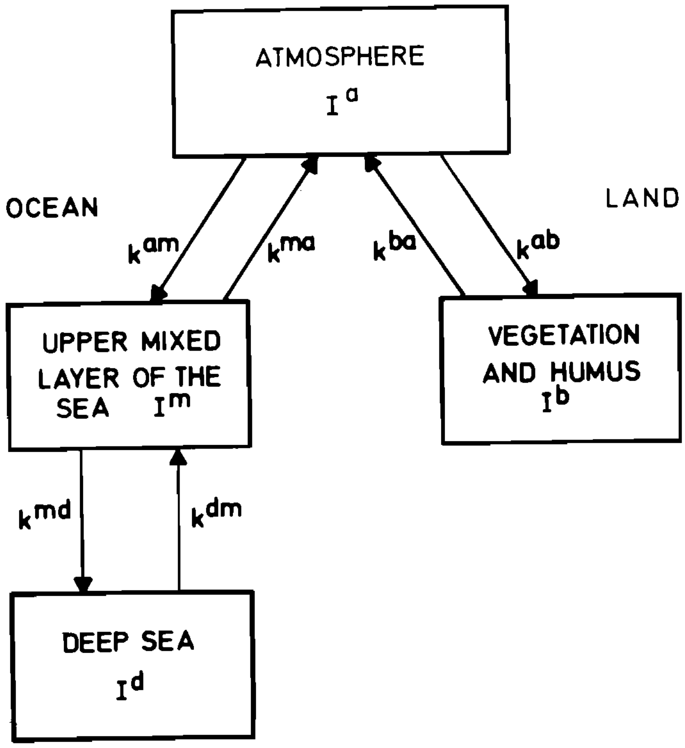

Figure 1. Natural reservoirs of carbon dioxide (based on [8]). I^x is the content of reservoir x, k^{xy} describes the transition from reservoir x to reservoir y.

This model may be characterized by an endomorphism (linear selfmapping) of the four-dimensional real space $\mathbb{R}^4$. The vectors in $\mathbb{R}^4$ are the states of the system, i.e., the CO_2 inventories of the different boxes. Let

$$\varphi: \quad \mathbb{R}^4 \rightarrow \mathbb{R}^4 \qquad (2)$$

be the mapping describing the system, and I_i be the state of the system at time i. Then $\varphi(I_i)$ is the state of the system at time t_{i+1}. The mapping φ can be described by a matrix A with respect to the canonical basis. Therefore, the transition from the state I_i to the state I_{i+1} reduces to a matrix multiplication

$$I_{i+1}^{t} = A \circ I_{i}^{t} + N_{i,i+1}^{t} \quad ^{\dagger} \tag{3}$$

where $N_{i,i+1} = (n_{i,i+1}, 0, 0, 0)$, and where the matrix A is according to eqs. (1) given by

$$A = \begin{pmatrix} 1-k^{ab}-k^{am} & k^{ba} & k^{ma} & 0 \\ k^{ab} & 1-k^{ba} & 0 & 0 \\ k^{am} & 0 & 1-k^{ma}-k^{md} & k^{dm} \\ 0 & 0 & k^{md} & 1-k^{dm} \end{pmatrix} . \tag{4}$$

2.1 The Equilibrium State

In this section we analyze the pre-industrial state of the CO_2 inventories of the four boxes i.e. the state with $n_{i,i+1} = 0$ for all i = 1,2,... when the carbon cycle was still in equilibrium. An equilibrium state $\tilde{I} \in \mathbb{R}^4$ is defined by

$$\tilde{I}^t = A \circ \tilde{I}^t \quad . \tag{5}$$

This means that the equilibrium state is given by the set of eigenvectors of matrix A with eigenvalue 1. We will show that matrix A has eigenvalue 1. First we will discuss the material conservation property of our model:

Theorem 1 (material conservation). Let $\Sigma: \mathbb{R}^4 \rightarrow \mathbb{R}$ be the linear mapping that maps each element from $\mathbb{R}^4$ into the sum of of its coefficients.[††] If $I \in \mathbb{R}^4$ is a state of the system (1),

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[†] The vector X is defined as a row vector. Therefore the transposed vector X^t is a column vector.

[††] This linear mapping is described by a multiplication of $I \in \mathbb{R}^n$ with the row matrix (1,1,...,1), i.e.,

$$\Sigma(I) = (1,1,\ldots,1) \circ I^t = \sum_{j=1}^{n} I_j \quad .$$

and if $I' = A \circ I^t$, then $\Sigma(I) = \Sigma(I')$ holds.

