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Optimal risk reduction and insurance for government infrastructure protection

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ABSTRACT

This study applies the ruin theory to the government's fiscal problem of infrastructure protection against a risk of destruction associated with reconstruction expenditure. In preparation for the possibility of large-scale expenditure, the government will take out an excess-of-loss (XL) insurance policy as an insured party and, at the same time, invest in physical disaster damage reduction facilities (i.e., "risk reduction") in advance. The government decides on the layers of risk to be covered by risk reduction, risk retention, and insurance, so as to maximize the expected value of fiscal resources, subject to the constraint that the probability of financial collapse (i.e., "ruin") is kept below a certain level. Furthermore, the study clarifies the composition of the impacts of risk reduction and insurance on ruin probability and numerically analyzes the optimization problem based on a hypothetical setting constructed using data from Vanuatu. The numerical example demonstrates a case in which the ruin probability rises sharply within specific short intervals of risk reduction and insurance attachment points, surpassing the constraint threshold, suggesting the need for careful selection of countermeasure combinations. It further clarifies that the lower the government's initial financial reserves, the more extensive the insurance coverage should be.

1. Introduction

This study applies the ruin theory, known as the Cramér-Lundberg model, to analyze a problem of infrastructure management under risks of destruction caused by catastrophic events such as disasters. The ruin theory has targeted the optimal design of the excess-of-loss (XL) insurance contract, which has been applied in the private sector to safeguard businesses against catastrophic or extraordinary financial losses arising from customer insolvency or default (e.g., Asmussen et al., 2000; Bai et al., 2013; Li and Young, 2019; Liang et al., 2020). For example, direct-writing insurance companies have made the XL contract as the insured with reinsurance companies.

This research is unique in two respects in the research lineage of the ruin theory. Firstly, in this model, the insured party in the XL insurance contract is the government. Traditionally, the government has played the role of insurer, providing a final safety net for the private sector. However, in this study, we target the risk of government financial collapse and formulate an optimization problem for the government. In policy discussion, a government's need for insurance has been pointed out (e.g., Cebotari and Youssef, 2020; Ghesquiere, 2007; Mechler et al., 2006), although there has not been an optimization model for designing layers that include the XL insurance contract for a government. Secondly, we incorporate into the model what we call "risk reduction", which is a physical damage reduction measure. Such risk reduction measures include embankments, dams, and drainage systems. These physical facilities are fundamentally different from financial vehicles in that their performance levels are identified by the upper limit of external forces from which they can prevent damage, and in that damages from smaller external forces are accordingly prevented. For example, suppose that the banks of a river are designed to prevent flooding in the event of a 50-year rainfall. In that case, it automatically follows that they also prevent flooding in the event of a 20-year rainfall. In contrast, financial contracts can be designed to be more flexible with respect to any range or interval of disaster magnitude, provided that the magnitude of the disaster is measurable and the contract is enforceable (e.g., Asimit et al., 2017). We take this asymmetry into account and analyse the optimal combination of risk reduction and the XL insurance.

The rest of the paper is organized as follows: Section 2 formulates the model; Section 3 derives several properties of ruin probability and

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sets up the decision-making problem; Section 4 illustrates the numerical example; and Section 5 concludes the study.

2. Model

2.1. Catastrophic event

We assume that a country is subject to catastrophic events, which may destroy part of infrastructure managed by the government. The government has limited funds and seeks to determine the most effective strategy for financing infrastructure protection and recovery.

The times of the catastrophic events are assumed to follow a Poisson process with intensity λ_0. The time unit is defined by a year. If an event happens, it causes damage of X, where X is a random variable. We assume that X follows an exponential distribution with expectation μ, i.e., the cumulative distribution function (CDF) is given by $F(x) = 1 - \exp(-x/\mu)$, where x represents a realized value of X, and we call this "X is exponentially (μ) distributed." Note that the lowercase letter x is used hereafter to denote the realized value of a random variable in several different CDFs or density functions.

2.2. Risk reduction

A risk reduction measure absorbs possible damages up to a size of s. s is a policy variable decided by the government, and is measured by the unit of damage. It comes at annual costs of $K(s)$ and has the effect that the damage caused by an event changes from X to $\bar{X}_s = \max(X - s, 0)$. We specify the cost function $K(s)$ by

$$K(s) = \alpha s^2. \tag{1}$$

The form of the cost function $K(s)$ is specified to ensure that the marginal cost of preventing damage at each disaster increases. The quadratic form was chosen because it is one of the simplest convex functions. The risk reduction policies include the construction of durable disaster-prevention facilities such as levees. In such cases, $K(s)$ represents the average cost per period, including maintenance expenses. The random variable $\bar{X}_s$ has a CDF as follows:

$$F_s(x) = \mathrm{Prob}\{\bar{X}_s \le x\} = \begin{cases} 0 & \text{for } x < 0, \\ F(x+s) & \text{for } x \ge 0. \end{cases} \tag{2}$$

Notice that the distribution of $\bar{X}_s$ has a probability mass of $F(s)$ at zero. Since events with a damage of zero are not counted, the intensity of the counting Poisson process changes from λ_0 to

$$\lambda(s) = \lambda_0 \cdot \mathrm{Prob}\{\bar{X}_s > 0\} = \lambda_0 \cdot \mathrm{Prob}\{X > s\} = \lambda_0(1 - F(s)). \tag{3}$$

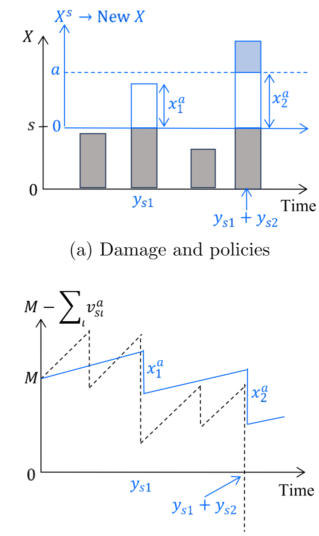
In the special case that X follows the exponential (μ) distribution, we have

$$\lambda(s) = \lambda_0 \exp(-s/\mu). \tag{4}$$

Let X_s be the damage size given that X exceeds s, X_s has a CDF as follows:

$$\begin{aligned} &\mathrm{Prob}\{X_s \le x\} = \mathrm{Prob}\{\bar{X}_s \le x \mid \bar{X}_s > 0\} = \mathrm{Prob}\{X \le x + s \mid X > s\} \\ &= \frac{F(x+s) - F(s)}{1 - F(s)} = 1 - \exp(-x/\mu) = F(x) \quad (\text{for } x \ge 0), \end{aligned} \tag{5}$$

which is the same as the distribution of the original X. This is due to the memoryless property of the exponential distribution. Hence, $\mathbb{E}[X_s] = \mu$. By a slight abuse of notation, we will therefore drop the index s and denote the damage distribution after the risk reduction measures were put in place also by X, bearing in mind that the introduction of these interventions has changed the intensity of the event process. Fig. 1a illustrates the relationship between the original damage X and the reduced damage X_s caused by policy s, referred to as the "New X". In the figure, the origin of the vertical axis shifts upward for the "New X", and the damage frequency decreases accordingly due to the lower arrival rate $\lambda(s)$.



(a) Damage and policies

(b) Process of capital level change

- In Fig. 1a, the black coordinate axes indicate times and damage magnitudes without a policy. The blue coordinate axes show times and damage magnitudes when the policy (s, a) is implemented. The origin of the axes shifts upward by s due to the risk reduction policy s.
- In Fig. 1a, the black-shaded portion of the damage represents the amount prevented by the risk reduction policy s. The first and third disasters are completely prevented and thus are not counted as disasters after the policy is introduced.
- In Fig. 1a, the blue shaded portion of the damage represents the amount covered by insurance. Therefore, it is removed from the damage the government incurs at the disaster time. Only the remaining white portion constitutes the government's loss at the disaster times.
- In Fig. 1b, the black dashed line represents the change in the capital level without a policy. The fourth disaster causes the capital level to turn negative, leading to ruin. The blue solid line represents the change in the capital level after the introduction of the policy (s, a).
- x_ι^a and $y_{s\iota}$ ($\iota = 1, 2$) represent the realized values of X^a and Y_s, respectively, at the time of the ι-th disaster, in the case with the policy (s, a).

Fig. 1. Damage, policies, and capital level change.

2.3. XL insurance contract

Let a be the attachment point of the insurance contract. If an event causes damage of X (defined by X_s), the insurer pays $X^{Ia} = \max(X - a, 0)$ to the policyholder, who is then left with the uninsured damage of $X^a = \min(X, a)$.

The annual premium $\pi(s, a)$ is calculated according to some premium principle. For simplicity, we take

$$\pi(s, a) = (1+\theta)\lambda(s) \cdot \mathbb{E}[X^{Ia}] = (1+\theta)\lambda(s) \cdot \mathbb{E}[X - X^a], \tag{6}$$

where θ (≥ 0) is a markup factor.

Having $F(x)$ as the CDF of X, the CDF of X^a is identified as

$$F^a(x) = \mathrm{Prob}\{X^a \le x\} = \begin{cases} F(x) & \text{for } x < a, \\ 1 & \text{for } x \ge a. \end{cases} \tag{7}$$

On the other hand, the insurer has to take the losses exceeding a, i.e., X^{Ia}, whose CDF is given by

$$F^{Ia}(x) = \mathrm{Prob}\{X^{Ia} \le x\} = \begin{cases} 0 & \text{for } x < 0, \\ F(a+x) & \text{for } x \ge 0. \end{cases} \tag{8}$$

For the case that X is exponentially (μ) distributed, we have for Eq. (7)

$$F^a(x) = \begin{cases} 1 - \exp(-x/\mu) & \text{for } x < a, \\ 1 & \text{for } x \ge a. \end{cases} \tag{9}$$

Considering that $\mathrm{Prob}\{X^a = a\} = 1 - F(a) = \exp(-a/\mu)$, the expectation of X^a is

$$\mathbb{E}[X^a] = \int_0^a \frac{1}{\mu} x \exp(-x/\mu)\, dx + a \cdot \exp(-a/\mu) = \mu\{1 - \exp(-a/\mu)\}, \quad (10)$$

and therefore, $\mathbb{E}[X^{Ia}] = \mu \exp(-a/\mu)$ is the expected claim of insurance that the insurer will pay for a single event. Note further that $\mathrm{Prob}\{X^{Ia} = 0\} = F(a) = 1 - \exp(-a/\mu)$ is the probability of an event occurring that does not result in a claim against the insurer. Therefore, the simple proportional annual premium is derived as

$$\pi(s,a) = (1+\theta)\lambda(s)\mu \exp(-a/\mu) = (1+\theta)\lambda_0\mu \exp\{-(s+a)/\mu\}. \quad (11)$$

It can be seen that s and a symmetrically affect $\pi(s,a)$.

2.4. Government financial process

Government financial assets are formed by an initial asset stock and subsequent cash flows. A positive sign in the financial asset position represents savings. We denote the assets at the time the government problem is set, i.e., the initial assets, by Z_0. When $Z_0 < 0$, the government has liabilities of $(-Z_0)$. In this model, we focus on the dual structure of sovereign debt observed particularly in developing countries. We assume that the liability $(-Z_0)$ consists of interest-bearing debt $(-z_0)$ and concessional debt $(-(Z_0 - z_0))$. Additionally, we assume that the government will continue to make interest payments only on the initial interest-bearing debt $(-z_0)$. In other words, all debt that changes after the initial point in time is concessional debt and does not involve interest payments. This assumption simplifies the model while also reflecting empirical facts: For many governments in developing countries today, concessional debt has become dominant in new borrowing (e.g., Diwan et al., 2023; World Bank, 2024), and market-rate debt is sticky (e.g., Cordella and Powell, 2021; Amegashie, 2023).

