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ON STOCHASTIC COMPUTER NETWORK CONTROL

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1. During the past year a problem concerning distributed control of a communication system for transmitting information has arisen at IIASA [1]. One can imagine a general network of nodes (terminals) and arcs (transmission lines) designed to meet different types of demands entering the system from outside, to be serviced at the corresponding terminals.

If by chance no (open) direct line is available to transfer a demand from an initial node i to its proper destination j, then the problem is to choose the best of a number of arcs (i,k) leading from i.

According to [1], this problem is posed in the following way. It is assumed that:

a) each line α is independently opened or closed with the given probabilities $p(\alpha)$ and $1 - p(\alpha)$;

b) if a demand D appears at some node k, only local information is available, i.e. one knows which line from i is open or closed;

c) the probability distribution of the system, which in case (a) is determined by all probabilities $p(\alpha)$, remains constant; in other words it does not depend on a routing (line choice) policy.

One can note a few weaknesses of the model (abc). First, according to the main problem this model has to work in a situation when a typical route from the initial node to the destination consists of more than one arc; but in that case assumption (a) concerning line independence does not hold.

Secondly, the existence of optimal control under which the system evaluation process becomes stationary[1] is a very

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[1]In other words, in proper system space there is the equilibrium (stable) point which is invariant under our (control) transformation.

interesting and difficult problem; but in model (abc) it is simply taken for granted (see assumption (c)).

In considering model (abc) the optimality criterion is to choose for the current demand the route which leads to the destination with maximum probability. Remember that according to assumption (b) one has to make a decision by knowing the probability (system state) distribution and local information at every node reached.

I am not familiar enough with a real communication network to know how realistic is assumption (b), but I believe that if the problem of the routing policy for random demand flow is actually important then a detailed knowledge of the whole system situation is indispensable. One can imagine, for example, that a system operator knows the whole network situation. In this case, the optimality criterion suggested in [1] does not work at all, but the problem of routing policy is still valid.

By the way, if we are given some probability distribution

$$P = P(x) \quad , \qquad x \in X \quad , \tag{1}$$

on some space X of all possible system states x, then the problem of the optimal route leading to the destination with (corresponding) maximum probability is a pretty good exercise in dynamic programming.

Note that under assumption (a) the system state may be described by a set x of all closed lines, and the probability distribution of the network system considered is

$$P(x) = \prod_{\alpha \notin x} p(\alpha) \prod_{\beta \in x} (1 - p(\beta)) \quad . \tag{2}$$

Let us therefore consider the arbitrary probability distribution (1) on state-space X. Any possible system state $x \in X$ indicates specifically which arcs are open or closed. Moreover it may show a route of each demand entering the system and so forth.

Let

$$\pi_{kj} \; (\cdot/z)$$

be the maximum probability of reaching destination j under the condition that the corresponding demand has appeared at terminal k and a part z of the current system state x $(z \subseteq x)$ is known.

After arriving at terminal k we learn something new about the system state x; say we know its part y,

$$z \subseteq y \subseteq x \quad ,$$

which specifically indicates a set of all open lines from k. Under the condition that the open line

$$\alpha : k \longrightarrow u(y) \tag{3}$$

is chosen, the new (conditional) probability of reaching destination j is

$$\pi_{u(y),j}(\cdot/y)$$

and

$$\pi_{kj}(\cdot/z) = \sum_{y \supseteq z} \pi_{u(y),j}(\cdot/y) \; P\{y/z\} \tag{4}$$

where summation is done over all disjoint outcomes y which may happen after arriving at terminal k, and

$$P\{y/z\} = \frac{\sum_{x \subseteq y} P(x)}{\sum_{x \subseteq z} P(x)} \quad . \tag{5}$$

For model (abc) we have to assume that z is nothing and y indicates exactly the set of all open lines from k. (Formally one can treat $y \subseteq x$ as a set of all closed lines from k.) In this case

$$\pi_{kj} = \sum_{y} \pi_{u(y),j} \; P\{y\} \quad ,$$

$$P\{y\} = \prod_{\alpha \notin y} p(\alpha) \prod_{\beta \in y} (1 - p(\beta)) \; .$$

