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STOCHASTIC RESERVOIR THEORY: AN OUTLINE OF THE STATE OF THE ART AS UNDERSTOOD BY APPLIED PROBABILISTS

A.A. Anis

E.H. Lloyd

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Stochastic Reservoir Theory: An Outline of the State of the Art as Understood by Applied Probabilists[1]

A.A. Anis[2] and E.H. Lloyd[3]

1. Introduction

The relation between water-storage systems in the real world and the system encompassed by the basic theory (due to Moran [15,16] may be described as follows in Table 1.

At first sight this model would appear to rest on simplifications that are so drastic as to render it devoid of mathematical interest or of practical potentialities. In fact, however, this is not the case. As far as purely mathematical developments are concerned, the interest that these simplifications have aroused is amply illustrated by the so-called "Dam Theory," in which the basic Moran theory was transformed largely by Moran [14] himself and by D. G. Kendall [10] into a sophisticated corpus of pure mathematics dealing with continuous-state, continuous-time stochastic processes. (For comprehensive contemporary surveys see Gani [6], Moran [15], and Prabha [22].)

It is true that the "reservoirs" in Dam Theory are of infinite capacity, that the release occurs at unit rate, and that the "water" involved posesses no inertia and no correlation structure of any kind, so that the inflow rate at $t + \delta t$ is statistically independent of the inflow rate at time t, however small the increment δt. This lack of realism does not detract from the beauty of the mathematics involved, but it does limit the possibilities of applying the theory to the real world.

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[1]This report is an expanded version of a seminar presented by the authors at IIASA, in May 1975.

[2]Ain Shams University, Cairo, Egypt.

[3]University of Lancaster, UK.

Table 1.

	Real World Situation	Basic Moran theory
Storage mechanism:	System of interconnected reservoirs	Single reservoir
Inflow process	Continuous in state and time Seasonal Autocorrelated	Discrete in state and time Non-seasonal Independent increments
Losses	Evaporation, seepage	Ignored
Capacity	Time-dependent due to silting	Constant
Release procedure	Continuous in state and time Related to past, current and predicted future contents	Discrete in state and time Constant rate of release

A second avenue of development, which is directed specifically towards engineering applicability, is also possible. This development views the basic Moran theory as an extremely ingenious abstraction from reality, so constructed as to allow modifications which are capable of removing practically all the restraints listed in Table 1 above. In particular, seasonality in inflow and outflow processes may be accommodated; the errors involved in approximating continuity by discreteness may be reduced to an acceptable level by working in appropriate units;

flexible release rules may be built in; and, most important, the original requirement of mutual independence in the sequence of inflows may be abandoned, and realistic autocorrelation structures incorporated by using Markovian approximations of arbitrary complexity (Kaczmarek [8], Lloyd [12]).

What this development of the theory produces is the probabilistic structure of the sequence of storage levels and overspills in the reservoir -- both structures being obtained in terms of the size of the reservoir, the inflow characteristics and the release policy. (See for example Odoom and Lloyd [19], Lloyd [12], Anis and El-Naggar [2], Gani [5], Ali Khan [1], Anis and Lloyd [3], Phatarfod and Mardia [21]; and review articles by Gani [7] and Lloyd [13].) Thus the effect of varying the release policy may be determined, with a view to optimizing various performance characteristics.

The theory can therefore be said to have reached a fairly satisfactory form, and regarded as a probabilistic model. The same cannot perhaps be said of the statistical estimation and modelling procedures needed for the practical application of the theory, and since the usefulness of the theory must depend on the reliability of the numerical estimates of the probability distribution and autocorrelation structure of the inflow process, it is the authors' opinion that particular attention ought now to be concentrated on these statistical problems. This is discussed further in Section 9.

2. The Basic Moran Model: Independent Inflows

In this section we describe the simplest version of the model, as outlined by the following diagram of the reservoir (see Figure 1). The "scheduling" or "programming" required by the fact that continuous time is being approximated by discrete time is shown in Figure 2.

