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LOCAL CONTROLLABILITY AND INFINITESIMAL GENERATORS OF SEMIGROUPS OF SET-VALUED MAPS

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September 1984
WP-84-70

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PREFACE

The concepts of tangent cones and derivatives of set-valued maps are used in this very important paper to

1. Show that any continuous set-valued map with compact convex values is the infinitesimal generator of a set-valued semigroup

2. Extend to differential inclusions the "variation equation" governing the evolution of the derivative of the solution set with respect to the initial conditions

3. Prove a generalization of the open mapping principle for maps defined on closed subsets of infinite-dimensional spaces

These results are then used to prove the local controllability of a differential inclusion around an equilibrium point.

It is shown that if a "linearized" differential inclusion (defined in some appropriate way) is controllable, then the initial differential inclusion is locally controllable. The method used does not involve algebraic or geometrical techniques, but only nonsmooth analysis.

This research was conducted within the framework of the Dynamics of Macrosystems study in the System and Decision Sciences Program.

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ABSTRACT

We study the semigroup properties of reachable sets of a differential inclusion

$$x' \in F(x)$$

using the derivative of the set-valued map which associates to each initial state the set of solutions. The results are applied to the local controllability problem.

Key words: differential inclusion, local controllability, reachable set, derivative of solution with respect to initial condition, semigroup properties of reachable sets, derivative of a set-valued map, generalized tangent cone.

CONTENTS

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CEREMADE

1. INTRODUCTION

In this paper we study the problem of local controllability of a system governed by a differential inclusion

(1) $\quad x' \in F(x)$

where F is a set-valued map from $\mathbb{R}^n$ into the subsets of $\mathbb{R}^n$. A particular case of (1) is the parametrized system (also called "control system")

$$x' = f(x,u(t)) \quad , \quad u(t) \in U$$

where U is a given set ; then F is defined by

$$F(x) = \{f(x,u) : u \in U\}$$

Let $\xi \in \mathbb{R}^n$, $T > 0$ be given. Denote by $S_T(\xi)$ the set of solutions to (1) issued from ξ and defined on the time interval $[0,T]$. The reachable set to (1) at time T from ξ is denoted by $R(T,\xi)$, i.e.

$$R(T,\xi) = \{x(T) : x \in S_T(\xi)\}$$

The system (1) is called locally controllable around ξ at time $T > 0$ if

(2) $\qquad \xi \in \text{Int}\, R(T,\xi)$

Under quite general assumptions (boundness and upper semicontinuity of F) the necessary condition for local controllability of (1) at ξ for small $T > 0$ is

(3) $\qquad 0 \in \overline{\text{co}}F(\xi)$

(the closed convex hull of $F(\xi)$), i.e. ξ has to be a weak equilibrium of the map F.

The purpose of this paper is to provide a sufficient condition for (2) when ξ is an equilibrium or a weak equilibrium of F.

We use the techniques of nonsmooth analysis and differential inclusions to answer this question in the following way:

We consider an adequate concept of tangent cone $C^{U,V}_{S_T(\xi)}(\xi)$ to the set of solutions $S_T(\xi)$ at the constant trajectory ξ and prove a kind of an Open Mapping Principle, which states that (2) follows from the "surjectivity" condition

(4) $\qquad \{w(T) : w \in C^{U,V}_{S_T(\xi)}(\xi)\} = \mathbb{R}^n$

(we specify in Section 3 the definition of $C^{U,V}_{S_T(\xi)}(\xi)$ and prove an abstract version of (4)).

How can (4) be verified? In the case when ξ is an equilibrium we proceed in the following way:

Under a Lipschitzeanity assumption, the differential inclusion (1) may be replaced by a "linear" approximation (along the solutions) around the equilibrium and we prove that if the linearized system is locally controllable at zero so does the initial system.

We have to explain now what do we mean by approximating a set-valued map. Set-valued analogues of linear operators are closed processes, i.e. set-valued maps whose graph are closed cones. Then the linearized inclusion along a trajectory $x \in S_T(\xi)$ is given by

$$(5) \qquad \begin{cases} w' \in dF(x(t),x'(t))(w) \\ w(0) = 0 \end{cases}$$

(where $dF(x(t),x'(t))$ is the intermediate derivative[1] of F at $(x(t),x'(t)))$. Then (3) follows from the more explicit condition

$$(6) \qquad \{w(T) : \liminf_{x \to \xi} (w \in W^{1,1}(0,T) : w'(t) \in dF(x(t),x'(t))(w(t)) ; \; w(0) = 0)\} = \mathbb{R}^n$$

which is nothing other than the regularized local controllability of the linear system (5).

Condition (4) can also be studied when ξ is an equilibrium in the following way: we consider the set-valued derivative $CF(\xi,0)$ associated with Clarke's tangent cone to graph of F at $(\xi,0)$ and the tangent cone (of convex analysis) $T_{coF(\xi)}(0)$ to the convex hull $coF(\xi)$ of $F(\xi)$ at zero. We prove that if F is Lipschitzean, then (4) holds when the reachable set to inclusion

$$\begin{cases} w' \in CF(\xi,0)(w) + T_{coF(\xi)}(0) \\ w(0) = 0 \end{cases}$$

at time T is the whole space $\mathbb{R}^n$.

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(1) Recall that the graph of the derivative of a differential map f at point a is the tangent space to the graph of f at the point $(a,f(a))$. Hence a natural way to extend the concept of derivative of a set-valued map F at a point $(a,b) \in$ graph (F) is to use the set-valued map $DF(a,b)$ whose graph is a tangent cone to graph (F) at (a,b). (In this paper we use Clarke's and intermediate tangent cone : see Section 2 for a precise definition.

The same techniques provide a sufficient condition for a solution x to the differential inclusion

$$x' \in F(t,x) \quad , \quad x(0) = x_0$$

to have $x(T)$ in the interior of the reachable set at some time $T > 0$. In particular, we derive from it the maximum principle of Pontriagin.

We proceed by investigating the analogies with the single-valued smooth case when the differential equation

$$(7) \qquad x' = f(x) \quad , \quad x(0) = \xi$$

has a unique solution, which can be written $r(t,\xi)$. The maps $r(t,\xi)$ form a semigroup in the sense that

$$r(0,\xi) = \xi \quad , \quad r(t+s,\xi) = r(t,r(s,\xi))$$

and f is the <u>infinitesimal generator</u> of this semigroup in the sense that

$$(9) \qquad f(\xi) = \frac{\partial}{\partial t} r(t,\xi)\big|_{t=0}$$

Furthermore, the derivative of $r(t,\cdot)$ with respect to the initial condition is given by the formula

$$(10) \qquad \frac{\partial}{\partial \xi} r(t,\xi)\cdot\eta = w(t)$$

where w is the solution to the linearized differential equation

$$(11) \qquad w'(t) = f'(r(t,\xi))w(t) \quad , \quad w(0) = \eta$$

This still holds true in the set-valued case (we have to replace the derivatives by intermediate derivatives for the statements below to hold true). We shall prove essentially that

a) the reachable sets $R(t,\xi)$ have the semigroup property.

b) the set-valued map F is the infinitesimal generator of $R(t,\cdot)$: $F(\xi)$ is the derivative of $t \to R(t,\xi)$ at $(0,\xi)$ in the direction 1.

c) the "derivative" of $\xi \to S_T(\xi)$ at a solution $x(\cdot)$ of (1) in the direction η is the set of solutions of

$$\text{(12)} \quad \begin{cases} w'(t) = dF(x(t),x'(t))(w(t)) \\ w(0) = \eta \end{cases}$$

This is a property which is the key to solve the local controllability problem in a straightforward way. This motivates not only the title of this paper but also provides some new properties of the reachable sets and the sets of solutions to differential inclusions studied by many authors (see for example Aubin-Cellina [2], Hermes [20], Blagodatskich [5], Haddad [19], Castaing-Valadier [9], Olech [22]).

We devote the second section to the study of the semigroup properties of the reachable sets. Section 3 deals with the controllability problem and the Open Mapping Principle. In Section 4 we provide several applications. Section 5 is devoted to the time dependent case and to the derivation of the maximum principle of Pontriagin.

The author wishes to thank St. Łojasiewicz (Jr.) for useful comments and an example stated in Section 4.

2. SEMIGROUP PROPERTIES OF THE REACHABLE SETS

Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a set-valued map. A function $x \in W^{1,1}(0,T)$ (Sobolev space) is called a solution to the differential inclusion

$$\text{(1)} \quad x' \in F(x)$$

if and only if $x'(t) \in F(x(t))$ almost everywhere.