Let us look at the general equation (4). Because the probability $P\{y/z\}$ does not depend on the control parameter $u = u(y)$ which is the next terminal after k, we have to have

$$\pi_{u(y),j}(\cdot/y) = \max \tag{6}$$

over all open lines from k indicated by y. Thus in order to determine at each step the corresponding optimal control parameter $u = u(y)$, it is sufficient to determine all probabilities

$$\pi_{ij}(\cdot/y) \quad .$$

In the case of finite system space X our step-by-step process is bounded, so actually

$$\pi_{ij}(\cdot/y) = \pi_{ij}(n/y)$$

for some finite n where

$$\pi_{ij}(n/y) \quad ; \qquad n = 1,2,\ldots, \tag{7}$$

is the corresponding probability of reaching j from i in not more than k steps.

The probabilities $\pi_{ij}(n/y)$ are the monotone increasing sequences, and

$$\pi_{ij}(\cdot/y) = \lim \pi_{ij}(n/y) \quad .$$

Obviously

$$\pi_{ij}(1/y) = \begin{cases} 0 \quad , & (i,j) \in y \\ 1 \quad , & (i,j) \notin y \end{cases}$$

if y means the set of all closed lines from i, and generally

$$\pi_{ij}(1/y) = \frac{\sum\limits_{\substack{x \supseteq y \\ x \not\ni (i,j)}} P(x)}{\sum\limits_{x \supseteq y} P(x)} \quad . \tag{8}$$

Similarly to (4)-(6) we have the following recurrent equation:

$$\pi_{ij}(n+1/z) = \sum_{y \supseteq z} \pi_{u(y),j}(n/y) \ P\{y/z\} \tag{9}$$

where $u = u(y)$ has to be the maximum point of the corresponding probability $\pi_{kj}(n/y)$ as a function of k, namely

$$\pi_{uj}(n/y) = \max \ \pi_{kj}(n/y) \tag{10}$$

over all open lines (i,k), $(i,k) \notin y$.

2. The optimal control routing described above depends on the corresponding probability distribution (1). We mentioned already the problem of stationary distribution which is invariant under the system transformations governed by the routing control.

Let us consider this problem in a case when the demand flow is of the Poisson type. Say a demand D appears during time interval Δt at the initial terminal i with the probability

$$\lambda_{ij}(D) \ \Delta t + o(\Delta t) \tag{11}$$

(where j is the corresponding destination) and is served (independently) at j according to exponential probability distribution with the parameter

$$\mu_j(D) \quad . \tag{12}$$

In this case, the system evaluation process

$$x = x(t) \quad , \qquad t \geq t_0 \quad , \tag{13}$$

is of the Markov type (with respect to the obvious system state description) with the transition probability matrix Q depending on our routing control, which itself depends on the choice of the probability distribution (1). Let us indicate such control as

$$u = u(P)$$

and set

$$Q = Q[u(P)] \quad ,$$

where P is the corresponding (a priori) probability distribution.

From well known ergodic properties one can be sure that for any P there is a (limit) stationary distribution P*:

$$P^* = P^* Q[u(P)] \quad . \tag{14}$$

The problem concerns a probability distribution P* such that

$$P^* = P^* Q[u(P^*)] \quad . \tag{15}$$

In the case of distribution $P = P^*$, the system process $x = x(t)$ governed by control $u = u(P)$ is stationary; in particular the probability state distribution $P = P^*$ remains constant.

Formula (14) gives us a non-continuous mapping

$$Q : P \longrightarrow P^*$$

of a convex set of all probabilistic vectors $P = \{P(x)\}$ into itself, and we have no special idea how to find the fixed point $P = P^*$ if such a point exists.

Note that under the control $u = u(P)$ with respect to any fixed distribution P the corresponding stationary distribution P* is the unique solution of the linear equation

$$P^* R[u(P)] = 0 \tag{16}$$

where

$$R[u(P)] = \{R_{xy}[u(P)]\}$$

denotes the transition densities matrix of the system's homogeneous ergodic Markov process (13) which can be easily determined by the initial parameters $\lambda_{ij}(D)$, $\mu_j(D)$ (see (11), (12)), and the routing control $u = u(P)$.