We assume "ruin" occurs when the government's financial asset stock falls below a threshold Z_R (< 0) (i.e., when the government's debt stock exceeds $(-Z_R)$). Therefore, there is a distance of $Z - Z_R$ between any asset position Z (≥ 0) and the ruin threshold. Thus, we shift the axis measuring the asset stock to the left, setting Z_R as the new origin. Let M denote the level of the asset stock measured on the new axis. The original axis's origin transforms to $M = -Z_R$ (> 0), and the initial asset position transforms to $M = Z_0 - Z_R$ (> 0). The fiscal policy continues when $M \geq 0$, and the ruin occurs when $M < 0$. This axis shift does not compromise the generality of the discussion.

The net expenditure flow from the initial time to the time of the first event is represented by V_s^a, which is given as

$$V_s^a = -\{w + rz_0 - K(s) - \pi(s,a)\}Y_s + X^a = -c(s,a)Y_s + X^a, \quad (12a)$$

$$\text{where} \quad c(s,a) = w + rz_0 - K(s) - \pi(s,a). \quad (12b)$$

w is a constant income per unit of time. r is the interest rate. We assume that r is constant and that interest is simple interest. If $z_0 > 0$, the government earns interest income. $K(s)$ and $\pi(s,a)$ are the costs for risk reduction and insurance premiums, respectively. $c(s,a)$ is the net revenue per unit of time. Y_s is the inter-event time of a Poisson process with intensity $\lambda(s)$, which has an exponential distribution with expectation $1/\lambda(s)$ and which is independent of X^a. In the following, we will denote $c(s,a)$ simply as c, bearing in mind that c is dependent on s and a. Similarly, we denote $\lambda(s)$ as λ. We will represent the CDF of V_s^a by $G_s^a(\cdot)$.

2.5. Recursive equation

The government has the assets M (≥ 0) at the initial point in time. As mentioned above, we define ruin as a financial state in which the sum of the initial assets and the flow of net income (i.e., minus net expenditure) at subsequent points in time falls into negative territory. The financial position (hereinafter often referred to as "capital" for simplicity, unless confusion arises) at the time of the n-th disaster is represented by $M - \sum_{\iota=1}^{n} v_{s\iota}^a$, where $v_{s\iota}^a$ is the net expenditure flow of the ι-th interval, namely, the realized value of V_s^a for the ι-th interval, as shown in Fig. 1b. If $M - \sum_{\iota=1}^{n'} v_{s\iota}^a \geq 0$ for $n' = 1, \cdots, n-1$ and $M - \sum_{\iota=1}^{n} v_{s\iota}^a < 0$, the government ruins at the n-th disaster. This case involves the fact that ruin does not occur at a non-disaster time, which is supported by the government's rational policy, where the combination of s and a in the range where $c(s,a)$ is positive. Accordingly, capital increases during non-disaster times and then jumps downward by X^a at the time of disaster. If, instead, c were non-positive, capital would monotonically decrease and eventually reach zero, namely, the ruin threshold, with probability 1. Therefore, as a necessary condition for obtaining a ruin probability less than 1, $c(s,a)$ needs to be positive. Namely,

$$(s^*, a^*) \in \Omega = \{(s,a) \mid K(s) + \pi(s,a) < w + rz_0\}, \quad (13)$$

where s^* and a^* refer to the optimal combination of the policy.

We are interested in the ruin probability. Under Eq. (13), ruin could happen only at disaster times. The ruin probability that is evaluated at the time that the level of capital is M is defined as

$$\Psi_s^a(M) = \mathrm{Prob}\left\{ \inf_{\iota} \left\{ M - \sum_{\iota=1}^{\infty} V_{s\iota}^a \right\} < 0 \right\}, \quad (14)$$

where $V_{s\iota}^a$ is the random variable of the net expenditure flow of the ι-th interval. $V_{s\iota}^a$ follows the same CDF as V_s^a, namely $G_s^a(\cdot)$, for any ι. With the probability $\Psi_s^a(M)$ defined on the infinite time horizon (e.g., Embrechts et al., 1999; Bazyari, 2023), namely, the probability of the ultimate ruin, we analyze the fiscal sustainability of the government.

The probability of ruin, Ψ_s^a, satisfies the following recursive equation:

$$\Psi_s^a(M) = 1 - G_s^a(M) + \int_{-\infty}^{\min(M,a)} \Psi_s^a(M - v)\, dG_s^a(v). \quad (15)$$

This equation corresponds to the equation shown in Corollary 3.3 of Asmussen and Albrecher (2010), but the form is slightly different. The recursive Eq. (15) focuses on the period from the present moment until the first disaster occurs. The term $\{1 - G_s^a(M)\}$ represents the probability of immediate ruin, which is caused by $V_s^a > M$ at the time of the first disaster. The integral term represents the probability of ruin at a time of a subsequent disaster, given no immediate ruin. The upper limit of the integral above reflects the limit $V_s^a \to X^a = \min(M,a)$ as $Y_s \to 0$, while the lower limit reflects that V_s^a can take arbitrary negative values, considering $c(s,a) > 0$ and that Y_s is unbounded.

2.6. Distribution of the net expenditure flow for one interval

Before analyzing $\Psi_s^a(M)$ numerically on Eq. (15), we explore the distribution of the net expenditure flow for one interval, V_s^a. Recall Eq. (12a), where Y_s and X^a are independent, the density of $G_s^a(v)$ is derived by convolution:

$$g_s^a(v) = (f_y * f^a)(v) = \int \left|\frac{1}{-c}\right| f_y\left(\frac{x-v}{c}\right) dF^a(x), \quad (16)$$

where $f_y(y)$ and $f^a(x)$ represent the probability density functions of Y_s and X^a, respectively:

$$f_y(y) = \lambda \exp(\lambda y), \quad (17a)$$

$$f^a(x) = \frac{dF^a(x)}{dx} = \frac{1}{\mu}\exp(-x/\mu). \quad (17b)$$

$Y_s = (X^a - V_s^a)/c$ and $Y_s \geq 0$ imply $X^a \geq V_s^a$. Hence, in the case where $-\infty < V_s^a < 0$, all values of X^a in its domain $[0,a]$ contribute to the integral in Eq. (16). However, in the case where $0 \leq V_s^a \leq a$, only values of X^a greater than V_s^a can contribute, restricting the lower integration limit. We have the case split as follows:

Case $v < 0$

$$g_s^a(v) = \int_0^a \left|\frac{1}{-c}\right| f_y\left(\frac{x-v}{c}\right) dF^a(x) + \left|\frac{1}{-c}\right| f_y\left(\frac{a-v}{c}\right) \cdot \mathrm{Prob}\{X^a = a\}$$

$$= \frac{\lambda}{\lambda\mu + c} \exp(\lambda v/c) \left\{ \frac{\lambda\mu}{c} \exp(-Ha) + 1 \right\}, \tag{18a}$$

Case $v \in [0, a]$

$$g_s^a(v) = \int_v^a \left| \frac{1}{-c} \right| f_y\left(\frac{x-v}{c}\right) dF^a(x) + \left| \frac{1}{-c} \right| f_y\left(\frac{a-v}{c}\right) \cdot \text{Prob}\{X^a = a\}$$

$$= \frac{\lambda}{\lambda\mu + c} \left\{ \frac{\lambda\mu}{c} \exp\left(\frac{\lambda v}{c} - Ha\right) + \exp(-v/\mu) \right\}, \tag{18b}$$

$$\text{where} \quad H = \frac{\lambda}{c} + \frac{1}{\mu}. \tag{18c}$$

Calculating the CDF, $G_s^a(v)$, by integration, we obtain

$$G_s^a(v) = \begin{cases} \left\{ \dfrac{c}{\lambda\mu + c} + \dfrac{\lambda\mu}{\lambda\mu + c} \exp(-Ha) \right\} \exp(\lambda v/c) & \text{for } v < 0, \\ 1 - \dfrac{\lambda\mu}{\lambda\mu + c} \left\{ \exp(-v/\mu) - \exp\left(\dfrac{\lambda v}{c} - Ha\right) \right\} & \text{for } v \in [0, a], \\ 1 & \text{for } v > a, \end{cases} \tag{19}$$

where $c = c(s, a)$ and $\lambda = \lambda(s)$ as defined by Eqs. (12b) and (4), respectively. We can confirm that $G_s^a(v)$ is continuous at $v = 0$ and $v = a$ and takes the following values:

$$G_s^a(0) = \frac{c}{\lambda\mu + c} + \frac{\lambda\mu}{\lambda\mu + c} \exp(-Ha), \quad G_s^a(a) = 1. \tag{20}$$

Notice that setting $s = 0$ and $a = \infty$ corresponds to the case with no risk reduction and no insurance, where $c = w + rz_0$. Denoting $w + rz_0$ as w', $G_s^a(v)$ takes the well-known form:

$$G_s^a(v) = \begin{cases} \dfrac{w'}{\lambda\mu + w'} \exp(\lambda v/w') & \text{for } v < 0, \\ 1 - \dfrac{\lambda\mu}{\lambda\mu + w'} \exp(-v/\mu) & \text{for } v \geq 0. \end{cases} \tag{21}$$

In this case, the integral Eq. (15) has the well-known explicit solution (e.g., Corollary 3.2 of Asmussen and Albrecher (2010)). If $w' > \lambda\mu$,

$$\Psi_s^a(M) = \frac{\lambda\mu}{w'} \exp\left\{ -M\left(\frac{w' - \lambda\mu}{w'\mu}\right) \right\}, \tag{22}$$

where $(w' - \lambda\mu)/w'\mu$ is the Lundberg exponent. If $w' \leq \lambda\mu$, then $\Psi_s^a(M) = 1$ for all M.

In the case with risk reduction and XL insurance, the ruin probability function $\Psi_s^a(\cdot)$ is obtained by solving Eq. (15) numerically.

3. Properties of ruin probability and decision-making problem

3.1. Properties of ruin probability

The problem represented by Eq. (15) can be classified into the following two cases. In the case where $a \leq M$, the maximum possible net expenditure in the current interval is capped at a that is smaller than the capital. Hence, $G_s^a(M) = 1$, and the government is free from the risk of immediate ruin. Eq. (15) reduces to

$$\Psi_s^a(M) = \int_{-\infty}^a \Psi_s^a(M - v)\, dG_s^a(v). \tag{23}$$

In the case where $a > M$, immediate ruin can occur. The full equation of Eq. (15) applies, namely

$$\Psi_s^a(M) = 1 - G_s^a(M) + \int_{-\infty}^M \Psi_s^a(M - v)\, dG_s^a(v). \tag{24}$$

Ruin probability has the following properties.