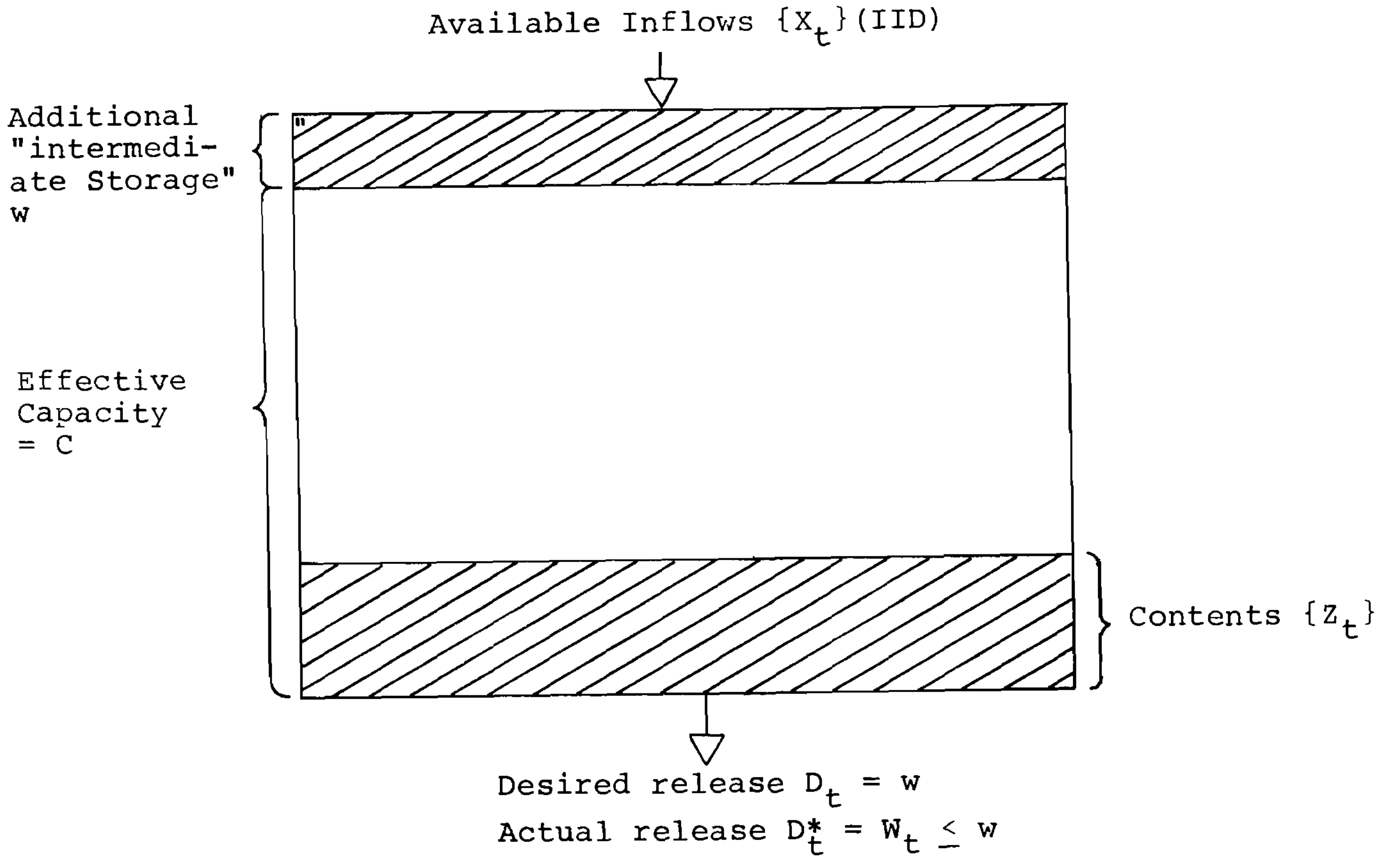

Figure 1.