Proof. Let $A(*,i)$ be the i-th column vector of A, and let $e_1,\ldots,e_4$ be the unit vectors of $\mathbb{R}^4$. Then $I = (I_1,\ldots,I_4)$ may be represented as a linear combination of the unit vectors:

$$I = \sum_{j=1}^{4} I_j \cdot e_j \quad .$$

Then we have

$$I'^t = A \circ I^t = A \circ \sum_{j=1}^{4} I_j \cdot e_j^t = \sum_{j=1}^{4} I_j \cdot A \circ e_j^t = \sum_{j=1}^{4} I_j \cdot A(*,j) \quad ,$$

and furthermore

$$\Sigma(I') = \sum_{j=1}^{4} I_j \cdot A(*,j)^t = \sum_{j=1}^{4} I_j \Sigma(A(*,j)^t) = \sum_{j=1}^{4} I_j = \Sigma(I) \quad ,$$

because we have

$$\Sigma(A(*,j)) = 1 \quad . \qquad \blacksquare$$

It can now be shown [9] that any non-negative square matrix, whose column vector sums are 1, has the eigenvalue 1 with an associated positive eigenvector. In other words, any system of form (1) that has the property of material conservation has eigenstates with eigenvalue 1. However, we will prove the following theorem.

Theorem 2. The matrix A has the eigenvector

$$\tilde{I} = (1,\ k^{ab}/k^{ba},\ k^{am}/k^{ma},\ k^{am}/k^{ma} \cdot k^{md}/k^{dm}) \qquad (6)$$

with the eigenvalue 1.

Proof. One sees immediately that $\tilde{I}$ satisfies (5), which is equivalent to the following system of equations:

$$\begin{aligned}
(-k^{ab} - k^{am}) \cdot X_1 + k^{ba} \cdot X_2 + k^{ma} \cdot X_3 &= 0 \\
k^{ab} \cdot X_1 - k^{ba} \cdot X_2 &= 0 \\
k^{am} \cdot X_1 + (-k^{ma} - k^{md}) \cdot X_3 + k^{dm} \cdot X_4 &= 0 \\
k^{md} \cdot X_3 - k^{dm} \cdot X^4 &= 0
\end{aligned} \quad (7)$$

. ■

Corollary 1. The eigenvectors of A with eigenvalue 1 determine a subspace of $\mathbb{R}^4$ with the dimension 1.

Proof. The proof follows immediately by solving the system (7). ■

To summarize the results obtained so far, the stable states of the system (1) are the eigenvectors of A with eigenvalue 1; because of Corollary 1, we obtain all stable states by multiplying (6) with a scalar factor.

So far we have considered only the transition from state I_i to the state I_{i+1}. We now consider the transition from state I_i to state I_{i+n}, $n > 1$. One sees immediately that the state I_{i+n} is obtained from the state I_i by an n-fold multiplication with the matrix A, i.e.

$$I_{i+n} = A^n \circ I_i = A \circ A \ldots \circ A \circ I_i; \; A^0 = E \quad . \qquad (8)$$

We can omit brackets because matrix multiplication is associative. The properties of A^n are described by the following theorem.

Theorem 3.

1. The matrix A^n fulfills the material conservation condition, i.e., from

$$I_{i+n} = A^n \circ I_i$$

it follows that

$$\Sigma(I_{i+n}) = \Sigma(I_i) \quad ,$$

where Σ is defined as in Theorem 1.

2. If $\tilde{I}$ is an eigenvector of A with eigenvalue 1, then $\tilde{I}$ is also an eigenvector of A^n with eigenvalue 1.

Proof (by complete induction).

1. We have $A^0 = E$, and E fulfills the material conservation condition. Assume

$$I \in \mathbb{R}^4, \; I_n = A^n \circ I, \; \Sigma(I_n) = \Sigma(I) \quad .$$

Then we have

$$I_{n+1} = A^{n+1} \circ I = A \circ I_n \quad ,$$

and therefore

$$\Sigma(I_{n+1}) = \Sigma(I_n) = \Sigma(I) \quad .$$

2. Let I be an eigenvector of A with eigenvalue 1. We have $E \circ \tilde{I} = \tilde{I}$; therefore $\tilde{I}$ is an eigenvector of A^0 with eigenvalue 1. Let $\tilde{I}$ be the eigenvector of A^n with eigenvalue 1. Then we have for A^{n+1}

$$A^{n+1} \circ \tilde{I} = A \circ A^n \circ \tilde{I} = A \circ \tilde{I} = \tilde{I} \quad .$$

Therefore, $\tilde{I}$ is an eigenvector of A^{n+1} with eigenvalue 1. ∎

Theorem 1 implies that the system which is in an equilibrium state remains in this equilibrium state. We now ask how a state $I \in \mathbb{R}^4$ that is not an equilibrium state will develop. We expect that it will evolve into an equilibrium state. In other words we ask how A^n behaves if n goes to infinity.