Property 1 (Existence and uniqueness). *If (s, a) is in the set Ω, there exists a unique function $\Psi_s^{a*}(\cdot)$ in the space of bounded functions $B[0, \infty)$ such that $\Psi_s^{a*}(\cdot)$ satisfies the fixed-point equation:*

$$\Psi_s^{a*}(M) = 1 - G_s^a(M) + \int_{-\infty}^{\min(M,a)} \Psi_s^{a*}(M - v)\, dG_s^a(v) \quad \text{for all } M \geq 0. \tag{25}$$

Proof. See Appendix A. □

Property 2. *Ruin probability $\Psi_s^{a*}(M)$ established as the fixed point by Eq. (25) has the following properties:*

1. (Interior solution). $0 < \Psi_s^{a*}(M) < 1$ *for all* M (≥ 0).
2. (Monotonicity). $\Psi_s^{a*}(M)$ *is strictly decreasing in* M.
3. (Limit). *If* $(s, a) \in \Omega$, *then* $\lim_{M\to\infty} \Psi_s^{a*}(M) = 0$.

Proof. See Appendix B. □

Property 3 (Sensitivity to policy parameters). *For $(s, a) \in \Omega$, the ruin probability $\Psi_s^{a*}(M)$ is differentiable with respect to the policy parameters s and a. The sensitivity satisfies*

$$\frac{d\Psi_s^{a*}}{d\chi} = \underbrace{(I - \mathcal{L}_{s,a})^{-1}}_{\text{Feedback multiplier}} \left[\underbrace{\frac{\partial T_{s,a}[\Psi_s^{a*}]}{\partial \chi}}_{\text{Direct effect}} \right] \quad \text{for } \chi = s, a, \tag{26}$$

where $T_{s,a} : B[0, \infty) \to B[0, \infty)$ is the operator defined by

$$T_{s,a}[\Psi_s^a](M) = 1 - G_s^a(M) + \int_{-\infty}^{\min(M,a)} \Psi_s^a(M - v)\, dG_s^a(v). \tag{27}$$

$B[0, \infty)$ is the Banach space of bounded functions on $[0, \infty)$ with the supremum norm. I denotes the identity operator, and $\mathcal{L}_{s,a} : B[0, \infty) \to B[0, \infty)$ is the linear operator defined by

$$\mathcal{L}_{s,a}[h](M) = \int_{-\infty}^{\min(M,a)} h(M - v)\, dG_s^a(v) \quad \text{for any } h \in B[0, \infty). \tag{28}$$

The direct effects are broken down as follows for s and a, respectively:

$$\frac{\partial T_{s,a}[\Psi_s^{a*}]}{\partial s}(M) = \underbrace{-\frac{\partial G_s^a(M)}{\partial s}}_{\text{CPIR effect}} + \underbrace{\int_{-\infty}^{\min(M,a)} \Psi_s^{a*}(M - v) \frac{\partial g_s^a(v)}{\partial s}\, dv}_{\text{Distribution shift effect}}, \tag{29a}$$

$$\frac{\partial T_{s,a}[\Psi_s^{a*}]}{\partial a}(M) = \underbrace{-\frac{\partial G_s^a(M)}{\partial a}}_{\text{CPIR effect}} + \underbrace{\int_{-\infty}^{\min(M,a)} \Psi_s^{a*}(M - v) \frac{\partial g_s^a(v)}{\partial a}\, dv}_{\text{Distribution shift effect}}$$

$$+ \underbrace{\mathbf{1}_{\{M > a\}} \cdot \Psi_s^{a*}(M - a) \cdot g_s^a(a)}_{\text{Boundary shift effect}}, \tag{29b}$$

where "CPIR" stands for "Change-in-Probability-of-Immediate-Ruin", and $\mathbf{1}_{\{\cdot\}}$ refers to the indicator function.

Proof. See Appendix C. □

The sensitivity formula (26) has the following interpretation. The "direct effect" measures how a marginal change in the parameter affects the ruin probability at the respective M, while holding the ruin probabilities at other capital levels constant. The "feedback multiplier" captures how changes propagate over time through the problem's recursive structure. The feedback multiplier exists and is bounded, as shown by Lemma 5 in Appendix C. Applying the Neumann series expansion, the multiplier is expressed as follows:

$$(I - \mathcal{L}_{s,a})^{-1} = I + \mathcal{L}_{s,a}^1 + \mathcal{L}_{s,a}^2 + \mathcal{L}_{s,a}^3 + \cdots, \tag{30}$$

where $\mathcal{L}_{s,a}^1 = \mathcal{L}_{s,a}$, and $\mathcal{L}_{s,a}^n$ $(n = 1, 2, \cdots)$ represents the effects propagated during the period initiated by the n-th disaster.

The direct effects are decomposed into several effects as shown by Eqs. (29a) and (29b). The "Change-in-Probability-of-Immediate-Ruin (CPIR) effect" measures how s (or a) affects the probability of surviving the first disaster. If $\partial G_s^a(M)/\partial s > 0$ (or $\partial G_s^a(M)/\partial a > 0$), namely, the distribution shifts toward lower net expenditure V_s^a, this term is negative, reducing ruin probability. The "distribution shift effect" captures how s (or a) affects the weighting of future ruin probabilities through changes in the density $g_s^a(v)$. In addition, for the attachment point a and in the case of $M > a$, the direct effect includes the "boundary shift effect", which captures how changing a affects the boundary of the integration region.

Impacts of s and a on the distribution of V_s^a, equivalently on the CDF G_s^a and the density g_s^a, are not one-directional. The parameter s affects the distribution G_s^a through two channels: 1) the frequency channel, where a higher s reduces disaster frequency $\lambda(s)$, and 2) the net revenue channel, where higher s affects net revenue $c(s,a)$ through an increased risk reduction cost $K(s)$ and a reduced insurance premium $\pi(s,a)$. The marginal impacts of s in both channels are represented by

$$\frac{\partial \lambda(s)}{\partial s} = -\frac{\lambda_0}{\mu}\exp(-s/\mu) \; < 0, \tag{31a}$$

$$\frac{\partial c(s,a)}{\partial s} = -\frac{\partial K(s)}{\partial s} - \frac{\partial \pi(s,a)}{\partial s} = -2\alpha s + (1+\theta)\lambda_0 \exp\{-(s+a)/\mu\}, \tag{31b}$$

respectively. The net revenue effect changes sign at $s^\circ(a)$, determined by the following equation:

$$s^\circ(a) = \frac{(1+\theta)\lambda_0 \exp\{-(s^\circ + a)/\mu\}}{2\alpha}. \tag{32}$$

When $s < s^\circ(a)$, the savings in insurance premiums exceed the increase in the risk reduction costs, so $\partial c/\partial s > 0$. The opposite holds for the reverse case.

The parameter a affects the distribution G_s^a through two channels: 1) the premium channel, where a higher a reduces insurance premium $\pi(s,a)$, increasing net revenue $c(s,a)$, and 2) the retained loss channel, where a higher a increases the maximum retained loss $X^a = \min(X,a)$, exposing the government to larger potential expenditures. The marginal impact of a on the premium is represented by

$$\frac{\partial c(s,a)}{\partial a} = -\frac{\partial \pi(s,a)}{\partial a} = (1+\theta)\lambda_0 \exp\{-(s+a)/\mu\} \; > 0. \tag{33}$$

3.2. Government's problem

At first glance, it may seem nonsensical to set the attachment point a above M because the government has already been ruined when X^a reaches a. However, an increase in a increases $c(s,a)$ by decreasing the premium payment $\pi(s,a)$ and makes the distribution of V_s^a more negative. Policies that increase a will be effective if, as a result, a disaster does not occur for a long period of time.

In general, governments do not focus solely on the risk of ruin, i.e., the possibility that their fiscal assets will fall below a certain threshold. Rather, governments have a greater objective of maximizing the benefits they provide to their citizens through the provision of various public goods and services. The expected amount of fiscal resources per unit time for such an objective is represented by

$$\mathbb{E}[B_s^a] = -\lambda(s)\mathbb{E}[V_s^a] = w + rz_0 - K(s) - \pi(s,a) - \lambda(s)\mathbb{E}[X^a], \tag{34}$$

where $K(s)$, $\pi(s,a)$, $\lambda(s)$, and $\mathbb{E}[X^a]$ are given by Eqs. (1), (11), (3), and (10), respectively. Note that, in this model, funds already committed for specific purposes have already been deducted from government revenue. Therefore, w represents revenue for which allocation has not yet been committed, and its use is subject to future decision. The expected discretionary annual budget B_s^a is determined after deducting all (expected) costs, taking into account the ruin probability $\Psi_s^a(M)$, where M is the initial capital relative to the actual borrowing limit.

The decision-making problem is formulated as follows:

$$\max_{(s,a)\in\Omega} \mathbb{E}[B_s^a] \quad \text{subject to} \quad \Psi_s^a(M) \le \Psi_0, \tag{35}$$

where Ψ_0 is the threshold that is set to constrain the ruin probability. This is a nonconvex optimization problem due to the constraint on ruin probability. Therefore, the optimal policy (s^*, a^*) must be solved numerically, while the following relationship can be derived analytically.

Property 4 (Inequality constraint and optimal solutions). *In cases where $\theta > 0$, when the ruin probability constraint is non-binding, the government opts for a risk reduction policy at a positive finite level, i.e., $0 < s^* < \infty$, and does not apply the XL insurance, i.e., $a^* = \infty$. When the ruin probability constraint becomes binding, the government may select a positive finite level for each policy.*

Proof. See Appendix D. □

Property 4 suggests that XL insurance is used only to lower the ruin probability. This is because the optimization aims to maximize the expected resources, and since the insurance premium includes a positive markup, it is higher than the expected insurance benefit. On the other hand, when the ruin probability constraint becomes binding, the effects described in Property 3 are incorporated into the policy's cost-effectiveness evaluation. When the marginal effect shown by Eq. (26) is positive, a finite attachment point may be selected. Regarding risk reduction, its cost does not include fixed costs, and the marginal costs increase continuously from zero. On the other hand, marginal benefits decrease continuously due to the exponential distribution of damages. The objective function is single-peaked with respect to s and has its maximum value. When the ruin probability constraint is binding, the effect by Eq. (26) is additionally considered.

4. Numerical analysis

4.1. Calculation method for ruin probability

First, we calculate a set of the ruin probabilities $\{\Psi_s^a(M)|$ for all $M\}$ for all $(s,a) \in \Omega$, and then solve the problem (35) to find the optimal combination (s^*, a^*). To obtain $\{\Psi_s^a(M)\}$, we solve the integral Eq. (15) through iteration and discretization. We incorporate the iteration as follows:

$$\Psi_{s,k+1}^a(M) = 1 - G_s^a(M) + \int_{-\infty}^{\min(M,a)} \Psi_{s,k}^a(M-v)\, dG_s^a(v), \tag{36}$$

where $k\ (=1,\cdots)$ indicates the iteration. Appendix A aims to prove Property 1, but also provides a proof showing that the iterative calculations converge by the contraction mapping theorem.

The function $\Psi_{s,k}^a(M)$ is discretized. We define a discrete and bounded grids, $\{i \mid i = 1,2,\cdots\}$ and $\{j \mid j = 1,2,\cdots\}$, for M and v, respectively. The length of unit interval is defined by δ for both grids. We set large integers, n_1 and n_2, and define $L_1 = (n_1 - 1)\delta$ and $L_2 = (n_2-1)\delta$. $(-L_1)$ corresponds to $v = -\infty$, and L_2 corresponds to $M = \infty$.