We denote by $S_T(\xi)$ the set of all solutions to the differential inclusion (1) defined on the time interval $[0,T]$ issued from ξ.

Existence theorems imply the nonemptiness of $S_T(\xi)$ for small $T > 0$ under several combinations of assumptions (see for instance, Aubin-Cellina [2] Chapters 2 and 4, Antosiewicz-Cellina [1], Bressan [8], Olech [22]).

(2.1) Definition: The set
$R(t,\xi) := S_T(\xi)(t) = \{x(t) : x \in S_T(\xi)\}$

where $t \in [0,T]$ is called the reachable set at time t from the initial condition ξ. ▲

The reachable sets satisfy the semigroup property. Namely if $R(t,\xi) \neq \emptyset$ for all $t \in [0,T]$ then

$$(2.2) \qquad \begin{cases} \text{(i)} & R(0,\xi) = \xi \\ \text{(ii)} & R(t+s,\xi) = R(t,R(s)\xi) \quad \text{if } t+s \leq T \end{cases}$$

We wish next to define the derivative of the map $t \to R(t,\xi)$ at $t = 0$ in the direction 1. By analogy with the single-valued case, this derivative will be called the infinitesimal generator of the semigroup.

(2.3) Definition: Let E,F be Banach spaces and $G : E \rightrightarrows F$ be a set-valued map Lipschitzean at $x,y \in G(x)$. By $dG(x,y)$ we denote the set-valued map from E into F defined by

$v \in dG(x,y)(u)$ if and only if

$$\lim_{h \to 0+} \text{dist}\left(v, \frac{G(x+hu) - y}{h}\right) = 0$$

The map $dG(x,y)$ is called the intermediate derivative of G at (x,y). ▲

(2.4) Remark: We assume in the definition 2.3 that whenever $G(x+hu) = \emptyset$ then $\text{dist}\left(v , \frac{G(x+hu)-y}{h}\right) = +\infty$.

(2.5) <u>Remark</u>: The graph of the map $dG(x,y)$ is the so-called intermediate tangent cone to graph(G) at (x,y) (see Frankowska [16], where the definition of intermediate tangent cone to an arbitrary set is given).

(2.6) <u>Definition</u>: Let $R(t,\cdot)$ be a family of set-valued maps satisfying the semigroup properties (2.2), Lipschitzean in t. We say that the map

$$\xi \to d_t R(0,\xi)\ (1)$$

from $\mathbb{R}^n$ into itself is the <u>infinitesimal generator</u> of the semigroup R. ▲

As for the single-valued case the question arises whether a given set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the infinitesimal generator of the semigroup R. The answer is positive when F is continuous bounded with compact convex values. We denote by $\overset{0}{B}(B)$ the open (respectively closed) unit ball.

(2.7) <u>Theorem</u>: Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a bounded set-valued map with compact values and $\xi \in \text{Int Dom}(F)$. Then

(i) If F is upper semicontinuous and $F(x)$ is convex for all x then

$$d_t R(0,\xi)(1) \subset F(\xi)$$

(ii) If F is continuous, bounded, then

$$F(\xi) \subset d_t R(0,\xi)(1).$$ ▲

<u>Proof</u>: If $v \in d_t R(0,\xi)(1)$ for all $h > 0$ there exists $x_h \in S_h(\xi)$ such that $x_h(h) \in R(h,\xi)$ and $\lim_{h \to 0+} (x_h(h) - \xi)/h = v$.
Since F is bounded for some $M > 0$ and all $h > 0$ we have:

$$|x_h(t) - \xi| \le tM \qquad \text{for} \quad t \in [0,h]$$

By the upper semicontinuity of F and the above inequality, for all $\varepsilon > 0$ there exists $h_o > 0$ such that $h \in]0,h_o]$ implies

$$F(x_h(t)) \subset F(\xi) + \varepsilon B \quad \text{for all} \quad t \subset [0,h]$$

Thus

$$\frac{1}{h}(x_h(h) - \xi) = \frac{1}{h} \int_o^h x_h'(t)dt$$

$$\in \frac{1}{h} \int_o^h (F(\xi)+\varepsilon B)dt$$

Since $F(\xi)+\varepsilon B$ is a closed convex subset the mean-value theorem (see for instance Aubin-Cellina [2] p.21) implies that

$$\frac{x_h(h) - \xi}{h} \in F(\xi) + \varepsilon B$$

and thus that $v \in F(\xi) + \varepsilon B$.

Since ε is arbitrary we have proved (i).

(ii) By Theorem of Filippov (see Filippov [15] and Aubin-Cellina [2] p.112) for all $\xi \in \text{Int Dom}(F)$ and $v \in F(\xi)$ there exists $T > 0$ and $x \in S_T(\xi)$ satisfying $x'(0) = v$. Therefore, the sequence $\frac{x(h) - \xi}{h}$ converges to v when $h \to 0_+$. Hence, $v \in d_tR(0,\xi)(1)$. ∎

(2.8) <u>Corollary</u>: If F is continuous, bounded with closed convex values and $\xi \in \text{Int Dom}(F)$ then

$$d_tR(0,\xi)(1) = F(\xi).$$ ▲

As in the case of ordinary differential equations, we need to study the differentiability of the solution map with respect to initial conditions.

Consider the solution map $S_T : \mathbb{R}^n \rightrightarrows W^{1,1}(0,T)$ and the intermediate derivative $dS_T(\xi,z)(\eta)$ of S_T at point (ξ,z) in the direction η.

(2.9) <u>Theorem</u>: Assume that F has closed graph and choose $z \in S_T(\xi)$. If the map F is Lipschitzean on an open neighborhood of $z([0,T])$, then

$$dS_T(\xi,z)(\eta) = \{w \in W^{1,1}(0,T) \; : \; w'(t) \in dF(z(t),z'(t))w(t), \; w(0) = \eta\}$$

▲

To prove the Theorem we need the following :

(2.10) <u>Lemma</u>: Let F be a set-valued map of closed graph and measurable functions

$$[0,T] \ni t \to \alpha_i(t) = (x_i(t),y_i(t)) \in \text{graph } F$$

$$[0,T] \ni t \to \phi(t) = (u(t),v(t) \in \mathbb{R}^n \times \mathbb{R}^n$$

be given. We assume that u is continuous and for a sequence $h_i \to 0+$ the following holds true

$$\lim_{i\to\infty} \operatorname{dist}(\phi(t), \frac{\text{graph}(F) - \alpha_i(t)}{h_i}) = 0 \tag{2.11}$$

If F is L-Lipschitzean around $\bigcup_i x_i([0,T])$ then there exist measurable functions

$$[0,T] \ni t \to \phi_i(t) = (u_i(t),v_i(t)) \in \mathbb{R}^n \times \mathbb{R}^n$$

satisfying

$$\left\{\begin{array}{l} \alpha_i(t) + h_i\phi_i(t) \in \text{graph } F \quad \text{a.e.} \\ u_i \text{ converge uniformly to } u \\ v_i \text{ converge to } v \text{ almost everywhere} \\ \|v_i(t)\| \leq \|v(t)\| + L\|u(t)\| + 1 \end{array}\right. \tag{2.12}$$

▲

Proof: For all natural number k set

$$V_{i,k}(t) = \frac{\text{graph } F - \alpha_i(t)}{h_i} \cap (\phi(t) + \tfrac{1}{k}B)$$

$$U_{i,k}(t) = u(t) \times \left(\frac{F(x_i(t) + h_i u(t)) - y_i(t)}{h_i} \cap L\|u(t)\|B\right)$$

$$M_{i,k} = \{t \in [0,T] : V_{i,k}(t) \neq \emptyset\}$$

$$W_{i,k}(t) = \begin{cases} V_{i,k}(t) & \text{if } t \in M_{i,k} \\ U_{i,k}(t) & \text{otherwise} \end{cases}$$

Since the graph of F is closed the set-valued map

$$[0,T] \ni t \to W_{i,k}(t)$$

is measurable for all i,k. Moreover, for all k and all $t \in [0,T]$ $V_{i,k}(t) \neq \emptyset$ when i is sufficiently large.

