A VIABILITY APPROACH TO LIAPUNOV'S SECOND METHOD

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PREFACE

In this paper, viability theorems are used to provide short proofs of extensions of Liapunov's second method to the case in which differential equations are replaced by differential inclusions, the Liapunov functions are only required to be continuous and viability constraints are present.

This research was conducted within the framework of the Dynamics of Macrosystems study in the System and Decision Sciences Program.

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ABSTRACT

The purpose of this note is to extend Liapunov's second method to the case of differential inclusions, when viability requirements are made and when the Liapunov functions are continuous.

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When f is a continuous single-valued map from an open subset Ω of $\mathbb{R}^n$ to $\mathbb{R}^n$ and V is a differentiable function defined on Ω, the Liapunov method derives from estimates of the form

(1) $\quad \forall x \in \Omega, \; < V'(x), f(x) > \; \leq \psi(V(x))$

informations on the behavior of a solution $x(\cdot)$ to the differential equation $x' = f(x)$, $x(0) = x_o$ given by inequalities of the form

(2) $\quad V(x(t)) \leq w(t)$

where w is a solution to the differential equation

(3) $\quad w'(t) = \psi(w(t)), \; w(0) = V(x_o)$

(see for instance Yoshizawa [1966]).

We shall extend this result when we replace the differential equation by a differential inclusion, when we require viability conditions and when we assume that V is only continuous (because "interesting" examples of functions V are derived from non-differentiable norms, for instance). We look for solutions $x(\cdot)$ to

(4) for almost all $t \in [0,T], x'(t) \in F(x(t)), x(0) = x_o$ given in K

satisfying

$$(5)\quad \begin{cases} \text{i) } \forall t \in [0,T], x(t) \text{ belongs to a closed subset } K \text{ (viability)} \\ \text{ii) } \forall t \in [0,T], V(x(t)) \leq w(t) \end{cases}$$

where $w(t)$ is a solution to the differential equation (3). For that purpose, we choose among the concepts of tangent cones to subsets and generalized directional derivatives of a function

- the contingent cone $T_K(x)$ to K at x, defined by

$$(6)\qquad T_K(x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0+} \frac{d(x+hv,K)}{h} = 0\}$$

introduced by Bouligand [1932], (see also Aubin-Cellina [1984] section 4.2, pp.176-177).

- the hypo-contingent derivative $D\_V(x)$ of V at x, defined by

$$(7)\qquad \forall v \in \mathbb{R}^n,\ D\_V(x)(v) := \limsup_{\substack{h \to 0+ \\ v' \to v}} \frac{V(x+hv') - V(x)}{h}$$

(see Aubin-Cellina [1984], section 6.1, p.287).

We shall prove the following:

Theorem 1:

Let V be a nonnegative continuous function defined on a neighborhood of the closed subset K and ψ be a nonpositive continuous function from $\mathbb{R}_+$ to $\mathbb{R}$ satisfying $\psi(0) = 0$. Let $x_o \in K$ be given.

(a) We assume that

$$(8)\qquad \begin{cases} F \text{ is upper semicontinuous with non-empty compact} \\ \text{convex values.} \end{cases}$$

If we replace estimate (1) by

$$(9)\qquad \forall x \in K, \exists v \in F(x) \cap T_K(x) \text{ such that } D\_V(x)(v) \leq \psi(V(x))$$

there exist $T > 0$ and solutions $w(\cdot)$, $x(\cdot)$ to the problem (3), (4) and (5).

(b) We assume that

(10) $\quad F$ is continuous with non-empty compact values

If we posit the stronger estimate

(11) $$\forall x \in K, F(x) \subset T_K(x) \text{ and } \sup_{v \in F(x)} D\_V(x)(v) \leq \psi(V(x))$$

there exist $T > 0$ and solutions $w(\cdot)$, $x(\cdot)$ to the problem (3), (4) and (5).

(c) We assume that

(12) $\quad F$ is Lipschitz on a neighborhood of K and has non-empty compact values and ψ is Lipschitz on a neighborhood of $[0,w_o]$

Then estimate (11) implies the existence of $T > 0$ such that any solution $(w(\cdot),x(\cdot))$ to (3) and (4) satisfies property (5).