Now, define $\psi_{s,k}^a(i)\ (i = 1,\cdots,n_2)$ as the approximation of $\Psi_{s,k}^a(M)\ (0 \le M < \infty)$ at iteration k. The approximant for Eq. (36) is

$$\psi_{s,k+1}^a(i) = 1 - G_s^a((i-1)\delta) + \sum_{j=1}^{J(i)-1} \psi_{s,k}^a(n_1 + i - j)\cdot g_s^a(-L_1 + (j-1)\delta)\cdot\delta, \tag{37}$$

where $J(i) = \lfloor n_1 + \min(i-1, a/\delta)\rfloor$. The iteration may start with $\psi_{s,1}^a = 0$ for all M. The iterative process terminates when the change between $\{\psi_{s,k+1}^a\}$ and $\{\psi_{s,k}^a\}$ is minimal. In the next step, we find the optimal combination (s^*, a^*) in the discretized (s,a)-space for the problem (35).

4.2. Numerical example

We apply the model to a hypothetical setting using data from Vanuatu, an island nation in the South Pacific. Vanuatu is highly vulnerable to numerous natural hazards due to its location within the Pacific Ring of Fire. It faces risks primarily from frequent tropical cyclones, earthquakes, and volcanic eruptions, along with tsunamis, floods, landslides, and droughts, making it one of the world's most disaster-prone countries (UNDRR, 2022). Table 1 lists the parameter values used in the calculations. In this numerical analysis, due to insufficient data, multiple assumptions were introduced in determining these values.

Fig. 2 shows the ruin probability surface. The ruin probability rises sharply within specific intervals of s and a, reaching 1. The ruin probability exceeds the threshold if at least one of the following conditions applies: The risk reduction is less than 0.67, or the insurance attachment point is greater than 4.0, implying that the government should implement both risk reduction and insurance measures. Fig. 3 shows the surface of the expected fiscal resources for policy combinations that

Table 1
Values of parameters.

Poisson intensity	λ_0	0.57
Mean damage	μ	5.75
Initial capital	M	2.66
Annual income	w	4.93
Interest rate	r	0.08
Interest-bearing debt	$-z_0$	11.21
Premium markup	θ	0.50
Risk reduction cost	α	0.02
Ruin probability limit	Ψ_0	0.08

- The units for μ, w, M, and z_0 are given as percentages relative to GDP.
- The values of the parameters regarding the fiscal condition are determined by combining data mainly from IMF (2024,b), CEIC Data (2018) and assumptions.
- The values of the parameters regarding hazards and disaster finance are determined by combining data mainly from CFE-DM (2023), World Bank, GFDRR (2017), UNDRR (2021), Nishizawa et al. (2019) and assumptions.

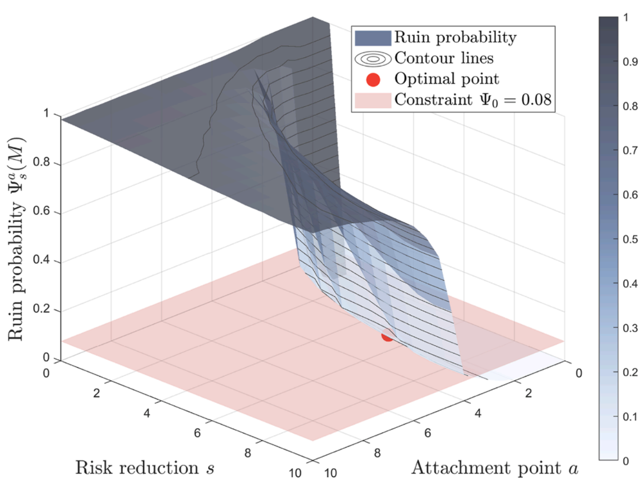


Fig. 2. Ruin probability (Basic Case).

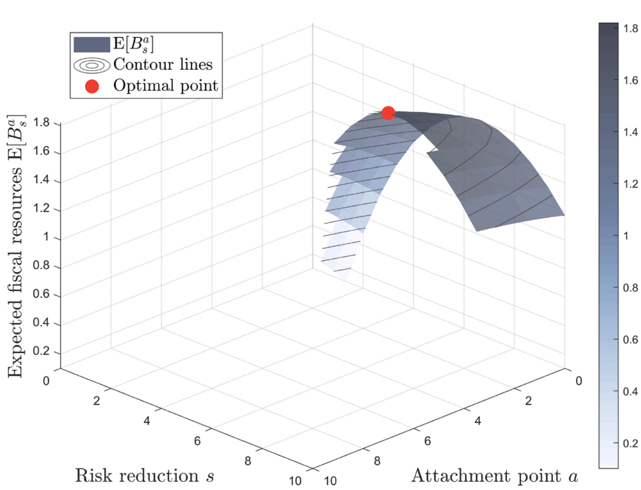


Fig. 3. Expected fiscal resources (Basic Case).

satisfy the ruin probability constraint. The optimal policy is determined as $(s^*, a^*) = (6.0, 3.0)$. The value of a^* also represents the thickness of the risk-retention layer.

Figs. 4 and 5 show how optimal policies change with parameter variations. When interpreting these changes, note that for risk reduction, larger values indicate greater investment, while for the attachment point, smaller values indicate broader insurance coverage. Fig. 4 shows changes in the optimal policies and the objective function values when the initial capital M varies. The optimal risk reduction is not so sensitive to changes in M (the left panel). The optimal attachment point is an increasing function of M (the second panel from the left). When the initial capital increases, the government becomes less dependent on insurance, reduces premium payments, and thereby increases the expected fiscal resources. Conversely, when the initial capital is small, the government needs to lower the attachment point and increase the thickness of the insurance layer. This numerical example suggests that when the initial capital is zero, the government should reduce the size of the risk-retention layer to 0.5% of GDP. The graph shape of the expected fiscal resources (the right panel) resembles that of attachment points because the amount of expected fiscal resources is strongly influenced by increases or decreases in insurance premium expenditures.

The response curve of ruin probability to M is non-monotonic and has been obtained as a bumpy curve (the third panel from the left). To interpret this result, we must bear in mind the following two points. First, this does not contradict the second property of Property 2, namely "Monotonicity". While s and a were fixed when the monotonicity was derived for Property 2, they are now chosen endogenously for each value of M to maximize the objective function $\mathbb{E}[B_s^a]$, taking different values for each M. Second, the bumpy curve results from our use of discrete optimization. Property 4 states that, if the ruin-probability constraint is non-binding, then $a^* = \infty$. The contraposition indicates that, if $a^* < \infty$, then the constraint is binding. Observing that a^* is far below the upper bound of its grid in this example case, we find that the ruin-probability constraint is binding. Therefore, if the problem were continuous optimization, $\Psi_s^a(M) = \Psi_0$ would hold across the entire test interval of M, illustrating the flat line in the figure. However, for the discrete optimization, "Ψ_0 is binding at (s^*, a^*)" means "$\Psi_{s^*}^{a^*}(M) \le \Psi_0$ (i.e., (s^*, a^*) is feasible) and $\Psi_{s^*}^{a^*+\Delta a}(M) > \Psi_0$ (i.e., the next grid point is infeasible)". The $\Psi_s^a(M)$-curve is obtained by meeting this condition. Let's take the case where $M = 5.32$ (the right endpoint of the test interval) as an example. At this point, $a^* = 3.5$ and $\Psi_s^a(M) = 0.024$ (the second and third panels from the left). However, if a takes the next-grid value of 4.0, $\Psi_s^a(M)$ becomes 0.28, significantly exceeding Ψ_0 $(= 0.08)$. While this result from discrete optimization may be problematic because the values of $\Psi_s^a(M)$ diverges significantly from those of continuous optimization, it provides insight into how steep the gradient of the ruin probability surface is around the optimal solution. Furthermore, this result suggests that the grid of policy variables (s, a) should be carefully designed.

Fig. 5 shows the responses of the four endogenous variables to increases and decreases in the four parameters. As the expected damage to the government increases, the government must expand the scale of both risk reduction and insurance (Panels (s^*, μ) and (a^*, μ)). The expected fiscal resources decrease with μ (Panel $(\mathbb{E}[B_s^a], \mu)$), but this reduction would be even larger without the optimal policy. Lower costs for risk reduction increase the optimal risk reduction and decrease the optimal insurance coverage (Panels (s^*, α) and (a^*, α)). Higher insurance costs lead to a higher optimal risk reduction (Panel (s^*, θ)). Finally, if the ruin probability constraint Ψ_0 becomes stricter, the government needs to lower the attachment point of insurance (Panel (a, Ψ_0)). The endogenous ruin probability $\Psi_s^a(M)$ also decreases as the constraint Ψ_0 becomes stricter (Panel $(\Psi_s^a(M), \Psi_0)$). In this analysis example, $\Psi_s^a(M)$ does not strictly coincide with Ψ_0 (the panels in the third row) for the reason stated in the previous paragraph. However, this result demonstrates that

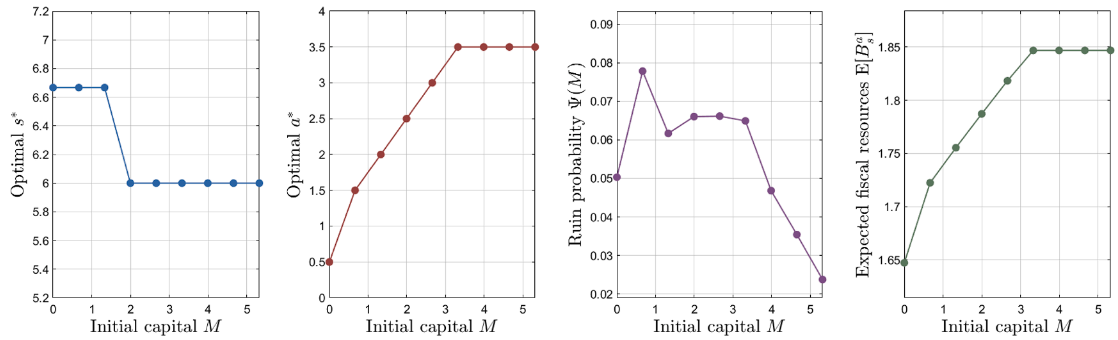
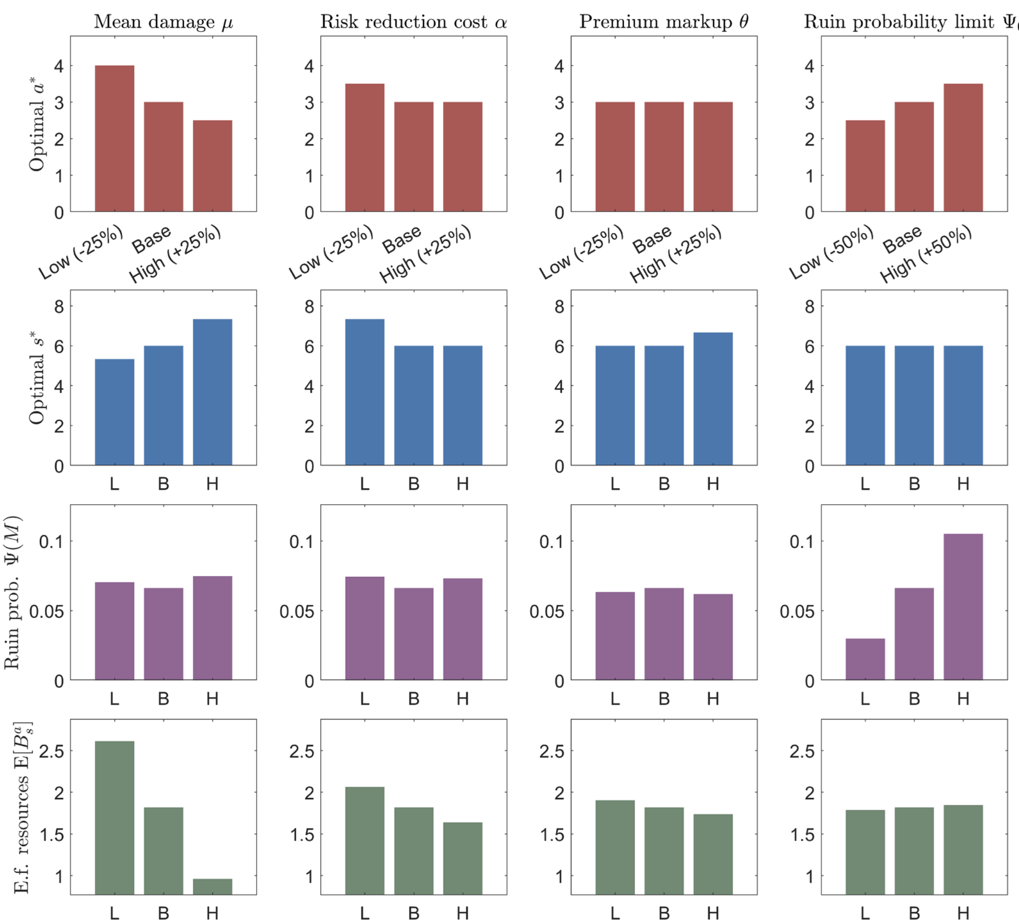