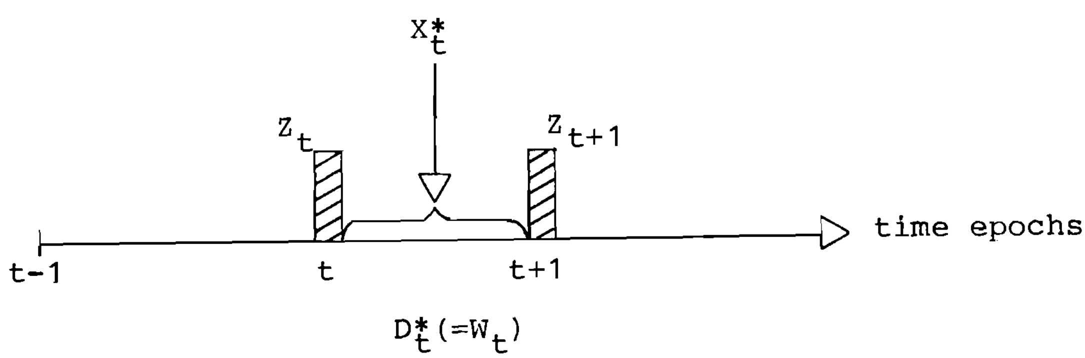

Figure 2.

Here X_t^* denotes the acceptable part of the inflow, in the light of the restraints imposed by the finite capacity of the reservoir. Similarly D_t^* (= W_t) is the feasible part of the desired outflow w, taking into account the fact that the reservoir may contain insufficient water to meet the whole of the demand.

In the Moran sequencing illustrated above, the outflow W_t is supposed to occur after the inflow has been completed. The inflow quantities $X_t = 0,1,\ldots,$ the outflow quantities $W_t = 0,1,\ldots,w$, and the storage quantities $Z_t = 0,1,\ldots,c$ are all quantized, all being expressed as integer multiples of a common unit. The "intermediate storage" indicated in the figure is required for the operation of the program.

The inflow process $\{X_t\}$ is supposed to be IID: that is, the random variables

$$\ldots,X_{t-1},X_t,X_{t+1},\ldots$$

are supposed to be mutually independent, and identically distributed.

Taking account of the finite size of the reservoir and of the sequencing imposed by the "program" we may formulate the following stochastic difference equation for the storage process $\{Z_t\}$:

$$Z_{t+1} = \min(Z_t + X_t, c + w) - \min(Z_t + X_t, w) \qquad (1)$$

which we shall abbreviate when necessary in the form

$$Z_{t+1} = h(Z_t, X_t) \quad . \qquad (2)$$

Here the constant, w, represents the <u>desired</u> outflow D_t. The <u>actual</u> outflow W_t is given by

$$W_t = \begin{cases} w & \text{if } Z_t + X_t \geq w \\ Z_t + X_t & \text{otherwise} \end{cases}$$

so that

$$W_t = \min(Z_t + X_t, w) \quad . \tag{2a}$$

The actually accepted inflow into the reservoir during $(t, t+1)$, as distinct from the available inflow X_t, is

$$X_t^* = \begin{cases} X_t \quad , & \text{if } Z_t + X_t \leq c + w \\ w + c - Z_t \quad , & \text{if } Z_t + X_t > c + w \end{cases}$$

and the quantity lost through overflow is

$$S_t = \max(Z_t + X_t - c - w, 0) \quad . \tag{3}$$

It is worth pointing out that, while presentations of this theory often concentrate on the determination of the storage process $\{Z_t\}$, both the outflow process $\{W_t\}$ and the spillage process $\{S_t\}$ are probably more important in applications, particularly when it is desired to optimize the release policy, since the function to be optimized will normally depend directly on these two processes.