Assume that the current system state x means that the serving demands $D_1, \ldots, D_m$, keep the corresponding lines

$$i_1 \rightarrow \cdots \rightarrow j_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$i_m \rightarrow \cdots \rightarrow j_m$$

and that the waiting demands $D_{m+1},\ldots,D_n$ occupy the lines

$$i_{m+1} \rightarrow \cdots \rightarrow j_{m+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$i_n \rightarrow \cdots \rightarrow j_n$$

(where possibly $i_k = j_k$; $k = m + 1,\ldots,n$). According to the routing control $u = u(P)$ each new arriving demand D has the certain route

$$i \rightarrow \cdots \rightarrow j \quad ,$$

so one can easily find out which new system state y is achievable from x during a short time interval Δt with a significant probability of $O(\Delta t)$. For example, a service of some D_k, $k \leq m$, may be finished, so the corresponding line

$$i_k \rightarrow \cdots \rightarrow j_k$$

will be open. This may happen with probability

$$\mu_{i_k j_k}(D_k)\ \Delta t + o(\Delta t)$$

and according to an additional routing prescription some of the waiting demands may be transferred further, which gives us the certain new system state y. Another significant possibility is for some new demand D to appear; then the corresponding line

$$i \rightarrow \cdots \rightarrow j$$

will be closed, which happens with probability

$$\lambda_{ij}(D)\ \Delta t + o(\Delta t) \quad .$$

Thus, the problem is to find a probabilistic solution P^* of the (non-linear) equation

$$P^* R \left[u(P^*)\right] = 0 \tag{17}$$

where $R\left[u(P^*)\right]$ is the matrix with components $R_{xy}\left[u(P^*)\right]$ which are the transition probability densities described above.

In any case, if one gets numerically a probabilistic solution P^* of equation (17), then under the control $u = u(P^*)$ one can be sure that the actual state distribution tends to P^*, which in the obvious sense is the equilibrium point; with respect to the state distribution $P = P^*$ we then have the maximum probability of reaching the destination.

3. As was mentioned, the optimality criterion considered for the routing control does not work when the system operator knows the complete situation, in other words when one has to control the system process (13) by knowing the corresponding system state $x = x(t)$.

Let us consider the routing control

$$u = u\left[x(t)\right] \tag{18}$$

according to which any appearing demand D has to be transferred in a proper way from the initial node i towards destination j. It may happen that for the current system state $x = x(t)$ it is impossible to transfer D from i to j. Let us call this situation a failure.

Let $Y_{ij}(x,D)$ be a set of all possible system states which are consistent with transmission of demand D from i to j under the condition that the current system state is x. The failure means that the corresponding set $Y_{ij}(x,D)$ is empty:

$$Y_{ij}(x,D) = 0 \quad . \tag{19}$$

Remember, we have a demand inflow of the Poisson type with the parameters $\lambda_{ij}(D)$ (see (11)), and it is easy to verify that a probability of failure during a short time interval $(t,t + \Delta t)$ is

$$\alpha(x)\ \Delta t + o(\Delta t) \quad ,$$

$$\alpha(x) = \sum_{\substack{i,j,D\ : \\ Y_{ij}(x,D)=0}} \lambda_{ij}(D) \quad . \tag{20}$$

The failure probability under the condition that some demand appears is

$$\pi(x) = \sum_{\substack{i,j,D\ : \\ Y_{ij}(x,D)=0}} \lambda_{ij}(D) \Bigg/ \sum_{i,j,D} \lambda_{ij}(D) \tag{21}$$

where x is the current system state. The appearing demand may be transferred from the initial node i to destination j in different ways; accordingly, the system will be transferred from x to one of the states $y \in Y_{ij}(D)$. It seems quite likely that one may be interested in minimizing the failure probability by choosing such a route leading from x to the new system state y, for which

$$\pi(y) \longrightarrow \min_{y \in Y_{ij}(x,D)} \quad . \tag{22}$$

A loss $\varphi_{ij}(D)$ may be associated with the failure of transmitting demand D from i to j. In this case, the loss average due to possible failure at system state x is

$$\Phi(x) = \sum_{\substack{i,j,D\ : \\ Y_{ij}(x,D)=0}} \varphi_{ij}(D)\ \lambda_{ij}(D) \Bigg/ \sum_{i,j,D} \lambda_{ij}(D) \quad , \tag{23}$$

and the optimality criterion may be generalized to the following:

$$\Phi(y) \longrightarrow \min_{y \in Y_{ij}(x,D)} \quad .$$

Of course there may be other operation goals. Say one is interested in minimization of the total loss expectation during a fixed time interval (t_0,T). Let the expectation of loss during a short time interval $(t,t+\Delta t)$ under condition