Definition 1. A sequence of $n \times n$ matrices $A_\rho = (a_{ik}^{(\rho)})$, $\rho = 1,2,\ldots,$ converges towards a limiting matrix $A = (\alpha_{ik})$:

$$\lim_{\rho \to \infty} A_\rho = A$$

if we have for each element a_{ik}, $i = 1,\ldots,n$, $k = 1,\ldots,n$

$$\lim_{\rho \to \infty} a_{ik}^{(\rho)} = \alpha_{ik}$$

in the usual sense. ■

With this definition we now establish the following theorem.

Theorem 4. The sequence $(A^n)_{n \in N}$ converges. Let

$$A^* = \lim_{n \to \infty} A^n \tag{9a}$$

and let X be the eigenvector of A with eigenvalue 1 and $\Sigma(X) = 1$. Then we have

$$A^* = (X^t, X^t, X^t, X^t) \quad . \tag{9b}$$

Proof. The proof needs the following definition.
Definition. A matrix A is called irreducible if there exists no matrix of the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B is a square sub-matrix obtained from A by permuting rows and columns.

Moreover, it needs the following theorem (see [10], p. 122).

Theorem. Let A be a non-negative irreducible square matrix with exactly t eigenvalues with amount λ_1; let X be an eigenvector of A with eigenvalue λ_1; let Y be an eigenvector of A^t with eigenvalue λ_1, and let

$$Y \cdot X^t = \sum_{i=1}^{n} X_i \cdot Y_i = 1 \quad .$$

Then we have

$$\lim_{n \to \infty} \frac{1}{t} \left(\frac{1}{\lambda_1^n} \cdot A^n + \ldots + \frac{1}{\lambda_1^{n+t-1}} \cdot A^{n+t-1} \right) = X^t \cdot Y \quad .$$

It can be seen immediately that A is irreducible. In addition, because of the material conservation property, we have $Y = (1,1,\ldots,1)$

$$A^t \circ Y^t = (Y \circ A)^t = Y^t \quad ,$$

which means that $Y = (1,\ldots,1)$ is an eigenvector of A^t with eigenvalue 1. Therefore, we have $Y \cdot X^t = 1$, and with the help of the theorem we get

$$A^* = X^t \circ Y = (X^t, X^t, X^t, X^t) \quad .$$

■

It should be noted that A^n converges uniformly and geometrically. Some properties of A^* are given by the following theorem.

Theorem 5. Let A, A^*, X be defined as above, and $I \in \mathbb{R}^4$ be an arbitrary state of the system. Then we have

1. $A^* \circ I = (\Sigma(I)) \cdot X$,

2. A^* has the material conservation property,

3. $A^* \circ I$ is an eigenvector of A with eigenvalue 1,

4. $A \circ A^* = A^*$, $A^* \circ A = A^*$,

5. If $\tilde{I}$ is an eigenvector of A with eigenvalue 1, then $\tilde{I}$ is also an eigenvector of A^* with eigenvalue 1.

Proof.

1. $A^* \circ I = A \cdot \sum_{j=1}^{4} I_j \cdot e_j = \sum_{j=1}^{4} I_j \cdot A^* \circ e_j = \sum_{j=1}^{4} I_j \cdot X = (\Sigma(I)) \cdot X$,

2. As $\Sigma(X) = 1$, we have $\Sigma(A^* \circ I) = \Sigma(\Sigma(I) \cdot X) = \Sigma(I) \cdot \Sigma(X) = \Sigma(X)$,