(An alternative "programming," which may be described as a net inflow scheme, and which dispenses with the need to introduce an intermediate storage zone, is shown in Figure 3. Here the total acceptable inflow X_t^* that occurs during $(t, t+1)$ and the total outflow W_t are assumed to be spread over the entire interval (with suitable modifications when a boundary is reached) both taking place at constant rate during that interval, so that they combine to form a single "inflow" of magnitude $X_t - W_t$, this being negative if $X_t < W_t$. The stochastic difference equation for $\{Z_t\}$ is then

$$Z_{t+1} = \min(Z_t + X_t - w, c) - \min(Z_t + X_t - w, 0)$$

$$= \min(Z_t + X_t, w + c) - \min(Z_t + X_t, w)$$

which coincides with the equation (1) obtained for Moran's own program. Similarly the actual outflow W_t and the spillage S_t are the same as under Moran's programming.)

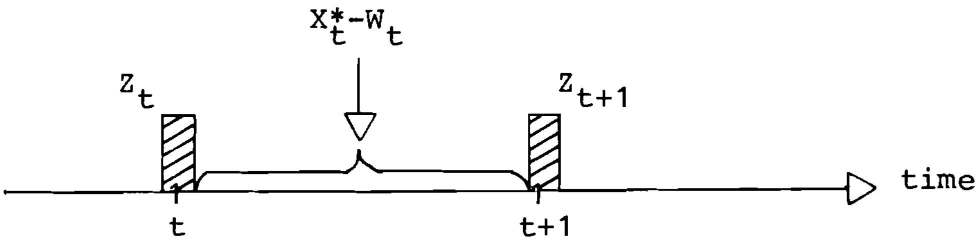

Figure 3.

3. The Storage Process $\{Z_t\}$ for the Basic Moran Model

As before, we take $\{X_t\}$ to be an IID process, and $D_t = w$. Then $\{Z_t\}$ is a lag-1 Markov Chain. For, since (by (2))

$$Z_{t+1} = h(Z_t, X_t)$$

where

$$P\{Z_{t+1} = r | Z_t = s, Z_{t-1} = s', Z_{t-2} = s'', \ldots,\}$$

$$= P\{h(s, X_t) = r | Z_t = s, Z_{t-1} = s', \ldots,\}$$

$$= P\{h(s, X_t) = r\} \quad , \tag{4}$$

the information relating to $Z_t, Z_{t-1}, \ldots,$ is suppressable on account of the assumed structure of $\{X_t\}$. Since (4) depends on s but does not depend on $s', s'', \ldots,$ the result follows. Thus

$$P(Z_{t+1} = r) = \sum_{s=0}^{c} P(Z_{t+1} = r | Z_t = s)\, P(Z_t = s) \quad ,$$

$$r = 0, 1, \ldots, c$$

or

$$\zeta_{t+1}(r) = \sum_s q(r,s)\zeta_t(s)$$

where

$$\zeta_t(s) = P(z_t = s) \quad ,$$

and

$$q(r,s) = P(Z_{t+1} = r | Z_t = s) \quad , \quad \text{for } r,s = 0,1,\ldots,c \quad .$$

In an obvious matrix notation this may be written

$$\underline{\zeta}_{t+1} = \underline{Q}\zeta_t \quad , \qquad t = 0,1,\ldots$$

whence

$$\zeta_t = \underline{Q}^t\zeta_o \quad . \tag{5}$$

Thus the distribution vector $\underline{\zeta}_t$ of storage at time t is determined in terms of the initial conditions $\underline{\zeta}_o$. In all "realistic" situations the transition matrix $\underline{Q}$ may be assumed to be ergodic, whence, for sufficiently large values of t, $\underline{\zeta}_t \simeq \underline{\zeta}$ where

$$\underline{\zeta} = \lim_{t\to\infty} \underline{\zeta}_t \quad , \tag{6}$$

$\underline{\zeta}$ being the "equilibrium distribution" vector which is the unique positive normalized solution of the homogeneous linear algebraic system

$$(\underline{Q} - \underline{I})\ \underline{\zeta} = \underline{0} \quad . \tag{7}$$

The matrix $\underline{Q}$, of order $(c+1) \times (c+1)$, has as its (r,s) element

$$q(r,s) = P(Z_{t+1} = r | Z_t = s)$$

$$= P\{h(s,X_t) = r\} \quad ,$$

and for any given function $h(\cdot)$, this is easily obtained in terms of the inflow distribution $P(X_t = j) = f(j)$, say, $j = 0,1,\ldots,2$.