Fix k. By a measurable selection theorem (see for instance, Wagner [29]) for all large i there exists a measurable selection

$$\phi_{i,k}(t) = (u_{i,k}(t), v_{i,k}(t)) \in W_{i,k}(t)$$

From the definition of $W_{i,k}(t)$ we obtain

$$\|\phi_{i,k}(t) - \phi(t)\| \leq 1/k \text{ if } t \in M_{i,k}$$

$$\left.\begin{array}{l} u_{i,k}(t) = u(t) \\ \|v_{i,k}(t)\| \leq L\|u(t)\| \end{array}\right\} \text{ otherwise}$$

Observe that for all t, i, k

$$\alpha_i(t) + h_i \phi_{i,k}(t) \in \text{graph } F$$

$$\|u_{i,k}(t) - u(t)\| \leq 1/k$$

$$\|v_{i,k}(t)\| \leq \|v(t)\| + \frac{1}{k} + L\|u(t)\|$$

Let $i_o = 0$. Using the induction arguments and (2.11) we define for all $k \geq 1$ numbers $i_k > i_{k-1}$ such that for all $i > i_k$ the Lebesgue measure

$$\mu(M_{i,k+1}) \geq T - 1/k$$

and set for all $i_{k-1} < i \leq i_k, t \in [0,T]$

$$\phi_i(t) = \phi_{i,k}(t)$$

It is clear that ϕ_i satisfy all the requirements of our lemma. ∎

Proof of Theorem 2.9: If $w \in dS_T(\xi,z)(\eta)$ then for all $h > 0$ there exist $w_h \in W^{1,1}(0,T)$ such that

$$z + hw_h \in S_T(\xi)$$

$$\lim_{h\to\infty} w_h = w \quad \text{in} \quad W^{1,1}(0,T)$$

Thus

(2.13) $$z'(t) + hw_h'(t) \in F(z(t) + hw_h(t)) \quad \text{a.e.}$$

and

(2.14) $$\lim_{h\to 0+} w_h = w \quad \text{in} \quad C([0,T])$$

(2.15) $$\lim_{h\to 0+} w_h'(t) = w'(t) \quad \text{a.e. in } [0,T]$$

Let L denote the Lipschitz constant of F. Then by (2.14), Lipschitzeanity of F and (2.13) for almost all t and for all small $h > 0$ we have

$$\operatorname{dist}\left(w'(t), \frac{F(z(t)+hw(t))-z'(t)}{h}\right) \leq$$

$$\operatorname{dist}\left(w_h'(t), \frac{F(z(t)+hw_h(t))-z'(t)}{h}\right) +$$

$$\|w_h'(t) - w'(t)\| + L\|w_h(t) - w(t)\| =$$

$$\|w_h'(t) - w'(t)\| + L\|w_h(t) - w(t)\|$$

This and (2.14), (2.15) imply

$$w'(t) \in dF(z(t),z'(t))w(t) \qquad \text{a.e.}$$

i.e.

$$dS_T(\xi,z)(\eta) \subset \{w \in W^{1,1}(0,T): w'(t) \in dF(z(t),z'(t))w(t), \; w(0) = \eta\}$$

To have the equality in the last inclusion we shall show that if

$$\begin{cases} w'(t) \in dF(z(t),z'(t))w(t) & \text{for almost all } t \in [0,T] \\ w(0) = \eta \end{cases}$$

then for all sequence $h_i > 0$ converging to zero there exists a sequence w_i converging to w in $W^{1,1}(0,T)$ such that $z+h_iw_i \in S_T(\xi+h_iw_i(0))$ and $w_i(0) = \eta$. Consider a sequence $h_i > 0$ converging to zero. By Lemma 2.10 applied to $x_i = z, y_i = z', u=w, v=w'$ there exists a sequence of measurable functions u_i converging uniformly to w and v_i converging to w' almost everywhere and such that

$$(2.16) \qquad \begin{cases} z'(t) + h_iv_i(t) \in F(z(t) + h_iu_i(t)) \text{ a.e.} \\ \|v_i(t)\| \leq \|v(t)\| + 1 \qquad \text{a.e.} \end{cases}$$

For all $t \in [0,T]$ set

$$\pi_i(t) = \eta + \int_0^t v_i(\tau)d\tau$$

and observe that

$$u_i - \pi_i \text{ converges to zero in } C(0,T)$$

$$\pi_i' \qquad \text{converges to } w' \text{ in } L^1(0,T)$$

By Lipschitzeanity of F and (2.16) for all large i we have

$$\operatorname{dist}((z' + h_i\pi_i')(t),\ F((z+h_i\pi_i)(t))) \leq$$

$$\leq Lh_i \| u_i(t) - \pi_i(t) \|$$

By Filippov Lemma (see Aubin-Cellina [2], p.120 or Filippov [17]), for all large i there exists $x_i \in S_T(\xi + h_i\eta)$ satisfying

$$\| (x_i' - z' - h_i\pi_i')(t) \| \leq$$

$$Mh_i[\, \|u_i - \pi_i\|_{L^1} + \|u_i(t) - \pi_i(t)\|]$$

where the constant M depends only on L,T. Set

$$w_i = (x_i - z)/h_i$$

Then $z + h_i w_i \in S_T(\xi + h_i\eta)$ and w_i converges to w in $W^{1,1}(0,T)$. ∎

3. THE LOCAL CONTROLLABILITY PROBLEM

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map, $\xi \in \mathbb{R}^n$ and $R(T,\xi)$ denote the reachable set from ξ at time T of the differential inclusion

$$(1) \qquad x' \in F(x)$$

We seek to know whether $\xi \in \operatorname{Int} R(T,\xi)$. If it holds true then we say that (1) is <u>locally controllable around ξ at time T</u>.

Our first result concerns a necessary condition for local controllability:

(3.1) Theorem: Assume that a set-valued map F from $\mathbb{R}^n$ into itself is bounded and upper semicontinuous at a point $\xi \in \mathbb{R}^n$. If the system (1) is locally controllable around ξ for all small time $T > 0$ then

$$0 \in \overline{\text{co}}F(\xi)$$

(the closed convex hull of $F(\xi)$). ▲

Proof: If $0 \notin \overline{\text{co}}F(\xi)$ then there exists $p \in S^{n-1}$ such that

$$\inf_{u \in F(\xi)} \langle p,u\rangle > 0$$

Since F is upper semicontinuous at ξ there exists $\beta > 0$ such that

$$\inf \{\langle p,u\rangle : u \in F(x) \ , \ x \in \xi + \beta B\} \geq 0$$

Let $M > 0$ be such that the image of F is contained in the ball MB. If $T \leq \beta/M$ then we have for all $x \in S_T(\xi)$

$$x([0,T]) \in \xi + MT\, B \in \xi + \beta B$$

and therefore

$$\langle p,x(T)\rangle = \langle p,\xi\rangle + \int_0^T \langle p,x'(t)\rangle\, dt \geq \langle p,\xi\rangle$$

But this means that $\xi \notin \text{Int}\, R(T,\xi)$. ■

We shall study next sufficient conditions for local controllability around the point of weak (or strong) equilibrium.

For this we shall use an Open Mapping Principle. We recall

(3.2) Definition (of Clarke's tangent cone): Let E be a Banach space, K be a subset of E and x be a point in the closure $\bar{K}$ of K. We say that v belongs to $C_K(x)$ (Clarke's tangent cone to K at x) if and only if for all sequence $x_i \in K$, $h_i > 0$ converging to x and zero respectively we can find a sequence v_i converging to v such that $x_i + h_i v_i \in K$. ▲

The set $C_K(x)$ is a closed convex cone (see Clarke [13]).

For studying the local controllability problem we need several weaker topologies on E. This is why we shall adapt Definition 3.2 to our case.

(3.3) Definition: Let U,V,E be Banach spaces and $E \subset U$, $E \subset V$. We assume that the topology of E is stronger than the ones of U and V. Let K be a subset of E and let $x \in \bar{K}^U$ (the closure of K in U). We say that $w \in C_K(x)$ if and only if for all sequence $x_i \in K$ converging to x in the space U there exists a constant $m = m(v)$ such that for all sequence $h_i > 0$ converging to zero we can find a sequence $v_i \in E$ converging to v in the space V, verifying for all large i

$$x_i + h_i v_i \in K \; ; \; \|v_i\|_E \leq m$$ ▲

The set $C_K^{U,V}(x)$ is a convex cone. Moreover,

$$C_K^{E,E}(x) = C_K(x) \subset C_K^{U,V}(x)$$

We denote by $\| \|_W$, $\| \|_C$ the usual norms of $W^{1,1}(0,T)$, $C(0,T)$ respectively.