3. X is an eigenvector of A with eigenvalue 1, and therefore also the vector $(\Sigma(I)) \cdot X$,

4. $A \circ A^* = (A \circ X^t, A \circ X^t, A \circ X^t, A \circ X^t) = (X^t, X^t, X^t, X^t) = A^*$. Let $A^* \circ A = (b_{ij})$, i = row index, j = column index. Then, because $\Sigma(A(*,j)) = 1$, we have

$$b_{ij} = (X_i, \cdots, X_i) \circ A(*,j) = X_i \cdot \Sigma(A(*,j) = X_i \quad ;$$

therefore $A \cdot A^* = A^*$,

5. Let $\tilde{I}$ be an eigenvector of A with eigenvalue 1. It follows from (3) that $\bar{I} = A^* \circ \tilde{I}$ is also an eigenvector of A with eigenvalue 1. The space of eigenvectors of A has the dimension 1; therefore we have $\bar{I} = c \cdot \tilde{I}$ with $c \neq 0$. Let $c \neq 1$; then $\Sigma(\bar{I}) = c \cdot \Sigma(\tilde{I})$ would be contradiction to the material conservation property of A^*. ■

The theorem implies that, if the system is in a stable state $\tilde{I}$ and no changes are imposed from outside, it remains in the stable state also in the asymptotic state. If a change is imposed (e.g., by external inputs, or by changes of the elements k^{xy} of matrix A), then the system will return asymptotically to the stable state. The speed of convergence to the stable state can be checked only numerically.

2.2 Unstable States

So far we have considered only material conserving states. In the following we consider situations where in the time interval (t_i, t_{i+1}) the amount $n_{i+1}^{(k)}$ of material is put into the k-th box; i.e., we consider the input vector

$$N_{i+1} = (n_{i+1}^{(1)}, n_{i+1}^{(2)}, n_{i+1}^{(3)}, n_{i+1}^{(4)}) \quad .$$

Let us assume that at time t_o the state of the system is given by $I_o \in \mathbb{R}^4$. Then the development of the system under the influence of inputs from outside is described by the following theorem.

Theorem 6. Let I_o be a stable state of the system at time t_o, and let $(N_{i+1})_{i \in \mathbb{N}}$ be a sequence of input vectors. Then state I_ℓ of the system at time ℓ is given by the following relation:

$$I_\ell = I_o + \sum_{i=1}^{\ell} A^{\ell-i} \circ N_{i+1} \quad . \tag{10}$$

Proof (by complete induction). ∎

We conjecture that if $(N_{i+1})_{i \in \mathbb{N}}$ is a sequence of input vectors with

$$\sum_{i=0}^{\infty} \Sigma(N_{i+1}) = N < \infty$$

the final state I^∞ of the system is also an eigenvector of A with

$$I^\infty = x \cdot X, \; x = N + \Sigma(I^0) \quad .$$

2.3 Uncertainty Considerations

We have so far assumed that all quantities under consideration are known precisely. In this section we will analyze several problems in connection with uncertainties about the system. Here we assume that the transition coefficients k^{xy} are known exactly (at least relative to the inventories of the several boxes).

First, we draw some general conclusions from the results of the foregoing sections. If we know that the system is in the equilibrium state, according to Theorem 5 we know the relative inventories precisely. In other words, if we have certain knowledge about the relative inventories in the equilibrium state, then our knowledge does not deteriorate. The same holds if the equilibrium state is disturbed by an input from outside of known total size; again we know, some time after the end of the disturbance, the relative inventories (which in fact are the same as before the disturbance occurred).

As a second source of uncertainties we will discuss the fact that the model is in reality better described by a system of linear differential equations. We used an approximation that is the better the smaller the time steps are. In our case, the transition coefficients k^{xy} measured in reciprocal years are very small, so the chosen time steps of one year seem to be justified. Nevertheless it might be preferable to solve the differential equation that corresponds to (3) as follows:

$$\frac{d}{dt} I(t) = A' \circ I(t) + N(t) \quad ; \quad A' = A-E \quad . \qquad (11)$$

The solution of the homogeneous system (11), i.e. for $N(t) = 0$, is given ([11], p. 163 ff.) by:

$$I(t) = EXP(A't) \circ I(t_0) \qquad (12)$$

where EXP is defined by

$$EXP(A't) := \sum_{n=0}^{\infty} \frac{(A't)^n}{n!} \quad . \qquad (13)$$

If we compare (12) with (3), we see that for $t = 1$ a development of (12) up to the first order gives exactly the expression (3) for $N(t) = 0$. Moreover, it can easily be seen that the equilibrium states of (3) are equilibrium states of (12), and that (13) has the material conservation property. There is still the question of the asymptotic state of (12) for any initial state I_0, given by

$$I(\infty) := \lim_{t \to \infty} \mathrm{EXP}(A't) \circ I_0 \quad . \qquad (14)$$

A solution of this problem for discrete points of time is given by the following theorem.