As a simple example, taking $w = 1$, we have:

$$q(0,0) = f(0) + f(1) \quad , \quad q(0,1) = f(0) \quad ,$$

$$q(r,s) = \begin{cases} f(r + 1-s) & , \quad s = 0,1,\ldots,r+1 \\ 0 & , \quad s = r+2,\ldots,c \end{cases}$$

$$(\text{for } r = 1,2,\ldots,c)$$

and

$$q(c,s) = f(c+1-s) + f(c+2-s) + f(c+3-s),\ldots,$$

$$s = 0,1,\ldots,c \quad . \qquad (8)$$

4. <u>The Yield Process in the Basic Moran Model</u>

The yield is the actual quantity released from the reservoir, defined by (2) as

$$W_t = \min(Z_t + X_t, w) \qquad (9)$$

$$= g(Z_t, X_t) \quad , \text{ say} \quad . \qquad (10)$$

Because of a) the "lumping" of Z-states implied by this function, and b) the addition process $Z_t + X_t$, the yield process $\{W_t\}$ is not Markovian. For most purposes it is best studied as a function defined on the Markov Chain $\{Z_t\}$, with conditional probabilities

$$P(W_t = r \mid Z_t = s) = P\{g(s,X_t) - r\} \quad . \tag{11}$$

This is, for each r and s, a well-defined function of the distribution of X_t, which allows us to evaluate (for example) the expected yield at time t as

$$E(W_t) = \sum_r rP\{W_t = r) = \sum_r \sum_s rP(W_t = r \mid Z_t = s) \; P(Z_t = s)$$

$$= \sum_r \sum_s rP\{g(s,X_t) = r\} \; P(Z_t = s) \quad .$$

For other purposes it may be convenient to use the fact that the pair $\{Z_t, X_t\}$ forms a bivariate lag-1 Markov Chain.

5. The Basic Moran Model with Flexible Release Policy

If one modifies the release policy D_t so that, instead of being a fixed constant w, D_t is, for example, a function of the current values of Z and of X--say a monotone non-decreasing function of $Z_t + X_t$--the effect of this on the theory outlined in Section 5 is merely to modify the transition matrix $\underline{Q}$, without altering the Markovian structure of $\{Z_t\}$. Equation (1), whether under Moran programming or net-inflow programming, is replaced by

$$Z_{t+1} = \min\,(Z_t + X_t, c + D_t) - \min\,(Z_t + X_t, D_t) \tag{12}$$

$$= g(Z_t, X_t, D_t) \quad ,$$

say. This still holds good, and Z_t maintains its simple

Markovian character, if D_t also depends on some additional random variable Y_t which may be correlated with X_t, provided that the sequence Y_t consists of mutually independent elements.

The actual outflow W_t is given by

$$W_t = \min (Z_t + X_t, D_t) \quad , \tag{13}$$

the analogue of (2), and the spillage becomes

$$S_t = \max (Z_t + X_t - D_t - c, 0) \quad . \tag{14}$$

If for example D_t is defined by the following Table 2:

Table 2.

$Z_t + X_t$	≤ 2	3	4	5	≥ 6
D_t	0	1	2	3	4

we may construct Table 3 with entries such as the following (for, say, c = 8) as found in Table 3 below. Row (a) indicates a situation giving no spillage, (b) one where five units are spilled, (c) one where several combinations of Z_t and X_t lead to the same value of Z_{t+1}, in which case we have

$$q(2,0) = P(Z_{t+1} = 2 | Z_t = 0) = P(X_t = 2 \ \underline{\text{or}}\ 3 \ \underline{\text{or}} \cdots \underline{\text{or}}\ 6)$$

$$= f(2) + f(3) + \cdots + f(6) \quad ,$$

whereas q(0,0) = f(0), a single term.