(3.4) Theorem: Assume that a set-valued map F from $\mathbb{R}^n$ into the compact subsets of $\mathbb{R}^n$ is Lipschitzean on a neighborhood of $\xi \in \mathbb{R}^n, F(\xi) \neq \emptyset$. If one of the following two assumptions holds true

(i) ξ is a weak equilibrium of F and $\{w(T) \; : \; w \in C_{S_T(\xi)}^{C,C}(\xi)\} = \mathbb{R}^n$

(ii) ξ is an equilibrium of F and $\{w(T) \; : \; w \in C_{S_T(\xi)}^{W,C}(\xi)\} = \mathbb{R}^n$

Then (1) is locally controllable around ξ at time T. ▲

To prove the above Theorem we shall use the following:

(3.5) Open Mapping Principle: Let E,U,V be Banach spaces $E \subset U \subset V$, K be a closed subset of E and $x_o \in \bar{K}^U$ (the closure of K in U). Let A be a continuously differentiable map from a neighborhood of K in V into $\mathbb{R}^q$. If

$$A'(x_o)C_K^{U,V}(x_o) = \mathbb{R}^q$$

then

$$A(x_o) \in \text{Int } A(K)$$ ▲

Proof: We assume for a moment that $A(x_o)$ does not belong to Int A(K) and we shall derive a contradiction. Then for all $n \geq 1$ there exists $y_n \in \mathbb{R}^q \setminus A(K)$ such that

$$\|A(x_o) - y_n\| \leq 1/2n^2$$

Since $x_o \in \bar{K}^U$ by continuity of A there exist $x_o^n \in K$ such that x_o^n converges to x_o in U and $\|A(x_o^n) - A(x_o)\| \leq 1/n^2$. By Ekeland's variational principle applied to the function $x \to \|A(x) - y_n\|$ on the complete subset K of E (see Ekeland [14], Aubin-Ekeland [4], Theorem 5.3.1, p.255) there exists $x_n \in K$ such that

(3.6)
$$\begin{cases} \text{(i)} \ \|A(x_n) - y_n\| + \frac{1}{n}\|x_n - x_o^n\|_E \leq \|A(x_o^n) - y_n\| \leq \frac{1}{n^2} \\ \text{(ii) For all } x \in K \end{cases}$$

$$\|A(x_n) - y_n\| \leq \|A(x) - y_n\| + \frac{1}{n}\|x_n - x\|_E$$

By (i) we know that x_n converges to x_o. Introduce a function $f:U \to \mathbb{R}$ by

$$f(v) := \|A(v) - y_n\|$$

By assumptions $A(x_n) - y_n \neq 0$. Let us set

$$p_n = \frac{A(x_n) - y_n}{\|A(x_n) - y_n\|} \in S^{q-1}$$

Then

$$f'(x_n) = A'(x_n)^* p_n$$

S^{q-1} is a compact, we can take a subsequence p_{n_i} converging to some $p \in S^{q-1}$. Let $w \in C_K^{U,V}(x_o)$ be such that $<p,A'(x_o)w> < 0$. From now on we set $x_i = x_{n_i}$, $p_i = p_{n_i}$, $y_i = y_{n_i}$. Since A is Fréchet differentiable on a neighborhood of x_i in K, for all $i > 0$ there exists $\eta_i > 0$ such that for all $v \in V$ of $\|v\|_V \leq \eta_i$

$$(3.7) \qquad \|A(x_i + v) - y_i\| - \|A(x_i) - y_i\| \leq <p_i,A'(x_i)v> + \frac{1}{n_i}\|v\|_V$$

Let m be the constant associated with w and $\{x_i\}$ by Definition 3.3. Consider any sequence $h_i \in]0,\eta_i/m[$ converging to zero.

Then there exists a sequence $w_i \in E$ converging to w in V such that

$$(3.8) \qquad \begin{cases} \|w_i\|_E \leq m \\ x_i + h_i w_i \in K \end{cases}$$

Setting $x = x_i + h_i w_i$ in (3.6)(ii) and using (3.7), (3.8) we obtain

$$-\frac{m}{n_i} \leq \frac{1}{h_i}(\|A(x_i + h_i w_i) - y_i\| - \|Ax_i - y_i\|) \leq$$

$$< p_i,A'(x_i)w_i>+ \frac{1}{n_i}\|w_i\|_E$$

Since $A'(x_i)$ is continuous and w_i converges to w in V, we obtain by passing to the limit in the last inequality when $i \to \infty$

$$<p,A'(x_o)w> \geq 0$$

which contradicts the choice of w and achieves the proof. ∎

Proof of Theorem 3.4: Let $\beta > 0$ be such that F is Lipschitzean on $\xi + \beta B$. Define

$$K := \{x \in S_T(\xi) : x([0,T]) \subset \xi + \beta B\}$$

Then K is closed in $W^{1,1}(0,T)$. By Filippov-Ważewski's relaxation theorem ξ belongs to the closure of K in the metric $\| \ \|_C$. If (i) holds use the open mapping principle with $E = W^{1,1}(0,T)$, $U = V = C(0,T)$ and $A : V \to \mathbb{R}^n$ defined by $Aw = w(T)$. If (ii) holds set $U = W^{1,1}(0,T)$ and E, V, A as in (i). ∎

We shall provide now the first consequence of the Controllability Theorem.

We recall first the definition of Kuratowski's lim inf.

(3.9) Definition: Let E, E_1 be Banach spaces and $Q : E \rightrightarrows E_1$ be a set-valued map, $\xi \in \overline{\mathrm{Dom}\ Q}$. The following set is called Kuratowski's lim inf of Q at ξ

$$\liminf_{x \to \xi} Q(x) := \bigcap_{\varepsilon > 0} \bigcup_{\delta > 0} \bigcap_{\substack{\|x - \xi\| < \delta \\ x \in \mathrm{Dom}\ Q}} (Q(x) + \varepsilon B)$$ ▲

Consider a solution $x \in S_T(\xi)$ of differential inclusion (1) and the linearized differential inclusion around $x(\cdot)$

$$\text{(3.10)} \qquad \begin{cases} w' \in dF(x(t), x'(t))w \\ w(0) = 0 \end{cases}$$

We denote by $SL_T(x)$ the set of all solutions of (3.10).

(3.11) Corollary: We posit the assumptions of Theorem 3.4 and consider the set-valued map $SL_T(\ )$ from $S_T(\xi)$ into $W^{1,1}(0,T)$. If ξ is an equilibrium and

$$\{w(T) : w \in \liminf_{\|x - \xi\|_W \to 0} SL_T(x)\} = \mathbb{R}^n$$

then the inclusion (1) is locally controllable around ξ at time T. ∎

Proof: By the results of Section 2 we know that $SL_T(x) = dS_T(\xi,x)(C)$ when x is sufficiently close to zero. That is for all $w \in SL_T(x)$ and for a sequence $h_i > 0$ converging to zero we can find a sequence $w_i \in W^{1,1}(0,T)$ converging strongly to w and such that $x + h_i w_i \in S_T(\xi + h_i w_i(0))$. Using the Lipschitzeanity condition on F exactly in the same way as in Section 2 we may assume that $w_i(0) = 0$. This implies that $dS_T(\xi,x)(0) \subset T_{S_T(\xi)}(x)$ (the Bouligand's contingent cone to $S_T(\xi)$ at x) where

$$T_{S_T(\xi)}(x) := \{w : \exists h_i \to 0+ \,, w_i \to w \text{ in } W^{1,1}(0,T), x + h_i w_i \in S_T(\xi)\}$$

Hence

$$\liminf_{\|x-\xi\|_W \to 0} dS_T(\xi,x)(0) \subset \liminf_{\|x-\xi\|_W \to 0} T_{S_T(\xi)}(x)$$

By a Theorem of Treiman [26] we have

$$\liminf_{\|x-\xi\|_W \to 0} T_{S_T(\xi)}(x) \subset C_{S_T(\xi)}(\xi)$$

Moreover

$$C_{S_T(\xi)}(\xi) \subset C^{W,C}_{S_T(\xi)}(\xi)$$

■

(3.12) Remark: If F is Lipschitzean with convex images on a neighborhood of x then the set-valued map $dF(x,y)$ has convex images. Indeed, if $v, v^1 \in dF(x,y)(u)$ then for all $h > 0$ there exist sequences v_h, v_h^1, u_h, u_h^1 converging to v, v^1, u, u as $h \to 0+$ respectively such that

$$y + hv_h \in F(x + hu_h)$$

$$y + hv_h^1 \in F(x + hu_h^1)$$

By Lipschitzeanity of F there exists a constant $c > 0$ and $z_h \in F(x + hu_h)$ such that $\|z_h - y - hv_h^1\| \leq ch\, \|u_h - u_h^1\|$.