Theorem 7. Let A, A^* and A' be as defined by (4), (9a) and (11) respectively. Let $A_e := \mathrm{EXP}(A')$. Then A_e has the material conservation property and

$$\lim_{n \to \infty} A_e^n = A^* \quad .$$

Proof. The proof follows immediately from the fact that any positive material conserving matrix has a positive eigenvector with the eigenvalue 1 and converges uniformly to a matrix with equal columns, every column being the positive eigenvector normalized to 1. ∎

We have shown that both the discrete and the differential equation system converge to the same equilibrium state. There remains the question how far the solutions of the two systems deviate. A simple numerical model (Table 1) indicates that the deviations become smaller the smaller the time intervals are.

Table 1. Comparison of exact and approximate solutions in a simple case.

Differential equation:

$$\frac{d}{dt} I(t) = - \lambda \cdot I(t) \qquad I(t) = I_o \cdot \exp(-\lambda t)$$

Difference equation:

$$\frac{I(n\Delta t) - I((n-1)\Delta t)}{\Delta t} = -\lambda \cdot I(t) \qquad I(n\Delta t) = I_o \cdot (t-\lambda\Delta t)^n$$

Example: $\lambda = 2,\ I_o = 1$

t	0	1	3	5	10	15	20	30	40
$\exp(-\lambda t)$	1	.819	.549	.368	.135	.050	.018	.002	.000
$(I-\lambda\Delta t)^n$ for $\Delta t=0.5$	1	.810	.531	.349	.122	.042	.015	.002	.000
$(I-\Delta t)^m$ for $\Delta t=1$	1	.800	.512	.328	.107	.035	.012	.001	.000

The solution of the inhomogeneous system of differential equations (11) is given by

$$I(t) = EXP(A' \cdot (t-t_o)) \cdot I_o + \int_{t_o}^{t_1} EXP(A'(t-s)) \circ N(s)\, ds \quad . \qquad (15)$$

This solution, being too awkward for computation, can be replaced by solving the system of simultaneous ordinary differential equations (11) with the Runge Kutta method. In Figure 2 we have plotted the solution, which is closer to the measured curve than the solution obtained with the help of equation (10).

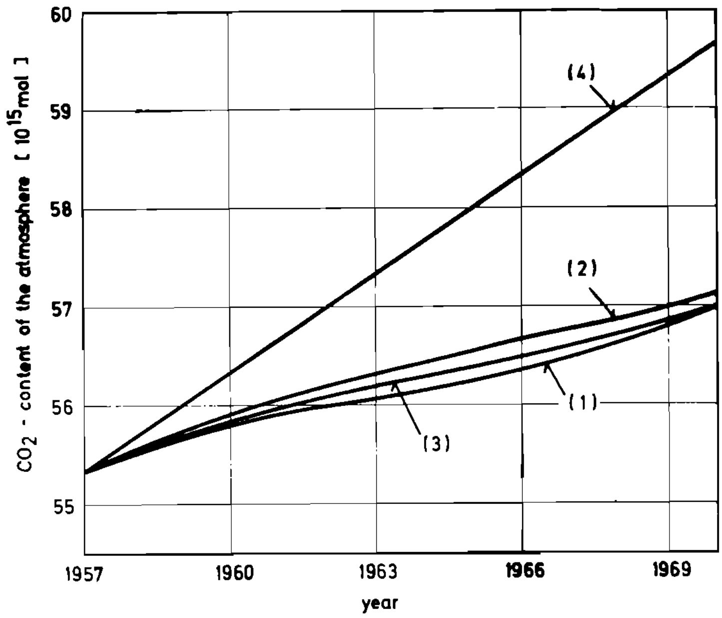

Figure 2. Comparison of measured and theoretical data for the CO_2 content of the atmosphere.

(1) Experimental data after Keeling [12]

(2) Theoretical data using formula (10)

(3) Theoretical data using the Runge-Kutta procedure for solving the system of differential equations (11)

(4) CO_2 from burnt fossil fuels kept in the atmosphere.