6. The Basic Moran Model Operating Seasonally

The effect of working with a multi-season year may be adequately illustrated in terms of a two-season year (See Table 4):

Table 3.

	Z_t	X_t	D_t	Z_{t+1}	W_t	S_t
(a)	4	2	4	2	4	0
(b)	5	12	4	8	4	5
. . .	. . .	. . .	. . .	. . .	. . .	. . .
(c)	0	2	0	2	0	0
(c)	0	3	1	2	1	0
(c)	0	4	2	2	2	0
(c)	0	5	3	2	3	0
(c)	0	6	4	2	4	0
. . .	. . .	. . .	. . .	. . .	. . .	. . .

Table 4.

	Year t		Year t + 1		Year t + 2	
	Season 0	Season 1	Season 0	Season 1	- - -	
Epoch:	(t,0)	(t,1)	(t,2) = (t + 1,0)	(t + 1,1)	(t + 1,2) = (t + 2,0)	
Storage distrib. vectors	$\underline{\zeta}(t,0)$	$\zeta(t,1)$	$\underline{\zeta}(t + 1,0)$	$\zeta(t + 1,1)$	$\underline{\zeta}(t + 2,0)$	...
Transition matrix	$\underline{Q}_0$	$\underline{Q}_1$	$\underline{Q}_0$	$\underline{Q}_1$		...

Here

$$\underline{\zeta}(t+1,1) = \underline{Q}_o\, \underline{\zeta}(t+1,0)$$

$$= \underline{Q}_o\, \underline{Q}_1\, \underline{\zeta}(t,1)$$

or, say,

$$\underline{z}(t+1) = Q\underline{z}(t)$$

where $Q = Q_oQ_1$, and $\underline{z}(t)$ is the storage distribution vector at the beginning of season 1 of year t. Clearly the year-to-year storage sequence at this season is a homogeneous Markov Chain, with transition matrix $Q = Q_oQ_1$, and the preceding theory applies.

7. Reservoir Theory with Correlated Inflows: Basic Version

Moran's basic theory is applicable as a first approximation, more or less without modification, to a reservoir providing year-to-year storage, in which there is a well-defined "inflow season" with no outflow, followed by a well-defined and relatively short "outflow interval," a situation approximated by the conditions on the Nile at Aswan. This approximation holds only to the extent that inter-year inflow correlations can be neglected, which may not be totally unreasonable for a one-year time scale. The approximation becomes increasingly inaccurate if one reduced the working interval from a year to a quarter, or a month, or less, and it becomes essential to provide a theory that allows the inflow process to have an autocorrelation structure. It was pointed out simultaneously in independent publications in 1963 by Kaczmarek [8] and by Lloyd that this could be done by approximating the actual inflow process by a Markov Chain.

In the simplest form of this theory we may consider a discrete-state/discrete-time reservoir, with stationary inflow process $\{X_t\}$, where X_t is a finite homogeneous lag-1

Markov Chain with ergodic transition matrix $B = (b_{rs})$ say, where $b_{rs} = P(X_{t+1} = r | X_t = s)$, with $r,s = 0,1,\ldots,n$ (X_t being assumed $\leq n$ for all t) and the distribution vector of X_t is $\underline{\xi}$, where $\underline{\xi}$ is the unique non-negative normalized solution of the homogeneous system

$$(\underline{B} - \underline{I})\ \underline{\xi} = \underline{0} \quad .$$

With a release policy D_t which may be a function of Z_t, Z_{t-1}, X_t and X_{t-1}, this stochastic difference equation for $\{Z_t\}$ is the same as in (12), with the supplementary information that $\{X_t\}$ is a Markov Chain. It is easy to show, by an argument entirely analogous to that employed in Section 3, that $\{Z_t\}$ is no longer a Markov Chain, but <u>the ordered pair $\{Z_t, X_t\}$ forms a bivariate lag-1 Markov Chain</u>, that is, that

$$P(Z_{t+1} = r, X_{t+1} = s | Z_t = i, X_t = j, Z_{t-1} = i', X_{t-1} = j', \ldots,)$$