$x(t) = x$ be

$$\sum_{\substack{i,j,D:\\ Y_{ij}(D)=0}} \varphi_{ij}(D)\left[\lambda_{ij}(D)\ \Delta t + o(\Delta t)\right] = \Phi(x)\ \Delta t + o(\Delta t)$$

where

$$\Phi(x) = \sum_{\substack{i,j,D:\\ Y_{ij}(x,D)=0}} \varphi_{ij}(D)\ \lambda_{ij}(D) \quad ; \tag{25}$$

then the expected value of the total loss has to be defined as

$$E \int_{t_0}^{T} \Phi(x(t))\ dt \quad . \tag{26}$$

(Note that in the case $\varphi_{ij}(D) = 1$ we deal with the expected number of failures.)

Standard dynamic programming may be applied to determine optimum functions $F_x(t)$ of t $(x \in X)$,

$$F_x(t) = \min E \left\{ \int_t^T \Phi(x(s))\ ds \Big/ x(t) = x \right\} , \tag{27}$$

where the minimum is taken over all possible Markov type routing controls

$$u = u(x,t)$$

(see, for example, [2]).

Obviously the optimal control has to be of the following type: if demand D appears at time t when the system is at state x, it has to be transferred in such a way that the corresponding new system state $y \in Y_{ij}(x,D)$ gives the minimum future loss:

$$F_y(t) \longrightarrow \min_{y \in Y_{ij}(x,D)=0} \quad . \tag{28}$$

Formally this control description may be verified by considering the conditional loss expectation with a fixed system trajectory up to moment τ of the n^{th} demand appearance under the condition that our routing control is actually optimal after moment τ:

$$E \int_t^T \Phi(x(S))\, ds$$

$$= EE \left\{ \int_t^\tau \Phi(x(s))\, ds + F_{x(\tau+0)}(\tau) \Big/ x(s),\ s \leq \tau \right\} .$$

That is, the routing policy (28) gives us the minimum $F_{x(\tau+0)}(\tau)$ because of our choice of system state $x(\tau + 0) = y$ which may be achieved from the previous state $x(\tau) = x$.

Let us consider an expected loss of the following general type:

$$E \int_{t_0}^\infty \Phi(x(t))\, c(t)\, dt \quad , \tag{29}$$

where $c(t)$, $t \geq t_0$ is some weight function. If

$$c(t) = \begin{cases} 1 \ , & t_0 \leq t \leq T \\ 0 \ , & t > T \end{cases} \quad ,$$

we are dealing with a loss of the previous type (28). Let us set $t_0 = 0$ and

$$c(t) = e^{-at} \quad , \qquad t > 0 \quad ; \tag{30}$$

this weight function may be treated as a discount factor.

In this particular discount case the dynamic programming objective functions $F_x(t)$ which give us the corresponding expected loss minimum have the following property:

$$\begin{aligned} F_x(t) &= \min E \left\{ \int_t^\infty e^{-as}\, \Phi(x(s))\, ds \Big/ x(t) = x \right\} \\ &= e^{-at} \min E \left\{ \int_0^\infty e^{-as}\, \Phi(x(s + t))\, ds \Big/ x(t) = x \right\} \\ &= e^{-at} F_x(0) \quad . \end{aligned} \tag{31}$$

Thus if we set

$$F_x = F_x(0)$$

the optimal control may be described as follows: demand D appearing at system state x has to be transferred from the initial node i to destination j in such a way that for the new system state y we have

$$F_y \longrightarrow \min_{y \in Y_{ij}(x,D)=0} \qquad (32)$$

(compare (22), (24), and (28)).

Let us consider this type of routing control for an arbitrary objective function F_x. We suggest defining the stationary optimal objective function

$$F_x \quad , \qquad x \in X \quad ,$$

as a function with respect to which the routing control is optimal in the following sense.