Fig. 4. Sensitivity of optimal policy to initial capital.



- The figure shows how the values of the four endogenous variables, a^*, s^*, $\Psi_s^a(M)$, and $\mathbb{E}[B_s^a]$, change when the values of the parameters, μ, α, θ, and ψ are each decreased or increased.
- "Base" and "B" denote the basic case. "Low (-25%)" and "L" ("High (-25%)" and "H") indicate cases where the parameter value is decreased (increased) by 25% from the basic case. The analysis for Ψ_0 uses a rate of 50% for the changes in the parameter.

Fig. 5. Sensitivity of optimal policy to parameters.

the ruin probability constraint exerts a constraining effect on the problem of maximizing expected financial resources.

5. Concluding remark

This paper proposed to apply the ruin theory to the problem of managing infrastructure exposed to damage risks. In the model, the government chooses the optimal combination of measures to prevent a part of disaster damage through disaster prevention investment, namely, risk reduction, and measures to secure post-disaster reconstruction costs through its own funds and insurance contracts. While the ruin theory has a vast body of research, this study is unique in that it considers the government as the insured party and examines the optimal XL insurance while taking risk reduction into account. By leveraging the memoryless property of the exponential distribution and so on, we were able to extend the model while maintaining the possibility of analytical treatment and to derive several properties, including one that clarified the composition of the policy effects.

This model is a useful tool for quantitatively discussing disaster prevention policies. The set of parameters used in the numerical example was determined based on the hypothetical scenarios, but at the rough scale, these parameters, such as those for fiscal and disaster damage sizes, reflect situations that actual developing country governments may face. The example demonstrated a case in which the ruin probability rises sharply within a specific short interval for both the level of risk reduction and the attachment point of insurance, surpassing the constraint threshold. This implies that, in general, governments need to be careful when determining a set of countermeasures. Additionally, it is important to recognize that, even when the government's objective function is based on the expected value of fiscal resources, it can be effective to use insurance that is not actuarially fair, when considering the ruin probability constraint. Moreover, the smaller the initial capital, meaning the less financial reserve the government has, the more extensive the insurance coverage should be, and the more limited the risk retention needs to be. In the real world, it is often observed that developing country governments do not implement sufficient disaster management measures due to their strained fiscal circumstances. However, this result suggests that such governments would be better off avoiding risk and covering potential risks through risk reduction and insurance, as this is more effective for fiscal sustainability. Without doubt, different strategies will be suggested in other parameter environments. This model has the potential to provide meaningful insights for practical disaster policy discussions. Going forward, it will be important to refine the parameter values and improve statistical validity.

This study offers many avenues for future research. The model in this study assumed that all debts incurred after the initial point in time were concessional. This assumption reflects reality to some extent, while it is theoretically interesting to develop a model in which interest rates are endogenously determined by the ruin probability. Additionally, it is a future topic to consider the case where ruin is more closely linked to sovereign default and to examine fiscal management after the default.

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Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

CRediT authorship contribution statement

Muneta Yokomatsu: Writing – review & editing, Writing – original draft, Visualization, Software, Data curation, Investigation, Formal analysis, Methodology, Conceptualization, Project administration; **Georg Pflug:** Writing – original draft, Software, Formal analysis, Methodology.

Declaration of competing interest

None.

Appendix A. Proof of Property 1

Consider the Banach space $B[0,\infty)$ of bounded functions $f, g : [0,\infty) \to [0,1]$, equipped with the supremum norm: $\|f-g\| = \sup_{M\geq 0} |f(M) - g(M)|$. This is a complete metric space.

Define the operator $T : B[0,\infty) \to B[0,\infty)$ by

$$T[\Psi_s^a(M)] = 1 - G_s^a(M) + \int_{-\infty}^{\min(M,a)} \Psi_s^a(M-v)\, dG_s^a(v). \tag{A.1}$$

We first verify that T maps $B[0,\infty)$ into itself. Since $\Psi_s^a(\cdot)$ takes values in $[0,1]$, and $G_s^a(\cdot)$ is a CDF, we have $0 \leq T[\Psi_s^a(\cdot)] \leq 1$. Moreover, if $\Psi_s^a(\cdot)$ is measurable, then $T[\Psi_s^a(\cdot)]$ is also measurable due to the properties of the Lebesgue integral. Hence, T maps $B[0,\infty)$ into itself. We proceed by partitioning the domain $[0,\infty)$ into two regions: Region 1, defined by $M \in [0,a)$, and Region 2 defined by $M \in [a,\infty)$.

Region 1: $M \in [0,a)$

For $M < a$, the operator T becomes

$$T[\Psi_s^a(M)] = 1 - G_s^a(M) + \int_{-\infty}^{M} \Psi_s^a(M-v)\, dG_s^a(v). \tag{A.2}$$

For any $\Psi_{s1}^a, \Psi_{s2}^a \in B[0,\infty)$ and any $M \in [0,a)$, we have

$$\begin{aligned} |T[\Psi_{s1}^a](M) - T[\Psi_{s2}^a](M)| &= \left| \int_{-\infty}^{M} [\Psi_{s1}^a(M-v) - \Psi_{s2}^a(M-v)]\, dG_s^a(v) \right| \\ &\leq \int_{-\infty}^{M} |\Psi_{s1}^a(M-v) - \Psi_{s2}^a(M-v)|\, dG_s^a(v) \\ &\leq \|\Psi_{s1}^a - \Psi_{s2}^a\| \cdot G_s^a(M). \end{aligned} \tag{A.3}$$

From Eq. (19), we have for $M \in [0,a)$

$$G_s^a(M) = 1 - \frac{\lambda\mu}{\lambda\mu + c}\left\{ \exp(-M/\mu) - \exp\left(\frac{\lambda M}{c} - Ha\right)\right\} < 1, \tag{A.4}$$

since the term in braces is strictly positive for $M < a$.

Let $\zeta_1 = \sup_{M\in[0,a)} G_s^a(M)$. Since $G_s^a(M) < 1$ for all $M \in [0,a)$ and G_s^a is continuous, we have $\zeta_1 < 1$. Therefore, T is a strict contraction on functions restricted to $[0,a)$ with contraction factor ζ_1. By the Banach fixed-point theorem (e.g., Salgado and Wise, 2022), there exists a unique function Ψ_s^{a*} on $[0,a)$ that satisfies $T[\Psi_s^{a*}](M) = \Psi_s^{a*}(M)$ for all $M \in [0,a)$.

Region 2: $M \in [a,\infty)$

For $M \geq a$, the operator T becomes

$$T[\Psi_s^a(M)] = 1 - G_s^a(M) + \int_{-\infty}^{a} \Psi_s^a(M-v)\, dG_s^a(v). \tag{A.5}$$

Since $G_s^a(M) = 1$ for $M \geq a$, this simplifies to

$$T[\Psi_s^a(M)] = \int_{-\infty}^{a} \Psi_s^a(M-v)\, dG_s^a(v). \tag{A.6}$$

Because $G_s^a(a) = 1$, which can not take the role of the contraction factor as $G_s^a(M)$ worked for Region 1, we prove by mathematical induction that given the unique solution $\Psi_s^{a*}(M)$ for $M \in [0,a)$, there exists a unique extension to $[a,\infty)$ satisfying the fixed-point equation.

Base Case: We have established that there exists a unique solution $\Psi_s^{a*}(M)$ for $M \in [0,a)$.

Inductive Hypothesis: Assume that for some integer $n \geq 1$, we have uniquely determined $\Psi_s^{a*}(M)$ for $M \in [0,na)$.

Inductive Step: We need to show that $\Psi_s^{a*}(M)$ can be uniquely extended to $M \in [na,(n+1)a)$.

For any $M \in [na,(n+1)a)$, we can decompose the integral based on the values of $M - v$. For $M \in [na,(n+1)a)$ and $v \in (-\infty, a]$,

- When $v > M - na$, $M - v < na$. Since $M \geq na$ and $v \leq a$, we have $M - v \geq M - a \geq na - a = (n-1)a \geq 0$. Thus, $M - v \in [0,na)$, so $\Psi_s^{a*}(M-v)$ is known by the inductive hypothesis.
- When $v \leq M - na$, $M - v \geq na$. Moreover, if $v \geq 0$, then $M - v \leq M < (n+1)a$. If $v < 0$, then $M - v > M$, so $M - v$ may exceed $(n+1)a$.

Therefore, we can write

$$T[\Psi_s^a](M) = \int_{-\infty}^{M-na} \Psi_s^a(M-v)\, dG_s^a(v) + \int_{M-na}^{a} \Psi_s^{a*}(M-v)\, dG_s^a(v). \tag{A.7}$$

To handle the case where $M - v$ may exceed $(n+1)a$, we define extended functions. For any function $\Psi_s^a \in B[na,(n+1)a)$, define its extension $\bar{\Psi}_s^a$ by

$$\bar{\Psi}_s^a(x) = \begin{cases} \Psi_s^a(x) & \text{if } x \in [na,(n+1)a), \\ \Psi_s^a((n+1)a - \epsilon) & \text{if } x \geq (n+1)a, \end{cases} \tag{A.8}$$

where $\epsilon > 0$ is arbitrarily small.

Define the operator $T_n : B[na,(n+1)a) \to B[na,(n+1)a)$ by

$$T_n[\Psi_s^a](M) = \int_{-\infty}^{M-na} \bar{\Psi}_s^a(M-v)\, dG_s^a(v) + C_{Tn}(M), \tag{A.9a}$$

$$\text{where } C_{Tn}(M) = \int_{M-na}^{a} \Psi_s^{a*}(M-v)\, dG_s^a(v) \tag{A.9b}$$

is a known function since $\Psi_s^{a*}(M-v)$ for $v \in (M-na, a]$ implies $M - v \in [0, na)$.