By convexity for all $\lambda \in [0,1]$

$$\lambda(y+h\,v_h) + (1-\lambda)z_h \in F(x+hu_h)$$

It implies that

$$y + h(\lambda v_h+(1-\lambda)v_h^1) \in F(x+h\,u_h) + ch\,\|u_h-u_h^1\|B$$

and therefore $\lambda v + (1-\lambda)v_1 \in dF(x,y)(u)$. Since λ is arbitrary in $[0,1]$ the proof follows. ∎

Corollary 3.9 involves an assumption which may be difficult to check: we have to study the sets of solutions to linearized differential inclusions around each solution x( ) closed to the equilibrium and take their lim inf.

Instead, we shall seek in the next section examples of "sublinearization" around the equilibrium whose solutions are in the tangent cone $C^{W,C}_{S_T(\xi)}(\xi)$.

4. CONTROLLABILITY OF SET-VALUED MAPS THROUGH THE SUBLINEARIZATION OF THE DIFFERENTIAL INCLUSION (1)

In this section we shall provide several applications of the results of Section 3.

We recall first

(4.1) <u>Definition</u>: Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map, Lipschitzean on a neighborhood of a point $x_o, y_o \in F(x_o)$. We denote by $CF(x_o,y_o)$ the set-valued map from $\mathbb{R}^n$ into $\mathbb{R}^n$ defined by

$v \in CF(x_o,y_o)(u)$ if and only if

$$\limsup_{\substack{(x,y)\to(x_o,y_o)\\(x,y)\in \text{graph } F\\ h\to 0+}} \text{dist}\left(v, \frac{F(x_o+hu) - y}{h}\right) = 0$$

▲

The graph of $CF(x_o,y_o)$ is the tangent cone to graph(F) at (x_o,y_o). So $CF(x_o,y_o)$ is a closed convex process satisfying

$$\text{graph } CF(x_o,y_o) \subset \text{graph } dF(x_o,y_o)$$

As in Section 2 we can prove that any solution w of "sublinearization"

$$(4.2) \qquad \begin{cases} w' \in CF(\xi,0)(w) \\ w(0) = 0 \end{cases}$$

belongs to $C_{S_T(\xi)}(\xi) \subset C^{W,C}_{S_T(\xi)}(\xi)$. Thus the controllability of (4.2) around zero at time T implies the controllability of (1) around ξ at time T.

We shall improve this result taking the larger sublinearization (4.4) defined below.

We denote by $T_{coF(\xi)}(0)$ the tangent cone to $coF(\xi)$ (convex hull of $F(\xi)$) at zero.

$$(4.3) \qquad T_{coF(\xi)}(0) := \text{cl} \bigcup_{h>0} \frac{1}{h}(coF(\xi) - 0)$$

It is a closed convex cone coinciding with Clarke's tangent cone and intermediate tangent cone to $coF(\xi)$ at zero.

Assume that ξ is an equilibrium and consider the sublinearization

$$(4.4) \qquad \begin{cases} w' \in CF(\xi,0)w + T_{coF(\xi)}(0) \\ w(0) = 0 \end{cases}$$

One can easily verify that the reachable set of (4.4) at time T is a cone. If it coincides with the whole space $\mathbb{R}^n$ we tell that (4.4) is controllable at time T. Our main theorem is

(4.5) <u>Theorem</u>: Assume that F is Lipschitzean on a neighborhood of an equilibrium ξ. Then the inclusion (1) is locally controllable around ξ at time T if the sublinearization (4.4) is controllable at time T. ▲

To prove the Theorem we need the following

(4.6) Lemma: Let $U:[0,T] \rightarrow\rightarrow \mathbb{R}^n$ be a measurable, integrably bounded set-valued map with closed images, and let $H(t) \in U(t)$ be a measurable selection. Then there exists a constant C depending only on $\int_0^T U$ and $H(\ )$ such that for all measurable $A \subset [0,T]$ and $a \in \int_A \mathrm{co}U$ we can find $\alpha(t) \in U(t)$ satisfying

$$a = \int_A \alpha(t)dt$$

$$\int_A \|\alpha(t) - H(t)\|dt \le C\|a - \int_A H(t)dt\|$$ ▲

We shall prove this Lemma in the Appendix.

Proof of Theorem 4.5: The proof is quite long and we shall proceed in several steps. Let us sketch first the main ideas.

(A) We shall begin by showing that the controllability of (4.4) at time T implies that for some finite set $K \subset F(\xi)$, $0 \in K$ the differential inclusion

$$(4.7)\qquad \begin{cases} w' \in CF(\xi,0)w + T_{\mathrm{co}K}(0) \\ w(0) = 0 \end{cases}$$

is controllable at time T. ■

(B) If (A) holds then by Theorem 3.4, we have to verify that any solution w of (4.7) belongs to $C^{W,C}_{S_T(\xi)}(\xi)$. For this we have to show that for any $x_i \in S_T(\xi)$, $\|x_i - \xi\|_W \to 0$ there exists $m > 0$ such that for all $h_i \to 0+$ we can find $z_i \in S_T(\xi)$ satisfying

$$(4.8)\qquad \|z_i' - x_i'\|_{L^1} \le h_i m \text{ for all large } i$$

(4.9) $$\frac{1}{h_i}(z_i - x_i) \to w \text{ uniformly on } [0,T]$$ ■

(C) For proving (B) we shall construct a sequence of functions $y_i \in W^{1,1}(0,T0$ such that

(4.10) $$y_i(0) = \xi$$

(4.11) $$\begin{cases} \text{For some constant } M \text{ independent of } \{h_i\} \text{ and for} \\ \text{all large } i \ \|y_i' - x_i'\|_{L^1} \leq Mh_i \end{cases}$$

(4.12) $$\begin{cases} \int_0^T \operatorname{dist}(y_i'(t), F(y_i(t))) = o(h_i) \\ \frac{1}{h_i}(y_i - x_i) \to w \text{ uniformly on } [0,T] \end{cases}$$

Once such a sequence is constructed we can apply, as in Section 2, the Filippov Theorem (see Aubin-Cellina [2] p.120) to deduce the existence of m, z_i satisfying (4.8) and (4.9). ■
Therefore, it remains to prove (A) and (C).

Proof of (A): We shall prove a more general

(4.13) Lemma: Let $A: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed convex process (a set-valued map whose graph is a closed convex cone), $Y \subset \mathbb{R}^n$ be a given compact set and $y \in Y$. Assume that the inclusion

(4.14) $$\begin{cases} w' \in A(w) + T_{coY}(y) \\ w(0) = 0 \end{cases}$$

is controllable at time T. Then there exists a finite set $K \subset Y$ such that inclusion

(4.15) $$\begin{cases} z' \in A(z) + T_{coK}(y) \\ z(0) = 0 \end{cases}$$

is controllable. ▲

Proof: The controllability of (4.14) implies implicitly that $\mathrm{Dom}(A) = \mathbb{R}^n$. Hence, from Robinson-Ursescu Theorem (see Robinson [24], Ursescu [28], Aubin-Ekeland [4]) we obtain

$$A \text{ is } \gamma\text{-Lipschitzean for some } \gamma > 0$$

Let w_i $i=0,\ldots,n$ be solutions of (4.14) satisfying (thanks to controllability of (4.14) and Caratheodory's Theorem)

$$0 \in \mathrm{Int}\,\mathrm{co}\{w_i(T) : i = 0,\ldots,n\}$$

Observe that for all $K \subset F(\xi)$ the reachable set $R^K(T)$ of the inclusion (4.15) at time T is a convex cone.