To summarize the discussion on the validity of the time discrete model we may state the following. The equilibrium state of the undisturbed system is the same for the discrete and for the continuous model. The speed with which the systems approach the equilibrium state, as well as the deviations, in general depend on the length of the time intervals relative to the magnitude of the exchange coefficients; in our special case, one-year time intervals seem to be justified. Our analysis was based primarily on the discrete model, because the analysis of this model appeared more illuminating, and because the problem of measurement errors could be handled much more easily in the discrete than in the continuous case.

3. NUMERICAL CALCULATIONS AND APPLICATIONS

As a first application of the theoretical results obtained, we will consider some carbon dioxide data reported in the literature and check their consistency in the sense that they fulfill condition (6).

Table 2a lists the transition coefficients and the inventories as given by Sawyer [5]. As can easily be seen these data are only partly consistent in the sense of formula (6); therefore, a consistent set of transition coefficients is also given in Table 2a. However, it should be noted that this set cannot be determined uniquely. We have changed the coefficients such that as few data as possible had to be changed, and that the inventory of the deep sea, the value of which is consistent with data reported in [4] and [6], remained unchanged.

Figure 3 shows the seven-box model developed by Machta [6] which takes into account the following reservoirs:

Stratosphere (1)	Mixed Layer Oceans (5)
Troposphere (2)	Marine (6)
Long Term Biosphere (3)	Deep Layer Oceans (7)
Short Term Biosphere (4)	

Without writing down the system of equations that corresponds to system (1) and that can be derived immediately from Figure 3, we give here only the equivalent to formula (6), i.e., the relative sizes of the inventories in the equilibrium state:

$$I = (1,\ k^{12}/k^{21},\ k^{12}/k^{21}\cdot k^{23}/k^{32},\ k^{12}/k^{21}\cdot k^{24}/k^{42},\ k^{12}/k^{21}\cdot k^{25}/k^{52},$$

$$k^{12}/k^{21}\cdot k^{25}/k^{52}\cdot k^{56}/k^{65},\ k^{12}/k^{21}\cdot k^{25}/k^{52}\cdot k^{57}/k^{75}) \quad . \qquad (6')$$

In Table 2b, the data of [6] are represented together with those data which would be consistent with formula (6'). One realizes the large differences with respect to the transition coefficient k^{dm}, or k^{75}, between Sawyer's and Machta's data respectively, which will be important for later considerations.

Table 2a. Relative inventories and transition coefficients for the four-box model according to Sawyer [5], and consistent with eq. (6).

	I^a	I^b	I^m	I^d	k^{ab}	k^{ba}	k^{am}	k^{ma}	k^{md}	k^{dm}
Sawyer [5]	1	1.2	1.2	58	$\frac{1}{33}$	$\frac{1}{40}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{300}$
Consistent with eq. (6)	1	1.21	1.2	58.06	$\frac{1}{33}$	$\frac{1}{40}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6.2}$	$\frac{1}{300}$

Table 2b. Relative inventories and transition coefficients for the seven-box model according to Machta [6], and consistent with eq. (6').

	I^1	I^2	I^3	I^4	I^5	I^6	I^7	k^{12}	k^{21}	k^{23}	k^{32}	k^{24}	k^{42}	k^{25}	k^{52}	k^{56}	k^{65}	k^{57}	k^{75}
Machta [6]	1	5.7	11.1	.7	30	.2	366.7	$\frac{1}{2}$	$\frac{1}{11.5}$	$\frac{1}{24.4}$	$\frac{1}{40}$	$\frac{1}{19}$	$\frac{1}{2}$	$\frac{1}{1.1}$	$\frac{1}{5.9}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{50}$	$\frac{1}{1600}$
Consistent with eq. (6')	1	5.75	11.2	.6	30.4	.22	366.7	$\frac{1}{2}$	$\frac{1}{11.5}$	$\frac{1}{24}$	$\frac{1}{47}$	$\frac{1}{19}$	$\frac{1}{2}$	$\frac{1}{1.25}$	$\frac{1}{6.6}$	$\frac{1}{273.8}$	$\frac{1}{2}$	$\frac{1}{132.8}$	$\frac{1}{1600}$