For any $\Psi_{s1}^a, \Psi_{s2}^a \in B[na,(n+1)a)$ and any $M \in [na,(n+1)a)$,

$$\begin{aligned} |T_n[\Psi_{s1}^a](M) - T_n[\Psi_{s2}^a](M)| &= \left| \int_{-\infty}^{M-na} [\bar{\Psi}_{s1}^a(M-v) - \bar{\Psi}_{s2}^a(M-v)]\, dG_s^a(v) \right| \\ &\le \int_{-\infty}^{M-na} |\bar{\Psi}_{s1}^a(M-v) - \bar{\Psi}_{s2}^a(M-v)|\, dG_s^a(v) \\ &\le \|\Psi_{s1}^a - \Psi_{s2}^a\|_{B[na,(n+1)a)} \cdot G_s^a(M-na). \end{aligned} \tag{A.10}$$

Since $M \in [na,(n+1)a)$, we have $M - na \in [0,a)$. We know that $G_s^a(x) < 1$ for all $x \in [0,a)$. Therefore, for any $M \in [na,(n+1)a)$,

$$G_s^a(M-na) < G_s^a(a) = 1. \tag{A.11}$$

This shows that T_n is a contraction on $B[na,(n+1)a)$ with the contraction factor $\zeta_n = \sup_{M \in [na,(n+1)a)} G_s^a(M-na) < 1$. By the Banach fixed-point theorem, there exists a unique fixed point $\Psi_s^{a*} \in B[na,(n+1)a)$ such that

$$\Psi_s^{a*}(M) = T_n[\Psi_s^{a*}](M) = \int_{-\infty}^{M-na} \bar{\Psi}_s^{a*}(M-v)\, dG_s^a(v) + C_{Tn}(M). \tag{A.12}$$

This unique fixed point satisfies

$$\Psi_s^{a*}(M) = \int_{-\infty}^{a} \Psi_s^{a*}(M-v)\, dG_s^a(v) \tag{A.13}$$

for all $M \in [na,(n+1)a)$. This completes the inductive step, showing that Ψ_s^{a*} can be uniquely extended from $[0,na)$ to $[0,(n+1)a)$.

Therefore, by the principle of mathematical induction, there exists a unique function $\Psi_s^{a*} \in B[0,\infty)$ satisfying the fixed-point Eq. (25).

Appendix B. Proof of Property 2

We will first derive three lemmas, and then, by combining them, prove the three properties of Property 2.

Lemma 1 (Non-triviality). $\Psi_s^{a*}(0) > 0$.

Proof. From Eq. (25) and $G_s^a(0) < 1$ from Eq. (19),

$$\Psi_s^{a*}(0) = 1 - G_s^a(0) + \int_{-\infty}^{0} \Psi_s^{a*}(0-v)\, dG_s^a(v) > 1 - G_s^a(0) > 0. \tag{B.1}$$

□

Lemma 2 (Monotonicity). $\Psi_s^{a*}(M_1) > \Psi_s^{a*}(M_2)$ *for all* $0 \le M_1 < M_2$.

Proof. The contraction mapping theorem used in the proof of Property 1 guarantees that the following iteration for $\Psi_{s,k}^a(M)$ converges for all M (≥ 0):

$$\Psi_{s,k+1}^a(M) = 1 - G_s^a(M) + \int_{-\infty}^{\min(M,a)} \Psi_{s,k}^a(M-v)\, dG_s^a(v), \tag{B.2}$$

where k $(= 1, \cdots)$ indicates the iteration cycle. To prove the lemma, we first prove the monotonicity of $\Psi_{s,k}^a(M)$ by applying mathematical induction as follows. Note that, in what follows, we will only deal with formulas that focus only on the case $M_1 < M_2 < a$, such as Eq. (B.4), but the other cases can be treated in the same manner to prove that the lemma holds.

Base Case: Assume $\Psi_{s,0}^a(M) = 0$ for all $M \ge 0$. Then,

$$\Psi_{s,1}^a(M) = 1 - G_s^a(M) + \int_{-\infty}^{M} \Psi_{s,0}^a(M-v)\, dG_s^a(v) = 1 - G_s^a(M). \tag{B.3}$$

Since $G_s^a(M)$ is a CDF and strictly increasing as shown by Eq. (19), $G_s^a(M_1) < G_s^a(M_2)$, thus $\Psi_{s,1}^a(M_1) > \Psi_{s,1}^a(M_2)$.

Inductive Hypothesis: Assume that for some integer $k \ge 1$, we have shown $\Psi_{s,k}^a(M_1) > \Psi_{s,k}^a(M_2)$ for all $0 \le M_1 < M_2$.

Inductive Step: The inductive hypothesis implies

$$\begin{aligned} \Psi_{s,k+1}^a(M_2) &= 1 - G_s^a(M_2) + \int_{-\infty}^{M_2} \Psi_{s,k}^a(M_2-v)\, dG_s^a(v) \\ &< 1 - G_s^a(M_2) + \int_{-\infty}^{M_2} \Psi_{s,k}^a(M_1-v)\, dG_s^a(v) \\ &= 1 - G_s^a(M_2) + \int_{-\infty}^{M_1} \Psi_{s,k}^a(M_1-v)\, dG_s^a(v) \\ &\quad + \int_{M_1}^{M_2} \Psi_{s,k}^a(M_1-v)\, dG_s^a(v) \\ &< 1 - G_s^a(M_2) + \int_{-\infty}^{M_1} \Psi_{s,k}^a(M_1-v)\, dG_s^a(v) \\ &\quad + \{G_s^a(M_2) - G_s^a(M_1)\} \\ &= 1 - G_s^a(M_1) + \int_{-\infty}^{M_1} \Psi_{s,k}^a(M_1-v)\, dG_s^a(v) \\ &= \Psi_{s,k+1}^a(M_1). \end{aligned} \tag{B.4}$$

This shows that $\Psi_{s,k+1}^a(M)$ is monotonically decreasing with respect to M. The induction has shown that $\Psi_{s,k}^a(M)$ is monotonically decreasing in M for all k.

Now, we take the limit as $k \to \infty$:

$$\begin{aligned} \lim_{k\to\infty} \{\Psi_{s,k}^a(M_1) - \Psi_{s,k}^a(M_2)\} &= \lim_{k\to\infty} \Psi_{s,k}^a(M_1) - \lim_{k\to\infty} \Psi_{s,k}^a(M_2) \\ &= \Psi_s^{a*}(M_1) - \Psi_s^{a*}(M_2) \ge 0. \end{aligned} \tag{B.5}$$

We cannot immediately conclude that the limit is strictly positive. To obtain a strict inequality, we will examine the limit more carefully. For any $\varepsilon > 0$, there exists k_ε such that for all $k \ge k_\varepsilon$, $|\Psi_{s,k}^a(M_1) - \Psi_s^{a*}(M_1)| < \varepsilon$ and $|\Psi_{s,k}^a(M_2) - \Psi_s^{a*}(M_2)| < \varepsilon$. From them, we have

$$-\varepsilon < \Psi_{s,k}^a(M_1) - \Psi_s^{a*}(M_1) < \varepsilon \;\Rightarrow\; \Psi_{s,k}^a(M_1) < \Psi_s^{a*}(M_1) + \varepsilon, \tag{B.6a}$$

$$-\varepsilon < \Psi_{s,k}^a(M_2) - \Psi_s^{a*}(M_2) < \varepsilon \;\Rightarrow\; -\Psi_{s,k}^a(M_2) < -\Psi_s^{a*}(M_2) + \varepsilon. \tag{B.6b}$$

Adding the two inequalities,

$$\Psi_{s,k}^a(M_1) - \Psi_{s,k}^a(M_2) < \Psi_s^{a*}(M_1) - \Psi_s^{a*}(M_2) + 2\varepsilon. \tag{B.7}$$

Now, we find that, if $\Psi_s^{a*}(M_1) = \Psi_s^{a*}(M_2)$, Eq. (B.7) would contradict the fact that $\Psi_{s,k}^a(M_1) > \Psi_{s,k}^a(M_2)$ by choosing ε small enough. Therefore, we conclude $\Psi_s^{a*}(M_1) > \Psi_s^{a*}(M_2)$, namely $\Psi_s^{a*}(M)$ is strictly decreasing in M. □

The following lemma concerns the limit of the ruin probability.

Lemma 3 (Limit). *For* $(s,a) \in \Omega$, *then* $\lim_{M\to\infty} \Psi_s^a(M) = 0$.

Proof. For any $\varepsilon > 0$, we will show that there exists M_ε such that for all $M > M_\varepsilon$, $\Psi_s^{a*}(M) < \varepsilon$.

First, for any $\varepsilon_0 > 0$, we define a sequence of capital levels $\{M_j \mid j = 0, 1, \cdots\}$ such that

$$1 - G_s^a(M_0) < \varepsilon_0 \quad \text{and} \quad M_j = M_{j-1} + \Delta \quad \text{for} \quad j \ge 1, \tag{B.8}$$

where $\Delta > 0$ is a fixed increment. For any capital level M_j, we have from the fixed-point Eq. (25)

$$\Psi_s^{a*}(M_j) = 1 - G_s^a(M_j) + \int_{-\infty}^{M_j} \Psi_s^{a*}(M_j - v)\, dG_s^a(v). \tag{B.9}$$

By Eq. (B.8), $1 - G_s^a(M_j) < \varepsilon_0$ holds for all $j \ge 0$, so we have

$$\Psi_s^{a*}(M_j) < \varepsilon_0 + \int_{-\infty}^{M_j} \Psi_s^{a*}(M_j - v)\, dG_s^a(v). \tag{B.10}$$

We can decompose the integral as follows:

$$\int_{-\infty}^{M_j} \Psi_s^{a*}(M_j - v)\, dG_s^a(v)$$

$$= \int_{-\infty}^{M_{j-1}} \Psi_s^{a*}(M_j - v)\, dG_s^a(v) + \int_{M_{j-1}}^{M_j} \Psi_s^{a*}(M_j - v)\, dG_s^a(v). \tag{B.11}$$

For $v \le M_{j-1}$, the monotonicity, stated by Lemma 2, implies $\Psi_s^{a*}(M_j - v) \le \Psi_s^{a*}(M_{j-1} - v)$, giving

$$\int_{-\infty}^{M_{j-1}} \Psi_s^{a*}(M_j - v)\, dG_s^a(v) \le \int_{-\infty}^{M_{j-1}} \Psi_s^{a*}(M_{j-1} - v)\, dG_s^a(v)$$

$$= \Psi_s^{a*}(M_{j-1}) - (1 - G_s^a(M_{j-1})) = \gamma_j \Psi_s^{a*}(M_{j-1}), \tag{B.12a}$$

$$\text{where}\quad \gamma_j = 1 - \frac{1 - G_s^a(M_{j-1})}{\Psi_s^{a*}(M_{j-1})} < 1. \tag{B.12b}$$

The equation between the first and second lines is obtained from the fixed-point Eq. (25). We see that γ_j works as a contraction factor. For the second integral of Eq. (B.11),

$$\int_{M_{j-1}}^{M_j} \Psi_s^{a*}(M_j - v)\, dG_s^a(v) \le G_s^a(M_j) - G_s^a(M_{j-1}). \tag{B.13}$$