is independent of i',j',i",j",...,. The vector:

$$\underline{\pi}_t = \begin{pmatrix} \underline{\pi}_t(0) \\ \underline{\pi}_t(1) \\ \cdot \\ \cdot \\ \cdot \\ \underline{\pi}_t(c) \end{pmatrix} \tag{15}$$

where

$$\underline{\pi}_t(r) = \begin{pmatrix} P[Z_t = r, X_t = 0] \\ P[Z_t = r, X_t = 1] \\ \cdot \\ \cdot \\ \cdot \\ P[S_t = r, X_t = n] \end{pmatrix} \qquad r = 0,1,\ldots,c \ , \tag{16}$$

represents the joint distribution vector of Z_t and X_t, and this is determined by a vector equation of the form

$$\underline{\pi}_{t+1} = \underline{M}\underline{\pi}_t \tag{17}$$

where $\underline{M}$ is the relevant transition matrix, of order $(c + 1)(n + 1) \times (c + 1)(n + 1)$. The structure of $\underline{M}$ is obtained from the stochastic equation for $\{Z_t\}$ and the inflow transition matrix. If for example we take the simple release policy $D_t = 1$, the equation (17) in partitioned form becomes

$$\pi_{t+1}(r) = \sum_{s=0}^{c} \underline{M}(r,s)\underline{\pi}_t(s) \quad , \tag{18}$$

where the $\underline{M}(r,s)$ are submatrices of order $(n + 1) \times (n + 1)$, which may most easily be defined in terms of the following representation of the inflow transition matrix B. Let

$$B = (\underline{b}_o, \underline{b}_1, \ldots, \underline{b}_n)$$

where $\underline{b}_s$ represents the s-column of $\underline{B}$, that is

$$\underline{b}_s = (b_{os}, b_{1s}, \ldots, b_{ns})', \quad s = 0,1,\ldots,n,$$

and let

$$\begin{aligned} B_o &= (\underline{b}_o, \underline{0}, \underline{0}, \ldots, \underline{0}) \quad , \\ \underline{B}_1 &= (\underline{0}, \underline{b}_1, \underline{0}, \ldots, \underline{0}) \quad , \\ &\cdot \;\; \cdot \;\; \cdot \;\; \cdot \\ \underline{B}_n &= (\underline{0}, \underline{0}, \ldots, \underline{0}, b_n) \quad . \end{aligned} \tag{19}$$

Then, in formal agreement with (8), we find

$$\underline{M}(0,0) = B_0 + B_1 \quad , \quad \underline{M}(0,1) = B_0 \quad ,$$

$$\underline{M}(r,s) = \begin{cases} B_{r+1-s} & , \quad s = 0,1,\ldots,r+1 \\ 0 & , \quad s = r,2,\ldots,c \end{cases} \tag{20}$$

$$\text{for } r = 1,2,\ldots,c$$

and

$$\underline{M}(c,s) = B_{c+1-s} + B_{c+2-s} + \cdots + B_n ,$$

$$s = 0,1,\ldots,c \quad .$$

Equation (17) has as its solution:

$$\underline{\pi}_t = M^t \underline{\pi}_o \quad , \quad t = 1,2,\ldots, \tag{21}$$

giving the joint distribution vector of Z_t and X_t in terms of the initial vector $\underline{\pi}_o$, this being the analogue for Markovian inflows of the equation (5) for independent inflows. The analogue of (6) is given by the statement that, for large values of t, $\underline{\pi}_t \cong \underline{\pi}$, where $\underline{\pi}$ is the "joint equilibrium distribution" vector which is the unique normalized non-negative solution to the homogeneous system

$$(M-I)\ \underline{\pi} = 0 \quad .$$

One extracts the distribution Z_t from the joint distribution by using the result that

$$P(Z_t = r) = \sum_{s=o}^{n} P(Z_t = r, X_t = s) \quad , \quad r = 0,1,\ldots,c$$

where, for given r, the terms $P(Z_t = r, X_t = s)$ are elements of the vector $\underset{\sim}{\pi}(r)$ of (16), this in turn being a subvector of the vector $\underline{\pi}_t$ given by (21).