We claim that it is enough to verify that for all $\varepsilon > 0$ and $i=0,\ldots,n$ we can find a finite subset K_i of Y and a solution z_i of (4.15) with $K = K_i$ satisfying

$$\|z_i(T) - w_i(T)\| \leq \varepsilon$$

Indeed, we can take then $K = \bigcup_{i=1}^{m} K_i$, so each z_i is a solution to (4.15) and K is finite. On the other hand, zero remains in the interior of $\mathrm{co}\{z_i(T) : i=0,\ldots,n\}$ when ε is small enough. Hence, zero belongs to the interior of $R^K(T)$. Since $R^K(T)$ is a cone, $R^K(T) = \mathbb{R}^n$. Fix then w_i and $\varepsilon > 0$. Then for some $v,u \in L^1(0,T)$ and almost all t

$$w_i'(t) = v(t) + u(t)$$

$$v(t) \in A(w_i(t))$$

$$u(t) \in T_{\mathrm{co}Y}(y) := \mathrm{cl} \bigcup_{h>0} \frac{\mathrm{co}Y - y}{h}$$

By Lemma 2.10 applied to $F \equiv \mathrm{co}Y$ and $\alpha_i \equiv (0,y)$ for any sequence $h_s > 0$ converging to zero there exists a sequence u_s converging to u almost everywhere and such that

$$y + h_s u_s(t) \in \mathrm{co}Y$$

$$\|u_s(t)\| \leq \|u(t)\| + 1$$

Let $\eta = \varepsilon/(1 + \gamma T \exp \gamma T)$ and s be so large that

$$\|u_s - u\|_{L^1} \leq \eta/2$$

The compact subset Y can be covered by p balls $x_j + h_s \frac{\eta}{2T} B$. We consider the finite subset $K_i = \{y, x_1, \ldots, x_p\} \subset Y$. Thus

$$y + h_s u_s(t) \in \mathrm{co} K_i + h_s \frac{\eta}{2T} B$$

By a measurable selecton Theorem (see Wagner [29]) there exists a measurable function $\bar{u}$ satisfying

$$y + h_s \bar{u}(t) \in \mathrm{co} K_i$$

$$\|u_s(t) - \bar{u}(t)\| \leq \eta/2T$$

Define

$$\bar{z}(t) = \xi + \int_0^t (v(\tau) + \bar{u}(\tau))d\tau$$

Then

$$\|\bar{z}(t) - w_i(t)\| \leq \int_0^t \|u(\tau) - \bar{u}(\tau)\| d\tau \tag{4.16}$$

Thus $\quad \mathrm{dist}[\bar{z}'(t), A(\bar{z}(t)) + \bar{u}(t)] \leq \gamma \|w_i(t) - \bar{z}(t)\| \leq$

$$\gamma(\|u - u_s\|_{L^1} + \|u_s - \bar{u}\|_{L^1}) \leq \gamma\,\eta$$

By Filippov's Theorem (see Aubin-Cellina [2], p.120) there exists $z_i \in W^{1,1}(0,T)$ satisfying

$$z_i'(t) \in A(z_i(t)) + \bar{u}(t)$$

$$z_i(0) = 0$$

such that

$$\|z_i - \bar{z}\|_C \leq e^{\gamma T} \gamma \eta T$$

Hence by (4.16) and the choice of $\bar{u}$

$$\begin{aligned}
&\|z_i(T) - w_i(T)\| \le \|z_i - \bar{z}\|_C + \|\bar{z} - w_i\|_C \\
&\le e^{\gamma T}\gamma\eta T + \|u_s - u\|_{L^1} + \|u_s - \bar{u}\|_{L^1} \le \\
&\le (e^{\gamma T}\gamma T + 1)\eta = \varepsilon
\end{aligned}$$

which ends the proof of Lemma 4.13. ∎

So the proof of (A) is also completed.

Proof of (C): Let K be such that the inclusion (4.7) is controllable at time T and let w be a solution of (4.7), i.e. there exists $v,u \in L^1(0,T)$ such that

$$(4.17) \qquad w'(t) = v(t) + u(t)$$

$$(4.18) \qquad v(t) \in CF(\xi,0)(w(t))$$

$$(4.19) \qquad u(t) \in T_{coK}(0)$$

Let $x_i \in S_T(\xi)$, $\|x_i - \xi\|_W \to 0$. We shall construct now a sequence $y_i \in W^{1,1}(0,T)$ satisfying (4.10) - (4.12).

The idea is to find $M_1 > 0$ and sets $B_i \subset [0,T]$ such that $\mu(B_i) \le M_1 h_i$ and $\frac{1}{h_i}(y_i' - x_i')$ is "close" to v on the complementary of B_i

$$\frac{1}{h_i}(y_i(t) - x_i(t)) \text{ is "close" to } \int_0^t (v(\tau) + u(\tau))d\tau$$

Step 1: We define here some "auxiliary" functions α_i. We introduce

$$A_i = \{t : \|x_i'(t)\| \le 1/i\}$$

Since $\|x_i'\|_{L^1} \to 0$ (because $x_i(\cdot)$ converges to an equilibrium ξ), we have

$$(4.20) \qquad \lim_{i \to \infty} \mu(A_i) = T$$

Inclusion 4.19 and Lemma 2.10 imply that there exists a sequence $u_i \in L^1(0,T)$ converging to u such that

$$h_i u_i(t) \in \text{co}K \qquad \text{a.e.}$$

Fix i and let q be an integer satisfying

$$(4.21) \qquad q \geq \frac{T}{h_i^2} \sup_{e \in \text{co}K} \|e\| + 1$$

and denote by I_j the interval $[\frac{(j-1)T}{q}, \frac{jT}{q}]$ $j = 1,\dots,q$.
By Lemma 4.6 applied to the map U(t) = K and $H \equiv 0$ there exsits a constant C > 0 depending only on $\int_0^T K$ and the functions $\alpha_i^j(t) \in K$ such that

$$(4.22) \qquad \int_{I_j \cap A_i} \alpha_i^j(t)dt = \int_{I_j \cap A_i} h_i u_i(t)dt$$

$$(4.23) \qquad \int_{I_j \cap A_i} \|\alpha_i^j(t)\|dt \leq C\|\int_{I_j \cap A_i} h_i u_i(t)dt\|$$

Set

$$\alpha_i(t) = \begin{cases} \alpha_i^j(t) & \text{if } t \in I_j \cap A_i \\ 0 & \text{otherwise} \end{cases}$$

$$(4.24) \qquad \begin{cases} \delta = \min\{\|e\| : e \in K \setminus \{0\}\} \\ B_i = \{t : \alpha_i(t) \neq 0\} \end{cases}$$

Then (4.23) implies

$$\int_0^T \|\alpha_i(t)\|dt = \sum_{j=1}^{q} \int_{I_j \cap A_i} \|\alpha_i^j(t)\|dt$$

$$\leq C \sum_{j=1}^{q} \int_{I_j \cap A_i} \|h_i u_i(\tau)\|d\tau \leq C\, h_i \|u_i\|_{L^1}$$

This and (4.24) yields

$$\delta\mu(B_i) \le C\, h_i \|u_i\|_{L^1}$$

It proves that for some constant $M_1 > 0$ independent of $\{h_i\}$ and all large i

(4.25) $$\mu(B_i) \le M_1 h_i$$

Moreover (4.21), (4.22) imply that for all $t \in [0,T]$ and some $j = j(t)$

$$\|\int_o^t \alpha_i(\tau)d\tau - \int_o^t h_i u_i(\tau)d\tau\| \le$$

$$h_i \int_{[0,t]\setminus A_i} \|u_i(\tau)\|d\tau + \int_{I_j} (\|\alpha_i(\tau)\| + h_i\|u_i(\tau)\|)d\tau$$

$$\le h_i \int_{[0,T]\setminus A_i} \|u_i\| + 2h_i^2 \quad \text{(because } \mu(I_j) \le T/m)$$

This and (4.20) imply the existence of $\varepsilon(h_i)$ such that for all t

(4.26) $$\begin{cases} \|\int_o^t \alpha_i(\tau)d\tau - \int_o^t h_i u_i(\tau)d\tau\| \le \varepsilon(h_i) \\ \lim\limits_{i\to\infty} \dfrac{\varepsilon(h_i)}{h_i} = 0 \end{cases}$$

<u>Step 2</u>: We shall construct now the functions y_i as in (C). Let L be the Lipschitz constant of F, and for all large i let $\gamma_i(t)$ be a measurable function satisfying

(4.27) $$\gamma_i(t) \in F(x_i(t))$$

(4.28) $$\|\gamma_i(t) - \alpha_i(t)\| \le L\|x_i(t) - \xi\|$$

(Remember that $K \subset F(\xi)$).