Let Q(F) be the transition probability matrix of system process $x = x(t)$ governed by the control defined above with respect to objective function F. Let $P = P(F)$ be the corresponding stationary probability

$$P = PQ(F)$$

which is the (unique) probabilistic solution of the linear equation

$$PR(F) = 0 \quad , \qquad (33)$$

where R(F) is the transition densities matrix. Then the expected value of the total loss during any time interval (t_0,T) is

$$E \int_{t_0}^{T} \Phi(x(s))\, ds = \left[\sum_x \Phi(x)\ P(x) \right] (T - t_0) \quad ,$$

and the optimal objective function F has to give us the minimum loss:

$$\sum_x \Phi(x)\, P(x) = \min_F . \tag{34}$$

Remember that $\Phi(x)$ was defined by formula (25), and the stationary probability distribution $P = \{P(x)\}$ depends on the objective function F because of relationship (33).

There is only a finite number of possible system states x and of all possible routes from any node i to any destination j. Thus we are actually dealing with a finite number of possible routing controls, and there is no question about the existence of an optimal objective function. But how do we determine one such function? Concerning this problem we wish to make a remark which may not be useless: namely, that under any homogeneous routing policy described by the corresponding operation function

$$F = F(x) \quad , \qquad x \in X \quad ,$$

we are dealing with the ergodic homogeneous Markov process

$$x = x(t) \quad , \qquad t \geq 0 \quad ;$$

and for any initial system state $x(0) = x$ we have

$$E\left\{\Phi(x(t)) \Big/ x(0) = x\right\} = \sum_y \Phi(y)\, P\left\{x(t) = y \Big/ x(0) = x\right\}$$

$$\xrightarrow[t \to \infty]{} \sum_y \Phi(y)\, P(y) \quad ,$$

where $P = P(y)$, $y \in X$, is the corresponding stationary system distribution and the convergence is uniform over all $x \in X$. (You will recall that there is only a finite number of different system states x.)

Now we have

$$G_x^a = E\left\{\int_0^\infty ae^{-at}\Phi(x(t))\ dt \Big/ x(0) = x\right\}$$

$$= \int_0^\infty ae^{-at}\ E\left\{\Phi(x(t))\Big/ x(0) = x\right\} dt$$

$$\xrightarrow[a \to 0]{} \sum_y \Phi(y)\ P(y) \quad ,$$

where $P = P(y)$, $y \in X$, is the corresponding stationary system distribution and the convergence is uniform over all operation functions F (you will recall that there is only a finite number of different routing strategies $u = u(x)$, $x \in X$).

Let a parameter "a" be a such that

$$|G_x^a - \sum_y \Phi(y)\ P(y)| \leq \varepsilon$$

and let

$$P^a = P^a(x) \quad , \qquad x \in X$$

be a stationary system distribution with respect to the routing policy determined by the operation function

$$F^a(x) = \min_u G_x^a \quad , \qquad x \in X \quad .$$

Obviously

$$\sum_x \Phi(x)\ P^a(x) \leq \min_u \sum_x \Phi(x)\ P(x) + \varepsilon \quad ,$$

and moreover

$$\sum_x \Phi(x)\ P^a(x) = \min_u \sum_x \Phi(x)\ P(x)$$

for all sufficiently small parameters a, $a \to 0$, so asymptotically ($a \to 0$) the operation function

$$F(x) = \min_{u} E \left\{ \int_{0}^{\infty} e^{-at} \Phi(x(t)) \, dt \Big/ x(0) = x \right\}$$

is stationary optimal (you remember there is only a finite number of different values $\sum \Phi(x) \ P(x)$).

One may believe that a similar property exists with respect to the operation function of the type

$$F(x) = \min_{u} E \left\{ \int_{0}^{T} \Phi(x(t)) \, dt \Big/ x(0) = x \right\}$$

$$(T \to \infty)$$

for which computation the standard dynamic programming methods may be applied.

References

[1] Bell, D. and A. Butrimenko. "An Adaptive Routing Technique for Channel Switching Networks." Laxenburg, Austria, IIASA RM-74-18, 1974.

[2] Bellman, R. Dynamic Programming. Princeton, New Jersey, Princeton University Press, 1959.