8. Reservoir Theory with Correlated Inflows: Elaborations

The theory described in Section 7 refers to a stationary lag-1 Markovian inflow process, and a constant-value release policy. The replacement of a constant release by a release D_t depending on Z_t, X_t, Z_{t-1} and X_{t-1} and possibly further random elements Y_t and Y_{t-1} is achieved by suitably modifying the matrix M. This does not alter the structure of the $\{Z_t, X_t\}$ process. (As has been pointed out by Kaczmarek [9] the storage process $\{Z_t\}$ becomes a lag-2 Markov chain provided that the equation (12) can be solved to give a unique value of X_t for each pair (Z_t, Z_{t+1}).)

Further, the introduction of seasonally varying inflow and outflow processes may be achieved by using the matrix product technique described in Section 6. As in the case of independent inflows, once one has obtained the distribution of Z_t one may obtain that of the actual release W_t and the spillage S_t, and utilize these if desired in optimization studies.

We have spoken so far of a lag-1 Markovian inflow. Generalizations to a multi-lag Markovian inflow are immediate: with, for example, a lag-2 Markov inflow one considers the three-vector $\underline{\eta}(t) = \{Z_t, X_t, X_{t-1}\}$. This will be a lag-2 (trivariate) Markov chain. Similarly we may accommodate a <u>multivariate</u> inflow process. Suppose for example we have a bivariate inflow process $\{X_t^{(1)}, X_t^{(2)}\}$, in which the components are cross-correlated as well as serially correlated. This inflow process will be specified by an appropriate transition matrix whose elements represent the conditional probabilities

$$P\{X_{t+1}^{(1)} = i, X_{t+1}^{(2)} = j \mid X_t^{(1)} = r, X_t^{(2)} = s\} \quad ,$$

using any suitable ordering convention, for example the following Table 5.

Table 5.

	$X^{(1)}$		Values at time t 0	1	...	n
		$X^{(2)}$	0 1...n	0 1...n	...	0 1...n
value at time t + 1	0	0 1 . . . n				
	1	0 . . . n				
	. . .					
	n	0 1 . . . n				

In this example the triplet $\{Z_t, X_t^{(1)}, X_t^{(2)}\}$ would be a (trivariate) lag-1 Markov chain (Lloyd [11], Anis and Lloyd [4].

9. Closing Remarks: Perspectives

In the authors' opinion, the probabilistic framework provided by the model described above is adequate for practical purposes, and the next steps ought to be concerned with the statistical problems of specifying families of few-parameter inflow transition matrix models and estimating their parameters. In the case of a nonseasonal univariate lag-1 inflow process, for example, it is likely that the available information will be best adapted to estimating the inflow distribution vector $\underline{p}$ (which is of course a standard and well-understood statistical procedure) and the first few autocorrelation coefficients, say ρ_1 and ρ_2. A model involving these directly would be particularly welcome. In the case of where an exponential autocorrelation function

$$\rho_k = \rho_1^k \quad , \quad k = 1,2,\ldots,$$

were thought to be appropriate, the Pegran [20] transition model

$$\underline{B} = \rho_1 \underline{I} + (1 - \rho_1)\ \underline{\xi}\ \underline{1}' \quad ,$$

that is

$$B(r,s) = \rho_1 \delta(r,s) + (1 - \rho_1)\ \xi_r$$

where

$$B(r,s) = P(X_{t+1} = r | X_t = s)$$

and

$$\delta(r,s) = \begin{cases} 1, & r = s \\ 0, & \text{otherwise} \end{cases}$$

satisfies these requirements. Further development along these lines, making m a discrete-state and discrete-time framework, and avoiding the difficulties inherent in the use of transformations of the normal autoregressive model, (Moran, [17]), would be highly desirable, particularly in the case of multivariate inflows.

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