We shall set $y_i'(t) = \gamma_i(t)$ on B_i. We have to define it on the complimentary of B_i. By Lemma 2.10 and inclusion (4.18) we can find a sequence of functions w_i converging to w uniformly and a sequence v_i converging to v in $L^1(0,T)$ such that

$$(4.29) \qquad (w_i(t),v_i(t)) \in \frac{\text{graph } F - (x_i(t),x_i'(t))}{h_i}$$

Hence, we define $y_i'(\ )$ by

$$(4.30) \qquad y_i'(t) = \begin{cases} \gamma_i(t) & \text{if} \quad t \in B_i \\ x_i'(t) + h_i v_i(t) & \text{otherwise} \end{cases}$$

and $y_i(\cdot)$ by

$$y_i(t) = \xi + \int_0^t y_i'(\tau)d\tau$$

Then by (4.25), (4.30)

$$\|y_i' - x_i'\|_{L^1} \leq h_i\|v_i\|_{L^1} + \int_{B_i} \|\gamma_i(t) - x_i'(t)\|dt$$

$$\leq h_i(\|v_i'\|_{L^1} + 2M_1 \sup_{e\in F(x_i(t))} \|e\|)$$

Since M_1 does not depend on $\{h_i\}$ there exists a constant M independent of $\{h_i\}$ such that for all large i

$$\|y_i' - x_i'\|_{L'} \leq Mh_i$$

Thus y_i satisfy (4.10), (4.11).

<u>Step 3</u>: We prove finally that y_i satisfies also (4.12). The relations (4.30), (4.24) imply

$$(4.31)\quad \begin{cases} y_i(t) - x_i(t) = \int_0^t (y_i'(\tau) - x_i'(\tau))d\tau = \\ \int_0^t h_i v_i(\tau)d\tau + \int_{[0,t]\cap B_i} (\gamma_i(\tau) - x_i'(\tau) - h_i v_i(\tau))d\tau \\ = \int_0^t h_i v_i(\tau)d\tau + \int_0^t \alpha_i(\tau)d\tau + \\ \int_{[0,t]\cap B_i} (\gamma_i(\tau) - \alpha_i(\tau)d\tau - \int_{[0,t]\cap B_i} (x_i'(\tau) + h_i v_i(\tau))d\tau \end{cases}$$

This and (4.28), (4.26) yield

$$\|y_i(t) - x_i(t) - h_i w_i(t)\| \leq \|\int_0^t h_i(v_i(\tau) + u_i(\tau))d\tau$$

$$- h_i w_i(t)\| + \varepsilon(h_i) + L\,\|x_i - \xi\|_C \mu(B_i) +$$

$$\int_{B_i} \|x_i'(\tau)\|d\tau + h_i \int_{B_i} \|v_i(\tau)\|d\tau$$

Remember that $\|x_i'(t)\| \leq 1/i$ for $t \in B_i$ $\|v_i - v\|_{L^1} \to 0$. Hence, using (4.25), (4.17) we obtain

$$\left\|\frac{y_i(t) - x_i(t)}{h_i} - w(t)\right\| \leq \|w_i - w\|_C + \|v_i - v\|_{L^1} +$$

$$+ \|u_i - u\|_{L^1} + \varepsilon(h_i) + L\,\|x_i - \xi\|_C M_1 + \frac{1}{i}M_1 + \int_{B_i} \|v_i(\tau)\|d\tau$$

Since $\quad \|w_i - w\|_C \to 0\;,\; \|x_i - \xi\|_C \to 0\;,\; \|v_i - v\|_{L^1} \to 0$

$\|u_i - u\|_{L^1} \to 0 \quad$ and $\quad \mu(B_i) \to 0$

the last inequality implies that

$$(4.32)\qquad \frac{y_i - x_i}{h_i} \to w \quad \text{uniformly on } [0,T].$$

Moreover, from (4.30), (4.29), (4.25) and Lipschitzeanity of F it follows that for all large i

$$(4.33)\qquad \begin{cases} \int_0^T \mathrm{dist}(y_i'(t),\ F(x_i(t)\ +\ h_i w_i(t)))dt\ = \\[2ex] \int_{B_i} \mathrm{dist}(\gamma_i(t),\ F(x_i(t)\ +\ h_i w_i(t))dt\ \le \\[3ex] \int_{B_i} Lh_i \|w_i(t)\| dt\ \le\ LM_1 h_i^2\ \sup\limits_{i,t}\|w_i(t)\| \end{cases}$$

By Lipschitzeanity of F for all large i we have

$$\mathrm{dist}(y_i'(t),\ F(y_i(t))\ \le$$

$$\mathrm{dist}(y_i'(t),\ F(x_i(t)\ +\ h_i w_i(t))) + L\|y_i(t)\ -\ x_i(t)\ -$$

$$h_i w(t)\|\ +\ Lh_i\|w_i(t)\ -\ w(t)\|$$

since $\|w_i\ -\ w\|_C \to 0$ (4.32) and (4.33) together imply

$$\int_0^T \mathrm{dist}(y_i'(t),\ F(y_i(t))dt\ =\ o(h_i)$$

so the proof is complete. ■

Remark: One must pay attention to the high sensitivity that the derivative $CF(\xi,0)$ inherits from the properties of the Clarke tangent cone. As an example, consider the closed unit ball in $\mathbb{R}^2$ and the set $A = B \cap \{0\} \times [1,+\infty]$. Then, although the set A is larger than B we have

$$C_B(0,1)\ = \mathbb{R} \times \mathbb{R}_-$$

$$C_A(0,1)\ =\ \{0\}$$

When a similar thing happens to $CF(\xi,0)$ it is often more appropriate (when it is possible) to consider a smaller differential inclusion

$$x'\ \in\ Q(x)$$

having the property:

$$(\xi,0) \in \text{graph } Q \subset \text{graph } F$$

The local controllability of a "smaller" inclusion then will imply the local controllability of the inclusion (1).

We shall illustrate this remark in the proof of the next Theorem and an example.

We show next how to derive from Theorem 4.5 the classical results on local controllability of parametrized systems, without assuming too much regularity. Let U be a compact topological space and let $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ be a continuous function. Assume that for some $(\xi,\bar{u}) \in \mathbb{R}^n \times U$, $f(\xi,\bar{u}) = 0$ and for some $\beta > 0$ and all $u \in U$

$$\frac{\partial f}{\partial x}(\cdot,u) \quad \text{is continuous on } \xi + \beta B$$

Consider the control system

$$(4.35) \qquad \begin{cases} x' = f(x,u) \quad , \quad u \in U \\ x(0) = \xi \end{cases}$$

We wish to study the local controllability of system (4.1) at a given time T.

(4.36) <u>Theorem</u>: If the sublinearized differential inclusion

$$(L) \qquad w' \in \frac{\partial f}{\partial x}(\xi,\bar{u})w + T_{co\ f(\xi,U)}(0)$$

is locally controllable around zero at time T then the system (4.15) is locally controllable around ξ at time T.

<u>Proof</u>: By Lemma 4.13 we may assume that U is finite and such that

$$(4.37) \qquad \forall u,v \in U,\ u \neq v \Rightarrow f(\xi,u) \neq f(\xi,v)$$

Set $\qquad F(x) = \{f(x,u) \ : \ u \in U\}.$

Then a simple computation gives

$$CF(\xi,0)(w) = \frac{\partial f}{\partial x}(\xi,\bar{u})w$$

and by Theorem 4.5 we complete the proof. ∎

(4.38) Remark: In Theorem 4.36 we did not compute directly the derivative $CF(\xi,0)$, but we used a "subsystem" with "nice" properties.

This can be illustrated by the following example suggested to us by St. Łojasiewicz (Jr.). Consider the control system in $\mathbb{R}^3$

$$\begin{cases} \dot{x} = u & u = [-1,1] \\ \dot{y} = v & v = [-1,1] \\ \dot{z} = \frac{x+y}{2} + \frac{x-y}{2} & w = [-1,1] \end{cases}$$

Then $\qquad F(x,y,z) = [-1,1] \times [-1,1] \times [\min(x,y), \max(x,y)]$

The direct computation of $CF(0,0)$ gives then $\{a \in \mathbb{R}^6 : a_1 = a_2 = a_6\}$. So Dom $CF(0,0) \neq \mathbb{R}^3$ and Corollary 4.12 can not be applied. But if we proceed as in the proof of the last theorem and consider only controls as in (4.38) then we are obliged to fix $w = w_o$. So our system becomes

$$\begin{cases} \dot{x} = u & \|u\| \leq 1 \\ \dot{y} = v & \|v\| \leq 1 \\ \dot{z} = (1+w_o)x/2 + (1-w_o)y/2 \end{cases}$$

and it is easy to see that it is controllable.