Therefore, from Eqs. (B.11), (B.12a), and (B.13), we have

$$\int_{-\infty}^{M_j} \Psi_s^{a*}(M_j - v)\, dG_s^a(v) < \gamma_j \Psi_s^{a*}(M_{j-1}) + (G_s^a(M_j) - G_s^a(M_{j-1})). \tag{B.14}$$

$G_s^a(M_{j-1})$ and $G_s^a(M_j)$ converge to 1 as j increases. Hence, for sufficiently large M_{j-1}, we can find a value η_j such that

$$\gamma_j \Psi_s^{a*}(M_{j-1}) + (G_s^a(M_j) - G_s^a(M_{j-1})) < \eta_j \Psi_s^{a*}(M_{j-1}), \tag{B.15}$$

where $\gamma_j < \eta_j < 1$. From Eqs. (B.10), (B.14), and (B.15), we have

$$\Psi_s^{a*}(M_j) < \varepsilon_0 + \eta_j \Psi_s^{a*}(M_{j-1}). \tag{B.16}$$

Iterating the right-hand side of the above inequality,

$$\begin{aligned}\Psi_s^{a*}(M_j) &< \varepsilon_0 + \eta_j \Psi_s^{a*}(M_{j-1})\\ &< \varepsilon_0 + \eta_j\big(\varepsilon_0 + \eta_{j-1}\Psi_s^{a*}(M_{j-2})\big) = \varepsilon_0 + \eta_j\varepsilon_0 + \eta_j\eta_{j-1}\Psi_s^{a*}(M_{j-2})\\ &< \cdots\\ &< \varepsilon_0\big(1 + \eta_j + \eta_j\eta_{j-1} + \cdots + \eta_j\eta_{j-1}\dots\eta_2\big) + \eta_j\eta_{j-1}\dots\eta_1\Psi_s^{a*}(M_0).\end{aligned} \tag{B.17}$$

Let $\eta = \sup\{\eta_j : j \ge 1\}$. Since $\eta_j < 1$ for all j, we have $\eta < 1$. Therefore,

$$\begin{aligned}\Psi_s^{a*}(M_j) &< \varepsilon_0(1 + \eta + \eta^2 + \ldots + \eta^{j-1}) + \eta^j \Psi_s^{a*}(M_0)\\ &= \varepsilon_0 \cdot \frac{1-\eta^j}{1-\eta} + \eta^j \Psi_s^{a*}(M_0) < \varepsilon_0 \cdot \frac{1}{1-\eta} + \eta^j.\end{aligned} \tag{B.18}$$

Since $\eta < 1$, $\eta^j \to 0$ as $j \to \infty$. For j sufficiently large such that $\eta^j < \varepsilon_0$, and defining $M_\varepsilon = M_j$,

$$\Psi_s^{a*}(M_\varepsilon) < \varepsilon_0 \cdot \frac{1}{1-\eta} + \varepsilon_0 = \varepsilon_0 \cdot \frac{2-\eta}{1-\eta}. \tag{B.19}$$

For any fixed $\eta < 1$, the term $(2-\eta)/(1-\eta)$ is a positive constant. Since ε_0 was arbitrary, by choosing a sufficiently small $\varepsilon_0 = \varepsilon \cdot (1-\eta)/(2-\eta)$, Eq. (B.19) ensures

$$\Psi_s^{a*}(M_\varepsilon) < \varepsilon \quad \text{for any } \varepsilon\ (>0), \tag{B.20}$$

which implies $\lim_{M\to\infty} \Psi_s^{a*}(M) = 0$. □

The first property (Interior solution) of Property 2 follows from combining Lemmas 1–3. The second (Monotonicity) and third (Limit) properties are identical to Lemmas 2 and 3, respectively.

Appendix C. Proof of Property 3

We employ the Fréchet derivative framework and the implicit function theorem in Banach spaces $B[0,\infty)$ of bounded measurable functions $f : [0,\infty) \to [0,1]$, equipped with the supremum norm $\|f\| = \sup_{M\ge 0} |f(M)|$.

Preliminaries: Recall that the ruin probability Ψ_s^{a*} is characterized as the unique fixed point of the operator $T_{s,a} : B[0,\infty) \to B[0,\infty)$ defined by Eq. (27). ($T_{s,a}$ is identical to T defined by Eq. (A.1) in Appendix A. In this section, the subscript "s,a" is added for convenience in sensitivity analysis.)

Definition 1 (Operator norm). For a linear operator $\mathcal{A} : B[0,\infty) \to B[0,\infty)$, the operator norm is defined as

$$\|\mathcal{A}\| = \sup_{h\ne 0} \frac{\|\mathcal{A}[h]\|}{\|h\|} = \sup_{\|h\|=1} \|\mathcal{A}[h]\|, \tag{C.1}$$

where h denotes any function with $h \in B[0,\infty)$.

Definition 2 (Fréchet derivative). Let X and Y be Banach spaces and let $F : X \to Y$ be an operator. F is Fréchet differentiable at $x_0 \in X$ if there exists a bounded linear operator $DF(x_0) : X \to Y$ such that

$$\lim_{\|h\|\to 0} \frac{\|F(x_0+h) - F(x_0) - DF(x_0)[h]\|}{\|h\|} = 0. \tag{C.2}$$

The operator $DF(x_0)$ is called the Fréchet derivative of F at x_0.

In addition, we define the linear operator $\mathcal{L}_{s,a} : B[0,\infty) \to B[0,\infty)$ as formulated by Eq. (28).

Lemma 4. *The linear operator $\mathcal{L}_{s,a} : B[0,\infty) \to B[0,\infty)$ is bounded with $\|\mathcal{L}_{s,a}\| \le 1$. The operator $(I - \mathcal{L}_{s,a})$, where I denotes the identity operator, is also a bounded linear operator.*

Proof. For any $h \in B[0,\infty)$ and any $M \ge 0$,

$$\begin{aligned}|\mathcal{L}_{s,a}[h](M)| &= \left|\int_{-\infty}^{\min(M,a)} h(M-v)\, dG_s^a(v)\right|\\ &\le \int_{-\infty}^{\min(M,a)} |h(M-v)|\, dG_s^a(v)\\ &\le \|h\| \cdot \int_{-\infty}^{\min(M,a)} dG_s^a(v)\\ &= \|h\| \cdot G_s^a(\min(M,a)) \le \|h\|.\end{aligned} \tag{C.3}$$

Taking the supremum over all $M \ge 0$,

$$\|\mathcal{L}_{s,a}[h]\| = \sup_{M\ge 0} |\mathcal{L}_{s,a}[h](M)| \le \|h\|. \tag{C.4}$$

By Definition 1, the operator norm is obtained as

$$\|\mathcal{L}_{s,a}\| = \sup_{h\ne 0} \frac{\|\mathcal{L}_{s,a}[h]\|}{\|h\|} \le \sup_{h\ne 0} \frac{\|h\|}{\|h\|} = 1. \tag{C.5}$$

Furthermore, since both I and $\mathcal{L}_{s,a}$ are linear operators, their difference $(I - \mathcal{L}_{s,a})$ is also linear. In addition,

$$\|I - \mathcal{L}_{s,a}\| \le \|I\| + \|\mathcal{L}_{s,a}\| \le 1 + 1 = 2. \tag{C.6}$$

Therefore, $(I - \mathcal{L}_{s,a}) : B[0,\infty) \to B[0,\infty)$ is a bounded linear operator. □

Note that Lemma 4 does not directly imply that $(I - \mathcal{L}_{s,a})$ is invertible. The invertibility will be established separately below, using a constructive argument.

Invertibility of $(I - \mathcal{L}_{s,a})$:

Lemma 5. *The operator $(I - \mathcal{L}_{s,a}) : B[0,\infty) \to B[0,\infty)$ is invertible for all $(s,a) \in \Omega$. That is, for any $b \in B[0,\infty)$, there exists a unique $h \in B[0,\infty)$ such that $(I - \mathcal{L}_{s,a})[h] = b$. Moreover, the inverse $(I - \mathcal{L}_{s,a})^{-1}$ is bounded.*

Proof. We establish invertibility by verifying injectivity and surjectivity and applying the bounded inverse theorem. We use a constructive approach analogous to the proof of Property 1 in Appendix A.

Given $b \in B[0,\infty)$, we seek $h \in B[0,\infty)$ satisfying

$$h(M) - \int_{-\infty}^{\min(M,a)} h(M-v)\, dG_s^a(v) = b(M) \quad \text{for all } M \ge 0. \tag{C.7}$$

Rearranging, this is equivalent to

$$h(M) = b(M) + \int_{-\infty}^{\min(M,a)} h(M-v)\, dG_s^a(v). \tag{C.8}$$

Define the operator $S : B[0,a) \to B[0,a)$ by

$$S[h](M) = b(M) + \int_{-\infty}^{\min(M,a)} h(M-v)\, dG_s^a(v). \tag{C.9}$$

Region 1: $M \in [0, a)$

For $M < a$, the operator S becomes

$$S[h](M) = b(M) + \int_{-\infty}^{M} h(M - v)\, dG_s^a(v). \tag{C.10}$$

For any $h_1, h_2 \in B[0, a)$ and any $M \in [0, a)$, we have

$$\begin{aligned} |S[h_1](M) - S[h_2](M)| &= \left| \int_{-\infty}^{M} [h_1(M - v) - h_2(M - v)]\, dG_s^a(v) \right| \\ &\le \int_{-\infty}^{M} |h_1(M - v) - h_2(M - v)|\, dG_s^a(v) \\ &\le \|h_1 - h_2\|_{B[0,a)} \cdot G_s^a(M). \end{aligned} \tag{C.11}$$

From Eq. (A.4), we can identify the contraction factor as $\zeta_0 = \sup_{M\in[0,a)} G_s^a(M) < 1$. Taking the supremum over $M \in [0, a)$ in Eq. (C.11),

$$\|S[h_1] - S[h_2]\|_{B[0,a)} \le \zeta_1 \|h_1 - h_2\|_{B[0,a)}. \tag{C.12}$$

Since $\zeta_1 < 1$, the operator S is a strict contraction. By the Banach fixed-point theorem, there exists a unique $h^* \in B[0, a)$ satisfying $h^* = S[h^*]$, namely, satisfying Eq. (C.7).

Region 2: $M \in [a, \infty)$

For $M \ge a$, the operator, the operator S becomes

$$S[h](M) = b(M) + \int_{-\infty}^{a} h(M - v)\, dG_s^a(v). \tag{C.13}$$

Because of $G_s^a(a) = 1$, which can not take the role of the contraction factor as $G_s^a(M)$ worked for Region 1, we apply the mathematical induction structured as follows.

Base Case: We have established that there exists a unique solution $h^*(M)$ for $M \in [0, a)$.

Inductive Hypothesis: Assume that for some integer $n \ge 1$, we have uniquely determined $h^*(M)$ for $M \in [0, na)$.

Inductive Step: We need to show that $h^*(M)$ can be uniquely extended to $M \in [na, (n + 1)a)$.