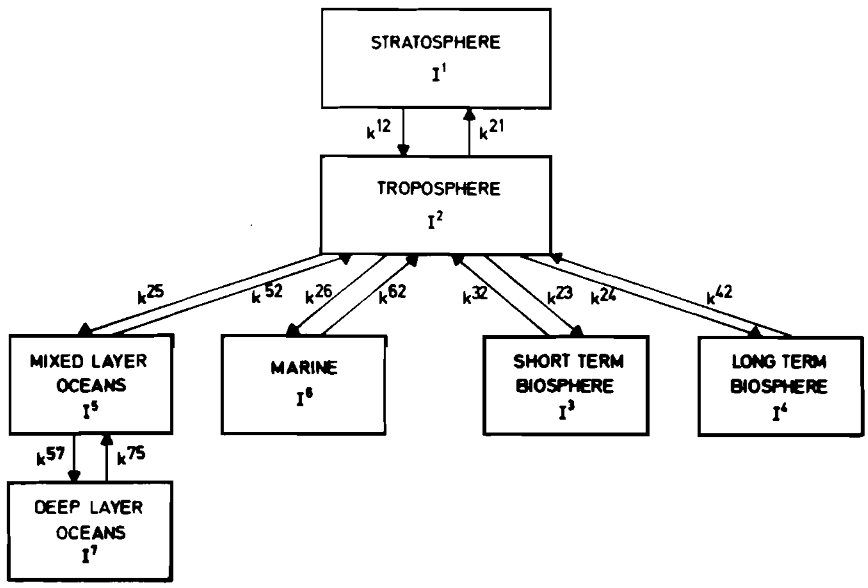

Figure 3. Natural reservoirs of carbon dioxide (based on [6]). I^x is the content of reservoir x, k^{xy} describes the transition from reservoir x to reservoir y.

As a second application, we analyze how fast the system will return to the equilibrium state after a disturbance in the sense of an input from outside has occurred. According to Theorem 5 the application of A^* to any state I gives an eigenstate, i.e., an equilibrium state. As in addition, according to Theorem 4, matrix A^* is the limiting matrix of A^n, we have to analyze how fast matrix A^n approaches matrix A^*.

In Table 3a, the consistent data for the four-box model after Sawyer, as represented in Table 2a, are used in order to tabulate the matrix A^n for increasing values of n. In Table 3b, the consistent data for the four-box model after Machta are used--the main difference compared to Sawyer's data being the value of k^{dm}. One concludes that A^n approaches A^* roughly after $1/k^{xy}$ steps (in our model, years), where k^{xy} is the smallest transition coefficient of the system. Generally in our model the speed of

Table 3a. Convergence of the matrix A^n towards A^* (defined by eq. (9a)) for the four-box model, based on Sawyer's data [5].

A				A^{100}			
.7697	.0250	.1667	.0000	.0241	.0466	.0200	.0153
.0303	.0975	.0000	.0000	.0564	.1598	.0374	.0156
.2000	.0000	.6720	.0033	.0240	.0370	.0216	.0188
.0000	.0000	.1613	.9967	.8954	.7567	.9210	.9502
A^{200}				A^*			
.0174	.0209	.0167	.0160	.0163	.0163	.0163	.0163
.0253	.0414	.0223	.0189	.0197	.0197	.0197	.0197
.0201	.0221	.0197	.0193	.0196	.0196	.0196	.0196
.9373	.9157	.9413	.9458	.9444	.9444	.9444	.9444

Table 3b. Convergence of the matrix A^n towards A^* for the four-box model, based on Machta's data [6].

A				A^{100}			
.7697	.0169	.0455	.0000	.0881	.0991	.0844	.0065
.0303	.9831	.0000	.0000	.1777	.3226	.1540	.0065
.2000	.0000	.9469	.0006	.3709	.3776	.3606	.0317
.0000	.0000	.0076	.9994	.3634	.2007	.4010	.9554
A^{1000}				A^*			
.0169	.0173	.0168	.0158	.0159	.0159	.0159	.0159
.0310	.0320	.0308	.0282	.0285	.0285	.0285	.0285
.0740	.0755	.0736	.0694	.0699	.0699	.0699	.0699
.8781	.8752	.8787	.8867	.8857	.8857	.8857	.8857

convergence to the equilibrium state is determined by the transition from the deep sea to the mixed layer of the sea. For Sawyer's data this means less than 300 years, for Machta's data less than 1600 years. As the consistency relations (6) or (6') are not sufficient to determine k^{dm} uniquely unless all other inventories and transition coefficients are known precisely, it would be extremely interesting, in view of the problems mentioned in the introduction, to have more and better data.