Remark:

If we assume furthermore that F is bounded, we can take $T = +\infty$ in the above theorem. This implies that $w(t)$ converges to some w_* when $t \to \infty$, where $w_* \in [0,V(x_o)]$ is a solution to the equation $\psi(w_*) = 0$. If $\psi(w) < 0$ for all $w > 0$, we then deduce that

(13) $$\lim_{t \to \infty} V(x(t)) = 0.$$

Proof of Theorem 1:

(a) We set:

(14) $$G(x,w,\lambda) := F(x) \times \psi(w) \times \{0\} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$$

We introduce the viability domain

(15) $\mathcal{K} := \{(x,w,\lambda) \in K \times [0,w_o] \times \mathbb{R}_+ | V(x) \leq w\}$,

which is a closed subset of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ (where $w_o > V(x_o)$). We observe that if

(16) $v \in T_K(x)$ satisfies $D\_V(x)(v) \leq \psi(V(x))$

then

(17) $(v,\psi(w),0)$ belongs to $T_{\mathcal{K}}(x,w,\lambda)$

Indeed, since v belongs to $T_K(x)$, there exist sequences of elements $h_n > 0$ and v_n converging to 0 and v such that

$$\forall n,\ x + h_n v_n \in K$$

By the very definition of $D\_V(x)(v)$, there exists a sequence of elements $a_n \in \mathbb{R}$ converging to $D\_V(x)(v)$ such that, for all $n \geq 0$,

$$V(x+h_n v_n) \leq V(x) + h_n a_n$$

If $\psi(V(x)) = w$, we take $b_n := a_n + \psi(V(x)) - D\_V(x)(w)$ if $D\_V(x)(w) > -\infty$ and $b_n = \psi(V(x)$ if $D\_V(x)(w) = -\infty$. If $\psi(V(x)) < w$, we take $b_n := \psi(w)$ and we deduce that $V(x+h_n v_n) \leq w + h_n \psi(w)$ for n large enough because V is continuous. In summary, b_n converges to w and satisfies

(18) $\forall n, x + h_n v_n \in K$ and $V(x+h_n v_n) \leq w + h_n b_n$

This shows that $(x+h_n v_n, w+h_n b_n, \lambda+h_n 0)$ belongs to $\mathcal{K}$ and thus, that $(v,\psi(w),0)$ belongs to $T_{\mathcal{K}}(x,w,\lambda)$.

We consider now trajectories $x(\cdot),w(\cdot),\lambda(\cdot)$ of the differential inclusion

(19) $$\begin{cases} \text{i)} & (x'(t),w'(t),\lambda'(t)) \in G(x(t),w(t),\lambda(t)) \\ \text{ii)} & (x(0),w(0),\lambda(0)) = (x_o,V(x_o),0) \end{cases}$$

which are viable in the sense that

(20) $\forall t \in [0,T], (x(t), w(t), \lambda(t)) \in K$

We then observe that $\lambda(t) = 0$, that $x(\cdot)$ is a solution to (4), that w is a solution to (3) and that (20) implies properties (5).

(b) If F satisfies assumptions (8) and (9), then G is also upper semicontinuous with compact convex values and $G(x,w,\lambda) \cap T_K(x,w,\lambda) \neq \emptyset$. Hence Haddad's viability theorem (see Haddad [1981], Aubin-Cellina [1984], Theorem 4.2.1, p.180) implies the existence of a solution to (19)-(20) on some interval.

(c) If F satisfies assumptions (10) and (11), then G is continuous with compact values and $G(x,w,\lambda) \subset T_K(x,w,\lambda)$. Hence the viability theorem of Aubin-Clarke [1977] (see also Aubin-Cellina [1984], Theorem 4.6.1, p.198) implies the existence of a solution to (19)-(20) on some interval.

(d) If F satisfies assumptions (11) and (12), then G is Lipschitz with compact values on a neighborhood of K and $G(x,w,\lambda) \subset T_K(x,w,\lambda)$. Hence the invariance theorem of Clarke [1975] (see also Aubin-Cellina [1984], Theorem 4.6.2, p.202) shows that any solution of (19) satisfies (20).

Remark:

We can solve in the same way the case when we consider

(21) $\begin{cases} \text{i) p nonnegative continuous functions } V_j \text{ defined on a neighborhood of K} \\ \text{ii) p nonpositive continuous functions } \psi_j \text{ from } \mathbb{R}_+ \text{ to } \mathbb{R}_+ \text{ satisfying } \psi_j(0) = 0 \end{cases}$

and when we replace condition (5) by

(22) $\begin{cases} \text{i) } \forall t \in [0,T], x(t) \in K \\ \text{ii) } \forall t \in [0,T], \forall j = 1,\ldots,p, V_j(x(t)) \leq w_j(t) \end{cases}$

where $w_j(\cdot)$ is some solution to the differential equation

$$(23) \qquad w_j'(t) = \psi_j(w_j(t)); \; w_j(0) = V_j(x_o)$$

We have to replace Liapunov estimates (9) by

$$(24) \qquad \begin{cases} \forall x \in K, \exists v \in F(x) \cap T_K(x) \text{ such that,} \\ \forall_j = 1,\ldots,p, \; D_-V_j(x)(v) \leq \psi_j(V_j(x)) \end{cases}$$

and estimate (11) by

$$(25) \qquad \begin{cases} \forall x \in K, F(x) \subset T_K(x) \text{ and } \forall_j = 1,\ldots,p, \\ \sup\limits_{v \in F(x)} D_-V_j(x)(v) \leq \psi_j(V_j(x)) \end{cases}$$