We partition the integral based on whether $h(M - v)$ is known or unknown as follows:

- For $v \in (M - na, a]$, since $M \ge na$ and $v \le a$, we have $M - v \ge na - a = (n - 1)a \ge 0$. Also, $M - v < M - (M - na) = na$. Thus $M - v \in [0, na)$, and therefore, $h^*(M - v)$ is known by the inductive hypothesis.
- For $v \in (-\infty, M - na]$, we have $M - v \ge na$. If additionally $v \ge 0$, then $M - v \le M < (n + 1)a$, so $M - v \in [na, (n + 1)a)$. If $v < 0$, then $M - v > M$, which may exceed $(n + 1)a$.

Therefore, we can write

$$S[h](M) = b(M) + \int_{-\infty}^{M-na} h(M - v)\, dG_s^a(v) + \int_{M-na}^{a} h^*(M - v)\, dG_s^a(v). \tag{C.14}$$

To handle the case where $M - v$ may exceed $(n + 1)a$, we define extended functions. For any $h \in B[na, (n + 1)a)$, define its extension $\bar{h}$ by

$$\bar{h}(x) = \begin{cases} h(x) & \text{if } x \in [na, (n + 1)a), \\ h((n + 1)a - \epsilon) & \text{if } x \ge (n + 1)a, \end{cases} \tag{C.15}$$

where $\epsilon > 0$ is arbitrarily small. Define the operator $S_n : B[na, (n + 1)a) \to B[na, (n + 1)a)$ by

$$S_n[h](M) = \int_{-\infty}^{M-na} \bar{h}(M - v)\, dG_s^a(v) + C_{Sn}(M), \tag{C.16a}$$

$$\text{where } C_{Sn}(M) = b(M) + \int_{M-na}^{a} h^*(M - v)\, dG_s^a(v) \tag{C.16b}$$

is the known function since $h^*(M - v)$ for $v \in (M - na, a]$ implies $M - v \in [0, na)$.

$$\begin{aligned} |S_n[h_1](M) - S_n[h_2](M)| &= \left| \int_{-\infty}^{M-na} [\bar{h}_1(M - v) - \bar{h}_2(M - v)]\, dG_s^a(v) \right| \\ &\le \int_{-\infty}^{M-na} |\bar{h}_1(M - v) - \bar{h}_2(M - v)|\, dG_s^a(v) \\ &\le \|h_1 - h_2\|_{B[na,(n+1)a)} \cdot G_s^a(M - na). \end{aligned} \tag{C.17}$$

Since $M \in [na, (n + 1)a)$, we have $M - na \in [0, a)$, so $G_s^a(M - na) < 1$. It follows that S_n is a contraction on $B[na, (n + 1)a)$ with the contraction factor $\zeta_n = \sup_{M\in[na,(n+1)a)} G_s^a(M - na) < 1$. Hence, by the Banach fixed-point theorem, there exists a unique fixed point $h^* \in B[na, (n + 1)a)$.

By induction, h^* is uniquely determined on all of $[0, \infty)$.

Verification of invertibility conditions: By the above construction, for any $b \in B[0, \infty)$, we have found a unique $h^* \in B[0, \infty)$ such that $(I - \mathcal{L}_{s,a})[h^*] = g$. Combined with the findings obtained as Lemma 4, we see that $(I - \mathcal{L}_{s,a})$ is continuous, injective (i.e., one-to-one), and surjective (i.e., onto). Furthermore, by applying the bounded inverse theorem (e.g., Corollary 2.12 in Rudin (1991)), it is guaranteed that the inverse operator $(I - \mathcal{L}_{s,a})^{-1}$ is also continuous and bounded. Hence, $(I - \mathcal{L}_{s,a})^{-1} : B[0, \infty) \to B[0, \infty)$ exists. □

Implicit function formulation: Define the residual operator $\mathcal{F} : \mathbb{R}_+ \times \mathbb{R}_+ \times B[0, \infty) \to B[0, \infty)$ by

$$\mathcal{F}(s, a, \Psi) = \Psi - T_{s,a}[\Psi]. \tag{C.18}$$

The fixed-point condition $\Psi_s^{a*} = T_{s,a}[\Psi_s^{a*}]$ is equivalent to

$$\mathcal{F}(s, a, \Psi_s^{a*}) = 0, \tag{C.19}$$

where 0 denotes the zero function in $B[0, \infty)$.

Lemma 6. *The Fréchet derivative of $\mathcal{F}$ with respect to Ψ is*

$$D_\Psi \mathcal{F}(s, a, \Psi)[h] = (I - \mathcal{L}_{s,a})[h], \tag{C.20}$$

which is independent of Ψ.

Proof. For any $\Psi, h \in B[0, \infty)$,

$$\begin{aligned} \mathcal{F}(s, a, \Psi + h) &= (\Psi + h) - T_{s,a}[\Psi + h] \\ &= (\Psi + h) \\ &\quad - \left(1 - G_s^a(M) + \int_{-\infty}^{\min(M,a)} (\Psi + h)(M - v)\, dG_s^a(v)\right) \\ &= \Psi - T_{s,a}[\Psi] + h - \int_{-\infty}^{\min(M,a)} h(M - v)\, dG_s^a(v) \\ &= \mathcal{F}(s, a, \Psi) + h - \mathcal{L}_{s,a}[h] \\ &= \mathcal{F}(s, a, \Psi) + (I - \mathcal{L}_{s,a})[h]. \end{aligned} \tag{C.21}$$

From the definition Eq. (C.2), and due to no higher-order terms in h,

$$\mathcal{F}(s, a, \Psi + h) = \mathcal{F}(s, a, \Psi) + D_\Psi \mathcal{F}(s, a, \Psi)[h]. \tag{C.22}$$

From Eqs. (C.21) and (C.22), we obtain (C.20). Note that this is linear in h and independent of Ψ, reflecting the affine structure of $T_{s,a}$ in Ψ. □

As Eq. (C.19) holds for all $(s, a) \in \Omega$, we differentiate this identity with respect to χ (= s, a) to have

$$\frac{\partial \mathcal{F}}{\partial \chi} + D_\Psi \mathcal{F}\left[\frac{d\Psi_s^{a*}}{d\chi}\right] = 0. \tag{C.23}$$

From Eq. (C.18), we have

$$\frac{\partial \mathcal{F}}{\partial \chi} = -\frac{\partial T_{s,a}[\Psi_s^{a*}]}{\partial \chi}. \tag{C.24}$$

By Lemma 6, $D_\Psi \mathcal{F} = I - \mathcal{L}_{s,a}$ holds for $h = d\Psi_s^{a*}/d\chi$. Substituting into Eq. (C.23), we have

$$-\frac{\partial T_{s,a}[\Psi_s^{a*}]}{\partial \chi} + (I - \mathcal{L}_{s,a})\left[\frac{d\Psi_s^{a*}}{d\chi}\right] = 0. \tag{C.25}$$

By Lemma 5, $(I - \mathcal{L}_{s,a})^{-1}$ exists and is bounded. Applying it to both terms, we obtain

$$\frac{d\Psi_s^{a*}}{d\chi} = (I - \mathcal{L}_{s,a})^{-1}\left[\frac{\partial T_{s,a}[\Psi_s^{a*}]}{\partial \chi}\right] \quad \text{for } \chi = s, a, \tag{C.26}$$

which is identical to Eq. (26). In addition, Eqs. (31a) and (31b) are obtained difectly by partially differentiating Eq. (27) with respect to s and a, respectively, and evaluating at $\Psi = \Psi_s^{a*}$.

To summarize, the application of the implicit function theorem requires the following three conditions, which we verified hold in our setting. 1) **Continuity**: The operator $T_{s,a}$ and its partial derivatives with respect to (s, a) are continuous in all arguments. This follows from the smoothness of G_s^a and g_s^a as functions of (s, a, v). 2) **Invertibility**: The operator $D_\Psi \mathcal{F} = I - \mathcal{L}_{s,a}$ is invertible for all $(s, a) \in \Omega$. This is guaranteed by Lemma 5. 3) **Interior domain**: We require $(s, a) \in \Omega$ to ensure differentiability. At boundary points where $c(s, a) = 0$, the analysis breaks down as the ruin probability approaches 1. Under these conditions, the implicit function theorem in Banach spaces (e.g., Theorem 15.1 of Deimling (1985)) guarantees that Ψ_s^{a*} is continuously differentiable in (s, a), with derivatives given by Eqs. (31a) and (31b).

Appendix D. Proof of Property 4

We examine the shape of the objective function of the maximization problem (35). In cases where s and a are continuous variables, the first and second partial derivatives of $\mathbb{E}[B_s^a]$ with respect to s and a are derived as follows:

$$\frac{\partial \mathbb{E}[B_s^a]}{\partial s} = -\frac{\partial K(s)}{\partial s} - \frac{\partial \pi(s,a)}{\partial s} - \frac{\partial \lambda(s)}{\partial s}\mathbb{E}[X^a]$$
$$= -2\alpha s + \lambda_0 \exp(-s/\mu)\{1 + \theta \exp(-a/\mu)\}, \tag{D.1a}$$

$$\frac{\partial^2 \mathbb{E}[B_s^a]}{\partial s^2} = -2\alpha - \frac{\lambda_0}{\mu} \exp(-s/\mu)\{1 + \theta \exp(-a/\mu)\} \ < 0, \tag{D.1b}$$

$$\frac{\partial \mathbb{E}[B_s^a]}{\partial a} = -\frac{\partial \pi(s,a)}{\partial a} - \lambda(s)\frac{\partial \mathbb{E}[X^a]}{\partial a} = \theta\lambda_0 \exp\{-(s+a)/\mu\} \ > 0, \tag{D.1c}$$

$$\frac{\partial^2 \mathbb{E}[B_s^a]}{\partial a^2} = -\frac{\theta\lambda_0}{\mu} \exp\{-(s+a)/\mu\} \ < 0. \tag{D.1d}$$

Eq. (D.1b) shows that $\mathbb{E}[B_s^a]$ is a concave function of s. $\partial \mathbb{E}[B_s^a]/\partial s$ changes sign at $s^{\circ\circ}(a)$, determined by the following equation:

$$s^{\circ\circ}(a) = \frac{\lambda_0 \exp(-s^{\circ\circ}/\mu)\{1 + \theta \exp(-a/\mu)\}}{2\alpha}. \tag{D.2}$$

When $s < s^{\circ\circ}(a)$, $\mathbb{E}[B_s^a]$ increases with s because the marginal benefits, indicated by the second term in the second line of Eq. (D.1a), exceed the marginal costs, indicated by the first term in the same line. When $s > s^{\circ\circ}(a)$, $\mathbb{E}[B_s^a]$ decreases with s for the opposite reason. It is found that $\mathbb{E}[B_s^a]$ is a single-peaked function of s and has its maximum value at $s^{\circ\circ}(a)$. When the ruin probability constraint is binding, the effect shown by Eq. (26) is additionally considered to determine the optimal point of s.

Eq. (D.1c) suggests that $a = \infty$ should be chosen if the constraint condition is not binding. $a = \infty$ represents the policy without an insurance contract. This solution is caused by θ (≥ 0) that makes the insurance premium larger than (or equal to) the expected insurance benefit. However, when the ruin probability constraint becomes binding, the effects described in Property 3 are taken into account. When the marginal effect shown by Eq. (26) is positive, the insurance contract with $a < \infty$ may be selected. If the insurance is fair, namely $\theta = 0$, the insurance is neutral to the level of $\mathbb{E}[B_s^a]$.

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