As a third application, we compare the data of the CO_2 content of the atmosphere in the years 1958 to 1970, measured by Keeling at Mauna Loa [12], with the theoretical values obtained from formula (10) on the basis of Sawyer's and Machta's data. To be able to do this we take for the CO_2 content of the atmosphere after Fairhall [13]

$$I^a_{1958} = 312 \text{ ppm (vol)} \quad , \; ^\dagger$$

and for the annual input n^a of CO_2 into the atmosphere as a result of the burning of fossil fuels (after Baxter [14])

$$n^a_{1958} = 0.248 * 10^{15} [\text{mol/a}]$$

$$n^a_{1970} = 0.425 * 10^{15} [\text{mol/a}] \quad .$$

The results of these calculations are represented in Fig. 2. This figure also represents the CO_2 content of the atmosphere that would result if all CO_2 from the burnt fossil fuels remained in the atmosphere (curve (3)). One sees that the material balance model (curve (2)) describes the measured data (curve (1)) much more accurately. It should be noted that curve (2) is obtained (within drawing accuracy) for both Sawyer's and Machta's data.

$^\dagger n^a$ [ppm (vol)] $= 5.64 * 10^{-15} * n^a$[mol].

As a last application, we ask what--according to our model --the asymptotic value of the carbon dioxide content of the atmosphere would be if all known fossil fuels were burnt. According to Zimen [15] this would correspond to a final cumulated input of $N = 600 * 10^{15}$ mol. If we start with $I_o^a = 51.4 * 10^{15}$ [mol] at pre-industrial time, i.e., before 1860 (see, e.g., [12], then we obtain with Sawyer's data

$$I_o^b = 62.2 * 10^{15},\ I_o^m = 61.7 * 10^{15},\ I_o^d = 2985.4 * 10^{15} [\mathrm{mol}] \quad .$$

This gives a total inventory I_o of

$$I_o = 3160 * 10^{15} \ [\mathrm{mol}] \quad .$$

To answer our question, we have to add to this inventory the CO_2 from the burnt fossil fuels and to distribute the total inventory according to eq. (6). The result is

$$I_\infty^a = 61.2 * 10^{15},\ I_\infty^b = 74 * 10^{15},\ I_\infty^m = 73.4 * 10^{15},$$

$$I_\infty^d = 3552 * 10^{15} \ [\mathrm{mol}] \quad .$$

This means that in the asymptotic state, $567 * 10^{15}$ [mol] of the $600 * 10^{15}$ [mol] go into the deep sea, and furthermore that the atmospheric content rises from 312 ppm (vol) as given today to 345 ppm (vol) in the asymptotic state.

These results, together with those for the speed of convergence, are especially interesting in view of recent proposals, namely the direct burial of the CO_2 from burnt fossil fuels in the deep sea (see, e.g., [16]). If such a scheme were feasible, then the figures given above indicate what fraction of the buried CO_2 will again go into the atmosphere and at what speed. Inversely, if all the CO_2 from burnt fossil fuels is released to the atmosphere, one gets an idea how long the CO_2 will stay in the atmosphere before it goes into the deep sea.

The latter argument indicates how important it is to have a precise knowledge about the exchange coefficients k^{xy}, especially about the coefficient k^{dm}. According to Sawyer we must live for about 200 years with an atmospheric CO_2 content that is higher than under equilibrium conditions, whereas Machta puts it at 1500 years.

The question whether or not a final equilibrium concentration of 345 ppm (vol) of the atmosphere would be tolerable goes beyond the scope of this paper.

4. CONCLUDING REMARKS

We have presented a mathematical analysis of an idealized model of the carbon cycle of the earth. The analysis suggests certain conclusions about the carbon cycle data in the literature and permits certain statements about the way in which disturbances may influence this cycle.

Approximations and uncertainties have also been considered; however, the area of handling the errors involved in the measurements and estimates of transition coefficients and inventories has been neglected. In fact, the authors are developing a statistical treatment of these problems along the lines of [7]; the main reason for not including it here is the lack of realistic information about measurement and estimation uncertainties. Any relevant information would be highly welcomed by the authors of this report.

It is clear that an analysis of more subtle effects, e.g., the seasonal variations of the carbon dioxide content of the atmosphere, as well as the difference between the northern and the southern hemispheres, calls for more detailed models (with respect to both the number of boxes and the time steps). This refinement is mandatory if on the basis of such models one tries to analyze questions such as that of imposing limits for carbon dioxide releases into the atmosphere, where geographical and seasonal differences clearly must be taken into account.

Work in this direction has been started, and it is the purpose of this report to give an idea of the power and flexibility of the material balance approach.

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