Therefore, the asymptotic properties of solutions to the differential inclusions (4) are concealed in the following function ψ_o defined by

$$(26) \qquad \psi_o(w) := \sup_{V(x)=w} \; \inf_{u \in F(x) \cap T_K(x)} D_-V(x)(u).$$

for set-valued maps F satisfying (8) or the function ψ_1 defined by

$$(27) \qquad \psi_1(w) := \sup_{V(x)=w} \; \sup_{u \in F(x)} D_-V(x)(u)$$

for set-valued maps F satisfying (10).

Hence any continuous function ψ larger than ψ_o (or ψ_1) will provide solutions $w(\cdot)$ to (3) estimating the value $V(x(t))$ on some trajectory of the differential inclusion (4).

For instance, we obtain the following consequence on asymptotic stability.

Corollary:

Let V be a nonnegative continuous function defined on a neighborhood of K and let x_o be given. Let F satisfy assumption (8). We assume further that $\lambda_o \in \mathbb{R}$ achieves the finite maximum in

$$(28) \qquad \rho_o := \sup_{\lambda\in\mathbb{R}} \inf_{w\geq 0} (\lambda w - \psi_o(w))$$

If $\rho_o > 0$ and $V(x_o) \leq \frac{\rho_o}{\lambda_o}(1-e^{-\lambda_o T})$, there exists a solution $x(\cdot)$ to the differential inclusion (4) satisfying

$$(29) \qquad \forall t \in [0,T],\ V(x(t)) \leq \begin{cases} \frac{\rho_o}{\lambda_o}(1-e^{\lambda_o(t-T)}) & \text{if } \lambda_o \neq 0 \\ -\rho_o(t-T) & \text{if } \lambda_o = 0 \end{cases}$$

If $\rho_o \leq 0$ and $\lambda_o < 0$, then there exists a solution $x(\cdot)$ to the differential inclusion (4) satisfying

$$(30) \qquad \forall t \geq 0,\ V(x(t)) \leq \frac{1}{\lambda_o}(\rho_o - c_o e^{\lambda_o t}) \text{ where } c_o = \rho_o - \lambda_o V(x_o)$$

Proof: We take $\psi(w) := \lambda_o w - \rho_o$

Remark:

Theorem 1 implies directly the asymptotic properties on U-monotone maps as they appear in corollaries 6.5.1 and 6.5.2, pp. 320-321 of Aubin-Cellina [1984].

Let $U : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ be a nonnegative function satisfying

$$(31) \qquad U(y,y) = 0 \text{ for all } y \in K$$

which plays the role of a semidistance (without having to obey the triangle inequality).

We assume that for all $x \in K$, $x \to U(x,y)$ is locally Lipschitz around K and we set

$$(32) \qquad U'(x,y)(v) := D_-(x\to U(x,y))(x)(v)$$

Let ϕ be a continuous map from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\phi(0) = 0$. We say that F is U-monotone (with respect to ϕ) if

$$(33) \qquad \forall x,y \in K, \forall u \in F(x), \forall v \in F(y), U'(x,y)(v-u) + \phi(U(x,y)) \leq 0$$

Let us assume that $c \in K$ is an equilibrium of F (a solution to

$0 \in F(c)$) and that $-F$ is U-monotone with respect to ϕ. Then we observe that by taking $V(x) := U(x,c)$, we have

$$(34) \qquad \psi_o(w) \leq \psi_1(w) \leq -\phi(w)$$

Let $w(\cdot)$ be a solution to the differential equation

$$(35) \qquad w'(t) + \phi(w(t)) = 0; \quad w(0) = U(x_o,c)$$

If F satisfies either (8) or (10), there exists a solution to the differential inclusion (4) satisfying

$$(36) \qquad U(x(t),c) \leq w(t) \text{ for all } t \in [0,T].$$

REFERENCES

Aubin, J-P, and A. Cellina

[1984] *Differential inclusions*, Springer Verlag, Berlin.

Aubin, J-P, and F.H. Clarke

[1977] Monotone invariant solutions to differential inclusions, J. London Math. Soc. 16, 357, 366.

Bouligand, G.

[1932] *Introduction à la géometrie infinitésimale directe* Gauthier-Villars, Paris.

Clarke, F.H.

[1975] Generalized gradients and applications. Trans.Am. Math.Soc. 205, 247-262.

Haddad, G.

[1981] Monotone trajectories of differential inclusions and functional differential inclusions with memory. Israel J.Math. 39, 83-100.

Yoshizawa, T.

[1966] *Stability theory by Liapunov's second method.* The Mathematical Society of Japan, Tokyo.