5. INTERIOR POINTS OF REACHABLE SETS

Consider a set-valued map from $[0,T] \times \mathbb{R}^n$ into compact subsets of $\mathbb{R}^n$ and the differential inclusion

$$(5.1) \qquad \begin{cases} x' \in F(t,x) \\ x(0) = \xi \end{cases}$$

Let z be a solution of (5.1), i.e. for almost all $t \in [0,T]$

$$z'(t) \in F(t,z(t))$$

$$z(0) = \xi$$

We are studying here the sufficient conditions for $z(T)$ to be an interior point of the reachable set $R(T,\xi)$ of (5.1) at time T.

Define

$$G(t,y) = F(t,z(t)+y) - z'(t)$$

Then $0 \in G(t,0)$. It means that zero is an equilibrium of $G(t,\cdot)$ for all $t \in [0,T]$. It is clear that $z(T) \in \text{Int}\, R(T,\xi)$ if and only if zero is an interior point of the reachable set at time T for the inclusion

$$\begin{cases} y' \in G(t,y) \\ y(0) = 0 \end{cases}$$

Our techniques can also be applied to this problem exactly in the same way. The only change is that we have to use the time dependent version of Filippov's results (for this, see Clarke [13] p.117), when F depends in a measurable way from the time variable t. Exactly in the same way as in Section 2 we prove

(5.2) <u>Theorem</u>: Assume that $F : [0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has compact images and for some $\beta > 0$, $k \in L^1(0,T)$, F is measurable in t ; $F(t,\cdot)$ is Lipschitzean of constant $k(t)$ on $z([0,T]) + \beta B$. Then $z(T)$ belongs to Int $R(T,\xi)$ if the sublinearizeation

$$(5.3) \qquad \begin{cases} w'(t) \in C\,F\,(t,z(t),z'(t))(w(t)) + T_{coF(t,z(t))}(\dot{z}(t)) \\ w(0) = 0 \end{cases}$$

is locally controllable around zero at time T.

<u>Remark</u>: The above Theorem and results of Section 4 imply also the Pontriagin Maximum Principle:

Let U be a compact topological space and $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ be a continuous function. Consider the control system

$$(5.4) \qquad \begin{cases} x' = f(t,x,u) \quad , \quad u \in U \\ x(0) = \xi \end{cases}$$

Let z be a solution of (5.4), u_* being a corresponding control.

(5.5) <u>Theorem</u>: Under the above assumptions assume for some $\beta > 0$, $k \in L^1(0,T)$ and all $u \in U$:

$f(t,\cdot,u)$ is $k(t)$ Lipschitzean on $z(t) + \beta B$

$\frac{\partial f}{\partial x}(t,\cdot,u)$ is continuous on $z(t) + \beta B$

If $z(T)$ belongs to the boundary of reachable set of (5.3) at time T then there exists a non-vanishing absolutely continuous function $p : [0,T] \to \mathbb{R}^n$ such that

$$p'(t) = -\frac{\partial f}{\partial x}(t,z(t),u_*(t))p(t)$$

$$\langle p(t),f(t,z(t),\ u_*(t))\rangle = \max_{u\in U} \langle p(t),f(t,z(t),u)\rangle$$

<u>Proof</u>: It is not restrictive to replace U by $U(t) \subset U$, where $U(t)$ is a compact and for all $t \in [0,1]$; $u,v \in U(t)$, $u \neq v$

$$f(t,z(t),u) \neq f(t,z(t),v)$$

Let $F(t,x) = f(t,x,U(t))$. Then we have

$$CF(t,z(t),z'(t))(w) = \frac{\partial f}{\partial x}(t,z(t),u_*(t))w$$

By Theorem 5.2 we know that (5.3) is not locally controllable at time T. Thus zero belongs to the boundary of the reachable set at time T of the inclusion

$$(5.6) \qquad \begin{cases} w' \in \frac{\partial f}{\partial x}(t,z(t),u_*(t))w + T_{coF(t,z(t))}(z'(t)) \\ w(0) = 0 \end{cases}$$

By duality, it implies the existence of a non-vanishing absolutely continuous function p satisfying

$$p'(t) = -\frac{\partial f}{\partial x}(t,z(t),u_*(t))p(t)$$

$$\sup_{v \in T_{coF(t,z(t))}} \langle p(t),v\rangle \leq 0$$

This achieves the proof.

APPENDIX

<u>Proof of Lemma 4.6</u>: This lemma is a consequence of a result of Frankowska-Olech [17]. We recall first that

$$\int_o^T U = \{\int_o^T u(t)dt \; : \; u(t) \in U(t)\}$$

The assumption that U is integrably bounded means that for some $k \in L^1(0,T)$

$$U(t) \subset k(t)B \qquad \text{a.e.}$$

Then for all measurable $A \subsetneq [0,T]$ $\int_A U$ is a convex compact set and $\int_A U = \int_A \text{co}U$ (see for example Olech [23]). We denote by $\text{ext} \int_A U$ the set of all extremal points of $\int_A U$. Then for all $b \in \text{ext} \int_A U$ there exists exactly one $u_b(t) \in U(t)$, satisfying

$$b = \int_A u_b(t)dt$$

In particular it implies that if $b \in \text{ext} \int_o^T U$ then for all measurable set $A \subset [0,T]$

$$(4.7)\quad \begin{cases} \text{(i)} & \int_A u_b(t)dt \in \text{ext} \int_A U \\ \text{(ii)} & \text{if for some } \alpha(t) \in U(t) \text{ we have} \\ & \int_A \alpha(t)dt \in \text{ext} \int_A U \\ & \text{then there exist } u(t) \in U(t) \text{ equal to } \alpha \text{ on} \\ & \text{the set } A \text{ and such that} \\ & \int_0^T u(t)dt \in \text{ext} \int_0^T U \end{cases}$$

(The above results can be easily deduced from Olech [23]).

Without any loss of generality we may assume that $H \equiv 0$. Then for all measurable $A \subset [0,T]$ we have

$$\int_A U \subset \int_0^T U$$

Let $\{K_\tau\}$ be a finite family of convex compact subsets of $\int_0^T U$ such that

$$(4.8)\quad \begin{cases} \bigcup_\tau K_\tau = \int_0^T U \\ \text{ext } K_\tau \subset \text{ext} \int_0^T U \cup \{0\} \\ \quad 0 \in \text{ext } K_\tau \end{cases}$$

Taking (if needed) a subdivision of K_τ we may assume that for some $c_1 > 0$ and all τ there exists an outer normal $n = n(\tau)$ to K_τ at zero (i.e. $\sup\{<z,n> : z \in K_\tau\} \leq 0$), $\|n\|=1$ such that for all $y \in K_\tau$

$$\sup_{z \in K_\tau} \{\|z\| : <z,n> \geq <y,n>\} \leq c_1\|y\|$$

Set $c = (2n-1)c_1$. We claim that c is the constant we are looking for. Fix a measurable $A \subset [0,T]$ and define

$$K_\tau^A = \mathrm{co}\{0\} \cup \{\int_A v(t)dt \; : \; \int_0^T v(t)dt \in \mathrm{ext}\; K_\tau \setminus \{0\}\}$$

Since $\int_A \mathrm{co}U = \mathrm{co\ ext} \int_A U$ by (4.7) we have

$$\int_A \mathrm{co}U = \bigcup_\tau K_\tau^A$$

Thus if $a \in \int_A \mathrm{co}U$ there exists τ_o such that

$$a \in K_{\tau_o}^A$$

Let $V(t) \subset U(t)$ be a (unique) measurable set-valued function such that

$$\int_0^T V = K_{\tau_o}$$

Then $\quad 0 \in V(t) \quad$ a.e. (Remember that $0 \in \mathrm{ext}\; K_{\tau_o}$)

and $\quad \int_A V = K_{\tau_o}^A$

Thereby for some $\bar{\alpha}(t) \in V(t)$ we have

$$a = \int_A \bar{\alpha}(t)dt$$

Set

$$\alpha(t) = \begin{cases} \bar{\alpha}(t) & \text{if } t \in A \\ 0 & \text{otherwise} \end{cases}$$

By a Lemma from Frankowska-Olech [17]

$$\int_A \|\bar{\alpha}(t)\| dt \leq (2n-1) \sup_{z \in K_{\tau_o}} \{\|z\| \; : \; \langle z,n\rangle \geq \langle a,n\rangle\}$$

and thus using (4.7) we obtain

$$\int_0^T \|\alpha(t)\| dt \leq (2n-1)c_1 \|a\| = c\|a\|$$

which ends the proof.

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